Faddeev - Popov ~~deta~~determinant

§ Simple example

$$I = \int_{-\infty}^{\infty} dx\,dy \; f(x,y)$$

assume that

$$f(x,y) = f(\sqrt{x^2+y^2})$$

then, $\exists$ rotational sym.

$$f(x,y) = f(x(\theta), y(\theta))$$

where $\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Faddeev - Popov method

(i) think of the identity below

sym. transf parameter

$$1 = \Delta(x,y) \int_0^{2\pi} d\theta \; \underbrace{\delta(y(\theta))}_{\text{"gauge" condition!}}$$

then, one can argue that

$$\Delta(x(\alpha), y(\alpha)) = \Delta(x,y) \; !$$

why ?

$$1 = \Delta\,(x(\alpha), y(\alpha)) \cdot \int_0^{2\pi} d\theta \;\; \delta\,(\,y(\overset{\theta'}{\overset{\shortparallel}{\alpha+\theta}})\,)$$

$$= \int_0^{2\pi} d\theta' \;\; \delta\,(\,y(\theta')\,)$$

$$\therefore \quad 1 = \Delta\,(x(\alpha), y(\alpha)) \cdot \Delta^{-1}\,(x, y) \quad \blacksquare$$

(ii) insert the identity inside the integral

$$I = \int dx\,dy \;\; \Delta(y) \int_0^{2\pi} d\theta \;\; \delta(y(\theta))\, f(x,y)$$

$$= \int dx'dy' \;\; \Delta(y') \int_0^{2\pi} d\theta \;\; \delta(y'(\theta))\, f(x',y')$$

& choose $(x', y') = (x(-\theta), y(-\theta))$

ⓐ $dx' \wedge dy' = dx(-\theta) \wedge dy(-\theta) = dx\,dy$

ⓑ $\Delta(y(-\theta)) = \Delta(y)$

ⓒ $f(x(-\theta), y(-\theta)) = f(x, y)$

$$I = \int dx\,dy\; \Delta(y) \underbrace{\int_0^{2\pi} d\theta}\; \delta(y)\, f(x,y)$$

$$= \left\{ \int_0^{2\pi} d\theta \right\} \times \left\{ \underbrace{\int dx\,dy\; \Delta(y)\, \delta(y)\, f(x,y)} \right\}$$

$$= \int dx\; \Delta(y=0)\, f(x, y=0)$$

(iii) how to compute the $\underbrace{\Delta(x,y)}_{\text{Faddeev-Popov determinant.}}$?

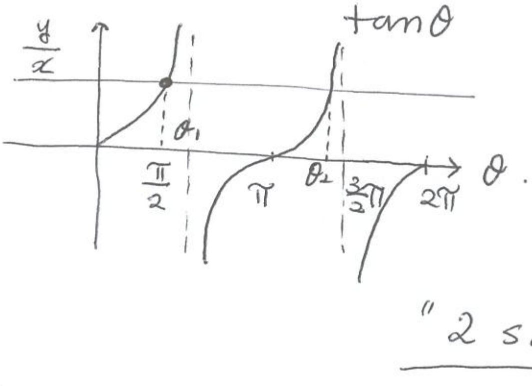
ⓐ $\delta(y(\theta)) = a_1\, \delta(\theta-\theta_1) + b_1\, \delta(\theta-\theta_2)$

where

$$\tan\theta_i = \frac{y}{x}$$

$$\Updownarrow$$

$$y\cos\theta_i - x\sin\theta_i = 0.$$



"2 solution exists"

ⓑ a_1 & b_1 ?

at $\theta = \theta_i$, $\quad x \cos\theta_i + y \sin\theta_i = \pm\sqrt{x^2+y^2}$

infinitesimal

near $\theta = \theta_i$, i.e., $\quad \theta = \theta_i + \varepsilon_i$

$$y(\theta) = y(\theta_i) - \varepsilon_i \big(\underbrace{y \sin\theta_i + x \cos\theta_i}_{= \pm\sqrt{x^2+y^2}} \big)$$

$$\frac{1}{\sqrt{x^2+y^2}} \sum_{i=1}^{2}{}' \delta(\varepsilon_i)$$

$$\shortparallel$$

$$\therefore \quad \delta(y(\theta)) = \delta(\varepsilon_1 \sqrt{x^2+y^2}) + \delta(-\varepsilon_2 \cdot \sqrt{x^2+y^2})$$

$$\int_0^{2\pi} d\theta \; \delta(y(\theta)) = \int d\varepsilon_1 \frac{1}{\sqrt{x^2+y^2}} \delta(\varepsilon_1 \cancel{\sqrt{x^2+y^2}}) + \int d\varepsilon_2 \, \delta(\varepsilon_2) \frac{1}{\sqrt{x^2+y^2}}$$

