High1 Workshop on Particle, String and Cosmology 2025

A New Genuine Multipartite Entanglement Measure:
from Qubits to Multiboundary Wormholes

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January 13, 2025

asia pacific center for theoretical physics

A New Genuine Multipartite Entanglement Measure: from Qubits to Multiboundary Wormholes

Today: 2411.11961

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→ (GIST) Soon

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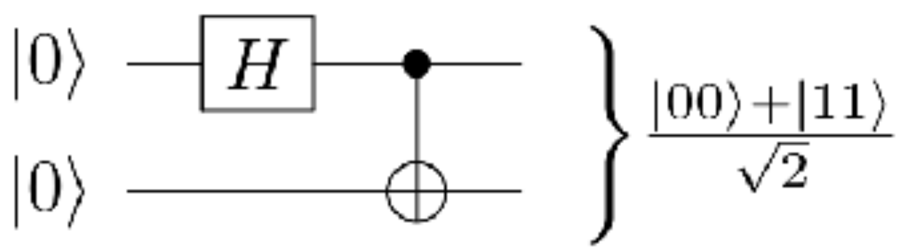

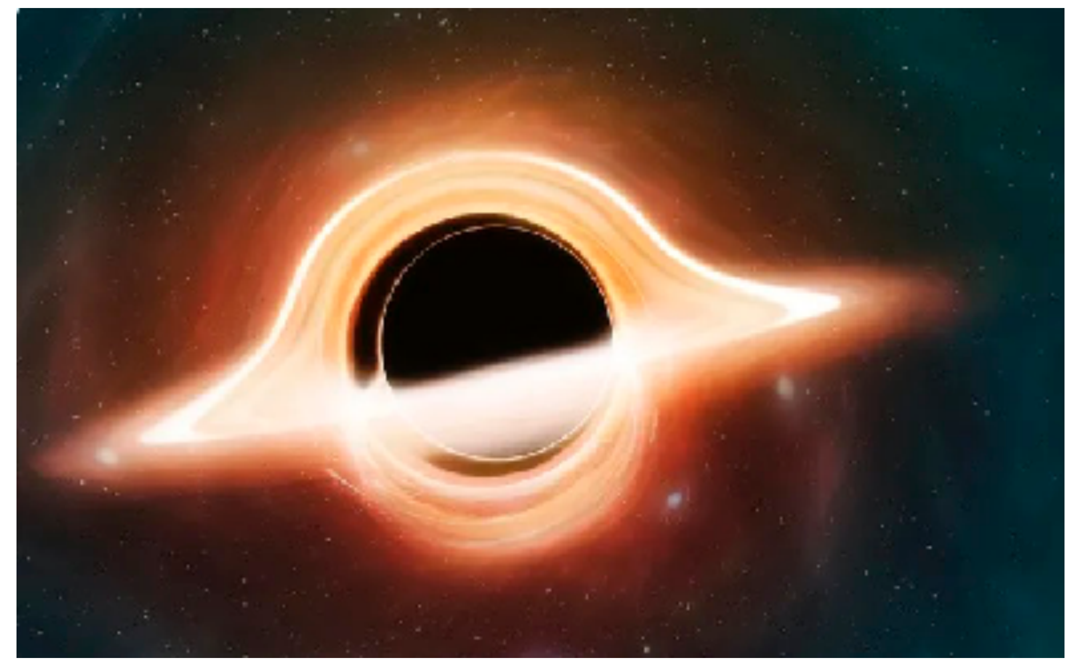

