Orthogonal Polynomial Method & Double Scaling Limit ①

remark

$\boxed{Q}$ where does the singular behavior of g come from?

fractional power

essential to have the double-scaling limit.

$\boxed{A}$ "$\lambda \sim 2\sqrt{a(g)}$" determines the singular behavior.
~~to answer the question, it's useful to consider~~

the orthogonal poly. method is extremely useful!
to show this

§ orthogonal polynomial

(i)

$$\int_{-\infty}^{\infty} d\lambda \; e^{-NV(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}$$

definition

$P_n(\lambda)$: n-th order polynomial

e.g. $V(\lambda) = \frac{1}{2}\lambda^2$ then, $P_n(\lambda)$ becomes Hermite polynomial.

$$e^{-NV(\lambda)/2} P_n(\lambda) \propto \psi_n(\lambda)$$

~~wavefunction~~ energy eigenstate of harmonic oscillator

(ii) recursion relation.

depends on "g"

$$\underline{\lambda P_n(\lambda) = P_{n+1}(\lambda) + S_n P_n(\lambda) + R_n P_{n-1}(\lambda)}$$

Q no other terms such as $P_{n-2}(\lambda)$?

$$\int_{-\infty}^{\infty} d\lambda \; e^{-NV(\lambda)} \lambda P_n(\lambda) P_{n-2}(\lambda) = \int_{-\infty}^{\infty} d\lambda \; e^{-NV(\lambda)} P_n(\lambda) \left(P_{n-1}(\lambda) + \begin{array}{c}\text{polynomial}\\ \text{of lowest deg}\end{array} \right)$$

$$= 0 ! \quad \text{identically}$$

$\therefore$ $\lambda P_n(\lambda)$ does not contain $P_{n-2}(\lambda)$ .... $P_0(\lambda)$

(iii) two relations that $R_n(\lambda)$ satisfy.

① $$\int d\lambda \; e^{-NV(\lambda)} \lambda P_{n-1}(\lambda) P_n(\lambda) = h_n(g)$$

$$\| $$

$$h_{n-1}(g) \cdot R_n(g)$$

Hence $R_n = h_n / h_0$

② $$\int d\lambda \; e^{-NV(\lambda)} \; \frac{d}{d\lambda} P_n \cdot P_{n-1} = n\, h_{n-1}$$

$$= N \int d\lambda \; e^{-NV(\lambda)} (\lambda + 4g\lambda^3) \cdot P_n \cdot P_{n-1}$$

$$= N R_n h_{n-1} + 4gN \int e^{-NV(\lambda)} \left(\cdots + R_{n+1} R_n P_{n-1} + R_n^2 P_{n-1} + R_n R_{n-1} P_{n-1} \right) P_{n-1}$$

$$= N \cdot R_n \left(1 + 4g \left(R_{n+1} + R_n + R_{n-1} \right) \right) h_{n-1}$$

$$\therefore \quad \boxed{\frac{n}{N} = R_n \left(1 + 4g \left(R_{n+1} + R_n + R_{n-1} \right) \right)}$$

(iv) partition function

$$Z[g] = \int d\lambda_1 \cdots d\lambda_N \; \Delta(\lambda_i) \; e^{-N(V(\lambda_1) + \cdots V(\lambda_N))}$$

by def

$$= \det{}^2 \left(\lambda_i^{j-1} \right)$$

$$= \det{}^2 \left(P_{j-1}(\lambda_i) \right)$$

$$= \left[\varepsilon^{j_1 \cdots j_N} P_{j_1}(\lambda_1) P_{j_2}(\lambda_2) \cdots P_{j_N}(\lambda_N) \right]^2$$

one can obtain that

$$Z[g] = N!\, h_0 h_1 \cdots h_N(g)$$

$$Z[g]\Big/Z[0] = \left(\frac{h_0(g)}{h_0(0)}\right)^N \left(\frac{R_1(g)}{R_1(0)}\right)^{N-1} \left(\frac{R_2(g)}{R_2(0)}\right)^{N-2} \cdots \left(\frac{R_{N-1}(g)}{R_{N-1}(0)}\right)$$

§ large N limit

(i) recursion relation in the $N \to \infty$ limit

$$\frac{n}{N} \equiv x . \qquad r_0\left(x = \frac{n}{N}, g\right) \equiv R_n(g)$$

then, the recursion relation can be written as

$$x = r_0(x,g)\Big(1 + 4g\,\underbrace{(r_0(x+\epsilon,g) + r_0(x,g) + r_0(x-\epsilon,g))}_{\text{leading term} \simeq 3 r_0(x,g)}\Big)$$

$$\therefore \quad r_0 \cdot (1 + 12 g r_0) = x \quad \text{or} \quad r_0(x,g) = \frac{-1 + \sqrt{1 + 48 g \cdot x}}{24 g}$$

$$r_0(x=1, g) = a^2(g)$$ end-point of eigenva

(ii) partition function.

$$\log\left(Z[g] \Big/ Z[0] \right) = N \log\left[h_0(g) \Big/ h_0(0) \right]$$

$$+ \; N \cdot \sum_{n=1}^{N-1} \left(1 - \frac{n}{N}\right) \log\left(R_n(g) \Big/ R_n(0) \right)$$