$$= \frac{2}{\sqrt{x^2+y^2}}$$

(c) $\quad 1 = \Delta(x, y) \cdot \dfrac{2}{\sqrt{x^2+y^2}}$

or $\quad \Delta(x,y) = \dfrac{\sqrt{x^2+y^2}}{2}$

(iv) the integral then becomes

$$I = \int_0^{2\pi} d\theta \; \underbrace{\int_{-\infty}^{\infty} dx \cdot \frac{\sqrt{x^2}}{2} \cdot f(\sqrt{x^2})}$$

$$= \int_0^{\infty} dr \cdot r \cdot f(r)$$

$$= \int_0^{\infty} r\,dr \int_0^{2\pi} d\theta \; f(r)$$

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§ Random Matrix Theory & Vandermonde det.

$$I = \int [DM] \; f(M)$$

e.g. $f(M) = e^{-\frac{1}{2}\mathrm{tr} M^2}$

$$f(M) = f(UMU^{+})$$

unitary transf.

note also that $[DM] = [DM']$

where $M' = UMU^{+}$

Use

Faddeev-Popov method!

(i) note that, ∀ a given Hermitian matrix,

$\exists \; U_0$ s.t. $\quad U_0 M U_0^{+} = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$.

real eigenvalue

gauge condition.

$$\Rightarrow \prod_{i \neq j} \delta\left((UMU^{+})_{ij}\right) \quad : \quad N(N-1) \text{ conditions}$$

$$1 = \Delta(M) \int \underbrace{[DU]}_{} \prod_{i \neq j} \delta((UMU^{+})_{ij})$$

well-defined measure

s.t. $DU = D(UU')$

one can again show that $\Delta(M) = \Delta(U'MU^{+})$

unitary matrix

(ii) the integral can be rewritten as

sym. transf.

$$I = \int [DM] \int [DU] \; \Delta(M) \prod_{i \neq j} \delta((UMU^{+})_{ij}) f(M)$$

Hermitian

change of variable $M \to M'$

& choose $M' = U^{+}MU$

$$= \int [DM] \int [DU] \; \Delta(M) f(M) \prod_{i \neq j} \delta(M_{ij})$$

since

$[DM'] = [DM]$,

$f(M) = f(U^{+}MU)$, & $\Delta(U^{+}MU) = \Delta(M)$

$$= \underbrace{\left\{ \int [DU] \right\}}_{vol(U(N))} \times \underbrace{\int [DM] \prod_{i \neq j} \delta(M_{ij}) \Delta(M) f(M)}$$

$$\int \prod_{i=1}^{N} dM_i{}^i \; \Delta(\cancel{M_i^i, M_i{}^j = 0} \; M)$$

$$M = \begin{pmatrix} M_1{}^1 & & 0 \\ & M_2{}^2 & \\ 0 & & \ddots M_N{}^N \end{pmatrix}$$

$$f(M)$$

$$= \int \prod_{i=1}^{N} d\lambda_i \, \Delta(\lambda_i) \, e^{-\frac{1}{2} \sum_{i=1}^{N} \lambda_i^2}$$

(iii) $\Delta(\lambda_i) = ?$

unique? → No. → $\exists$ $U(1)^N$ family

$U_0 M U_0^\dagger = \Lambda$ where $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$.

consider U near U_0, i.e.,

$$U = e^{iT} U_0$$

infinitesimal Hermitian matrix.

$$\cong (1 + iT) U_0.$$

$$\prod_{i\neq j}\delta\left((UMU^\dagger)_i{}^j\right) = \prod_{i\neq j}\delta\left[(\Lambda + i[T,\Lambda])_i{}^j\right]$$

$i \neq j$ $\qquad\qquad$ $i \neq j$

$$= \prod_{i\neq j}\delta\left(T_i{}^j(\lambda_i-\lambda_j)\right)$$

then

$$\int \underbrace{[DU]}_{\tilde{\prod\limits_i} dT_i{}^i \,\times\, \prod\limits_{i\neq j} dT_i{}^j} \prod_{i\neq j}\delta\left(iT_i{}^j\cdot(\lambda_i-\lambda_j)\right) = \left[\prod_{i<j}(\lambda_i-\lambda_j)^{-2}\right]\times \text{const}$$

related to non-uniqueness of U_0

indep. of $\{\lambda_i\}$

which implies that $\Delta(\lambda_i) \propto \prod_{i<j}(\lambda_i-\lambda_j)^2$

$$I = (\text{const}_N)\times\int\prod_{i=1}^{N}d\lambda_i\,\underbrace{\prod_{i<j}(\lambda_i-\lambda_j)^2}\,e^{-V(\lambda_i)}$$

~~Van De~~ Vandermonde determinant!