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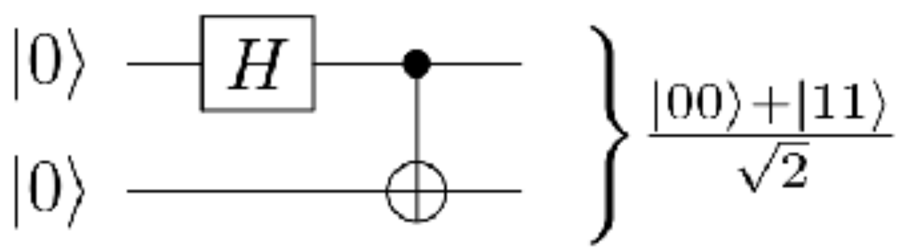
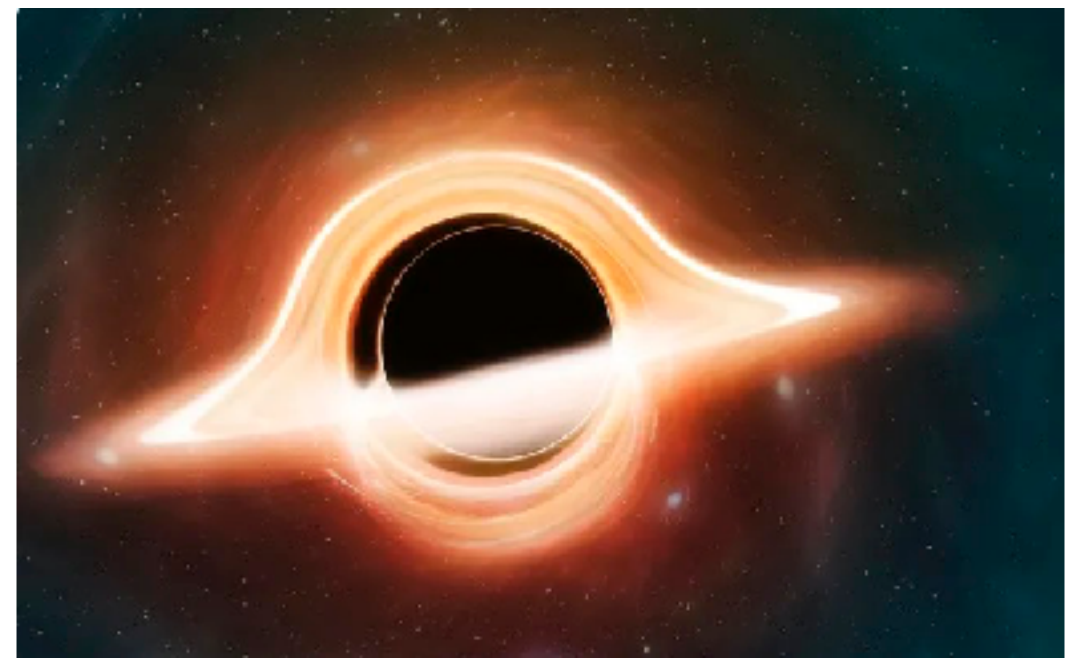
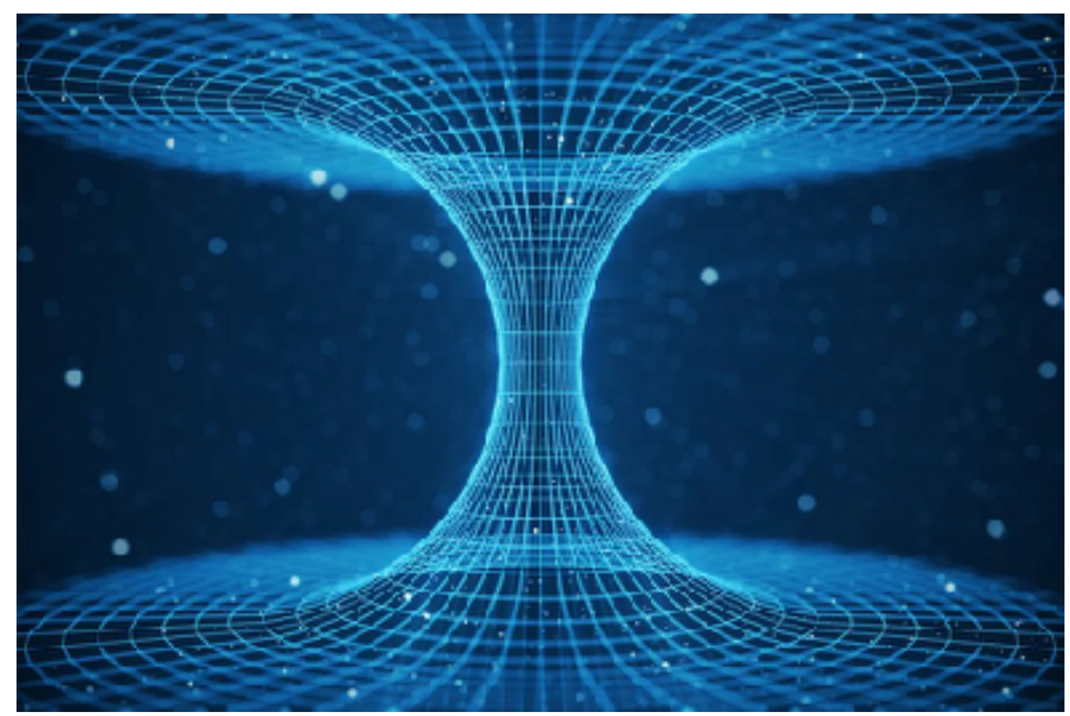




$$|\psi\rangle = \frac{1}{\sqrt{16}}\big(|0000000\rangle + |0001011\rangle + |0011001\rangle + |0110010\rangle + |0110100\rangle\big)$$

$$+\frac{1}{\sqrt{16}}\big(|1000110\rangle + |1010011\rangle + |1100001\rangle + |1101010\rangle + |1111111\rangle\big)$$

$$-\frac{1}{\sqrt{16}}\big(|0011110\rangle + |0100111\rangle + |0101101\rangle + |1001100\rangle + |1010101\rangle + |1111000\rangle\big)$$