N→∞ limit

$$\cong \; N^2 \int_0^1 dx \; (1-x) \log\left(r_0(x,g) \Big/ \underbrace{r_0(x,0)}_{=\,x} \right)$$

$$- \; E^{h=0}(g).$$

one can explicitly show that

$$\int_0^1 dx \; (1-x) \log\left(r_0(x,g) \Big/ x \right) \quad \text{where} \quad r_0(x,g) = \frac{-1 + \sqrt{1+48gx}}{24g}$$

$$= \; \frac{1}{2} \log a^2(g) \; - \; \frac{1}{24} \left(9 - a^2(g)\right) \left(a^2(g) - 1\right) \quad \square$$

(iii) § singular behavior

(i)

$$-E^{h=0}_{\rm sing}(g) = \int_0^1 dx\ (1-x)\ \underbrace{\log r_0(x,g)}$$

~~fractional power of x & g~~

note again that

$$\underbrace{r_0\,(1+12g\,r_0)} = x$$

$$= W(r_0)$$

$W'(r=r_c)=0.$

If $W(r_0) = W(r_c) + \frac{1}{2}W''(r_c)\,(r_0-r_c)^2 + \underbrace{\cdots}$

~~scalable~~ negligible in a certain limit

we will discuss this limit later ??

$$x - W(r_c) \cong \frac{1}{2}W''(r_c)\,(r-r_c)^2$$

$$\to \sqrt{\frac{2\,(x-W(r_c))}{W''(r_c)}} = r_0 - r_c \qquad \text{i.e.,}\quad r_0(x) \cong r_c\left(1 + a(g)\sqrt{x-b(g)}\right)$$

need to check non-singular in g

then, $$\int_0^1 dx\ (1-x)\ a\log r_0(x,g) \sim \int_0^1 dx\ (1-x)\ a(g)\sqrt{x-b(g)}$$

$$\sim (1-b(g))^{\frac{5}{2}}$$

? singular behavior ?

(ii) $W'(r_0) = 1 + 24 g r_0 = 0$ when $r_0 = r_c$.

$\therefore \quad r_c = -1/24g$

$\therefore$ the limit that approximate $W(r_0) \cong W(r_c) + \frac{1}{2} W''(r_c)(r_0 - r_c)^2$

is $r_0 \to r_c$

$$\frac{-1 + \sqrt{1 + 48 g x}}{24 g} \cong -1/24g \qquad \therefore \quad x \sim -1/48g$$

Since we are interested in $g \to g_c = -1/48$, $x \sim 1$

(iii) $W''(r_c) = 24 g$

$$W(r_c) = r_c + 12 g r_c^2 = -\frac{1}{24g} + \left(-\frac{1}{24g}\right)^2 12 g$$

$$= -\frac{1}{48g}$$

Hence $r_0(x) \cong r_c + \left(1 + \underbrace{(-24g)\sqrt{\frac{1}{12g}\left(x + \frac{1}{48g}\right)}}\right)$

$$= \sqrt{-2\left(x + \frac{1}{48g}\right)} \cdot \sqrt{-24g}$$

$$= \sqrt{2\left(-\frac{1}{48g} - x\right)} \sqrt{-24g}$$

$$- E^{h=0}_{\text{sing}}(g) \sim \int_0^1 dx\,(1-x)\cdot \underbrace{\sqrt{2\left(-\frac{1}{48g}-x\right)}\cdot\sqrt{-24g}}_{=\cancel{\sqrt{\frac{-1}{24g}}}\cdot\sqrt{(1+48gx)}}$$

singular terms only & $g \to g_c = -1/48$

$$\cong \frac{4}{15}\cdot\underbrace{\left(-\frac{1}{48g}\right)^2}_{\sim 1}\left(1+48g\right)^{\frac{5}{2}}$$

$48g = -1 + h$

(iv) $\left\{ \begin{array}{l} x \sim 1 \\ \& \\ g \to g_c \end{array} \right\}$ & "$\lambda \to 2a(g)$" end-point of eigenvalues.

one can verify that (proof in a <u>separate note</u>)

$$\rho_0(\mu) = \frac{1}{\pi}\int_0^1 dx\,\frac{1}{\sqrt{4r_0(x,g)-\mu^2}}\,\theta\left(4r_0(x,g)-\mu^2\right)$$

density of eigenvalues (large N)

change of variable $x \longrightarrow r_0(x,g)$. monotonic function of x

$$dx = W'(r)\,dr$$

$$W(r) = x$$
$$W(a^2(g)) = 1$$

$$\rho(\mu) = \frac{1}{\pi} \int_{\mu^2/4}^{a^2(g)} dr \, \frac{W'(r)}{\sqrt{4r - \mu^2}}$$

$\boxed{x \simeq 1} \longleftrightarrow \begin{cases} W'(r) \simeq 0 & \text{when } g \sim g_c = -1/48 . \\ r \cong r_c . \end{cases}$

Hence, the above integral ~~get~~ has a contribution only from $\mu^2 \cong 4 r_c = \cancel{\tfrac{1}{24}}\ 8$

$\therefore \quad \boxed{\mu \cong \pm 2\sqrt{2}}$ $\cdots$ the <u>end-point of eigenvalues</u>

$$a^2(g) = \frac{-1 + \sqrt{1+48g}}{24g} \cong 2$$

$$\therefore \quad \lambda_{end} = \pm 2 a(g) \cong \pm 2\sqrt{2}$$