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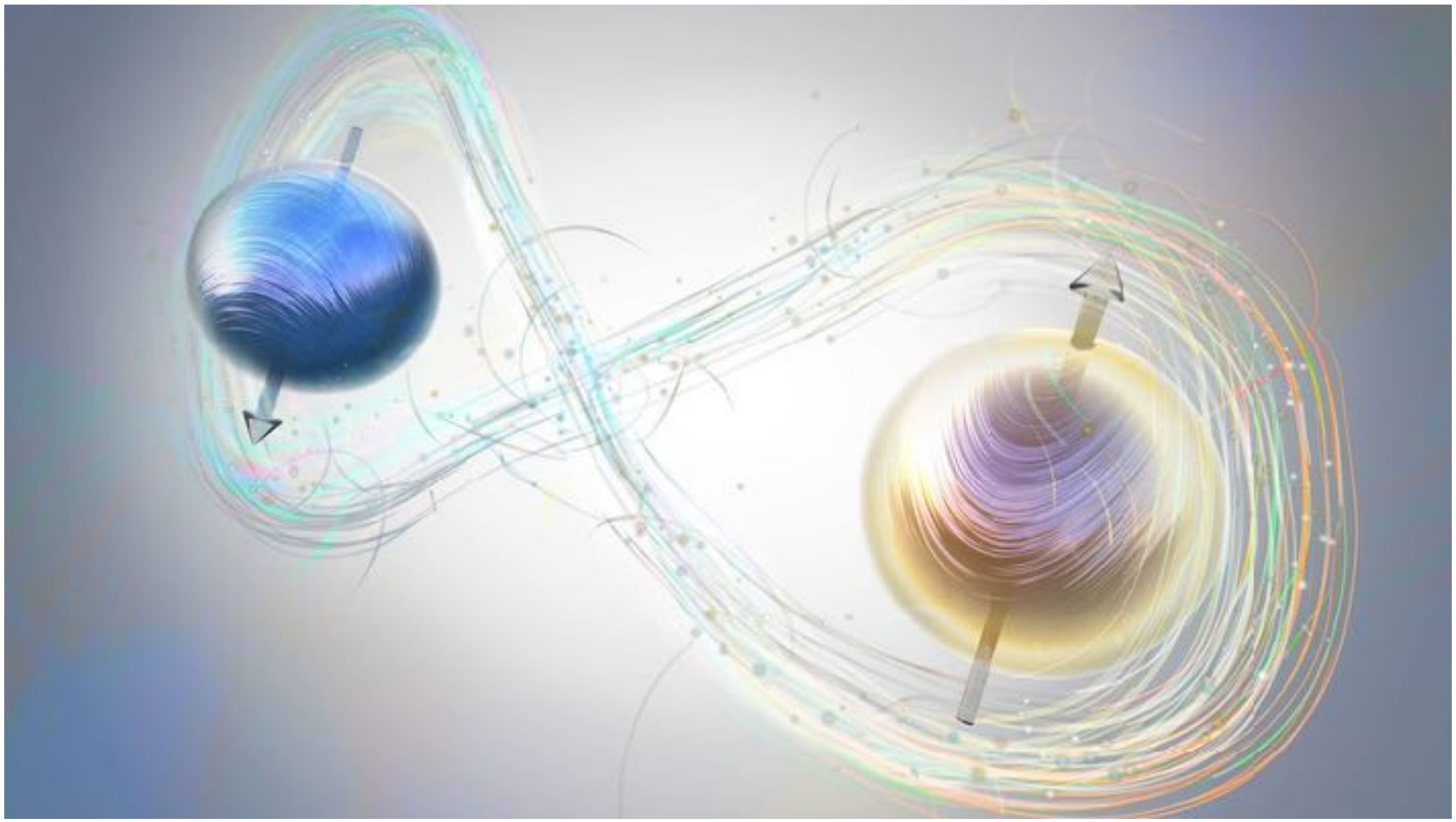

$$|\mathrm{Bell}\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$

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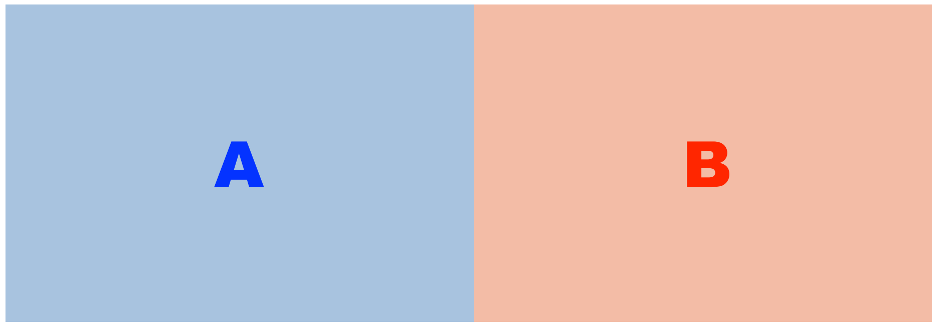

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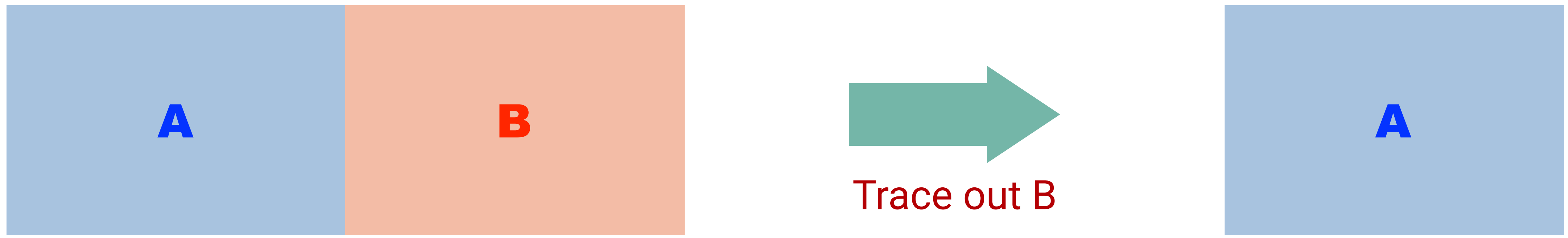
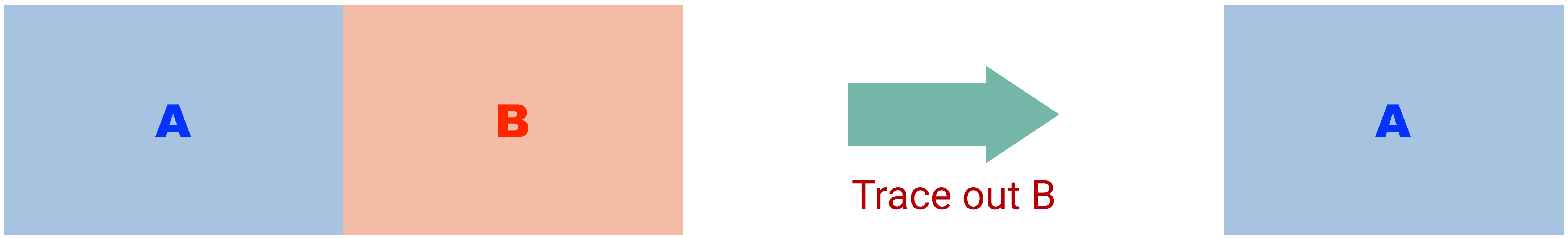


$$|\mathrm{Bell}\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$

$\rho_A = \mathrm{Tr}_B(\rho)$ maximally mixed state



$$|\mathrm{Bell}\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$$

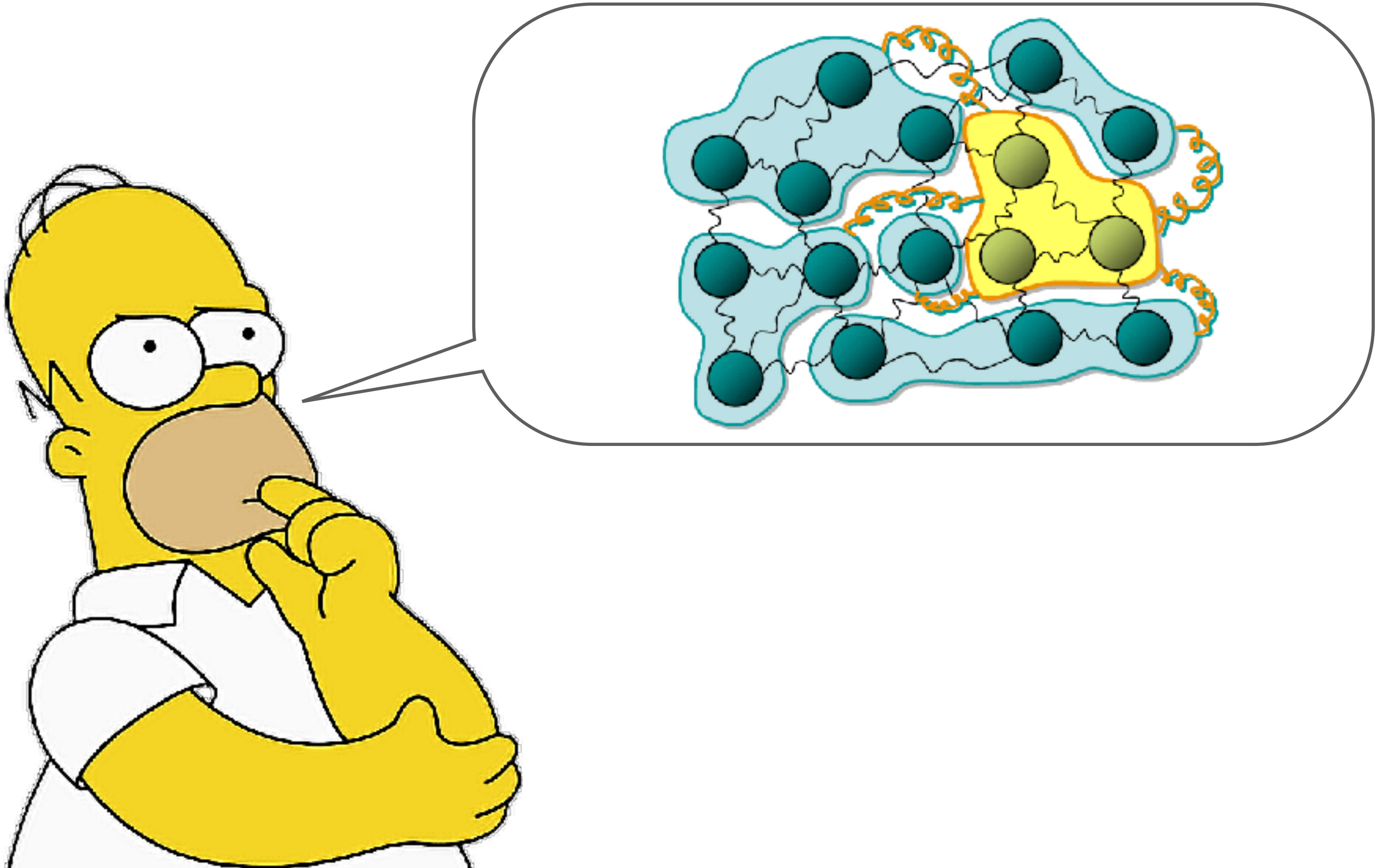
$\rho_A = \mathrm{Tr}_B(\rho)$ maximally mixed state

MAXimally mixed state saturates the entanglement entropy.

$$S_A = -\,\mathrm{Tr}(\rho_A \log \rho_A)$$

Good Measure
for bi-partite Entanglement

How about Multi-partite Entanglement?



Which state is more "special"?

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\Big(|000\rangle + |111\rangle\Big) \qquad \textit{vs} \qquad |\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\Big(|100\rangle + |010\rangle + |001\rangle\Big)$$

Which state is more “special”?

$$|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\big(|000\rangle + |111\rangle\big) \qquad vs \qquad |\mathrm{W}\rangle = \frac{1}{\sqrt{3}}\big(|100\rangle + |010\rangle + |001\rangle\big)$$

or, something else?

We need new MEASURE for

Multi-partite Entanglement!!

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Measure for Bi-partite Entanglment

$$S_A = -\,\mathrm{Tr}(\rho_A \log \rho_A)$$

→

Measure for Multi-partite Entanglment

We need new MEASURE for Multi-partite Entanglement!!

Measure for Bi-partite Entanglment

$$S_A = -\,\mathrm{Tr}(\rho_A \log \rho_A)$$

Measure for Multi-partite Entanglment

Maximally Entangled State

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_n |n\rangle \otimes |n\rangle$$

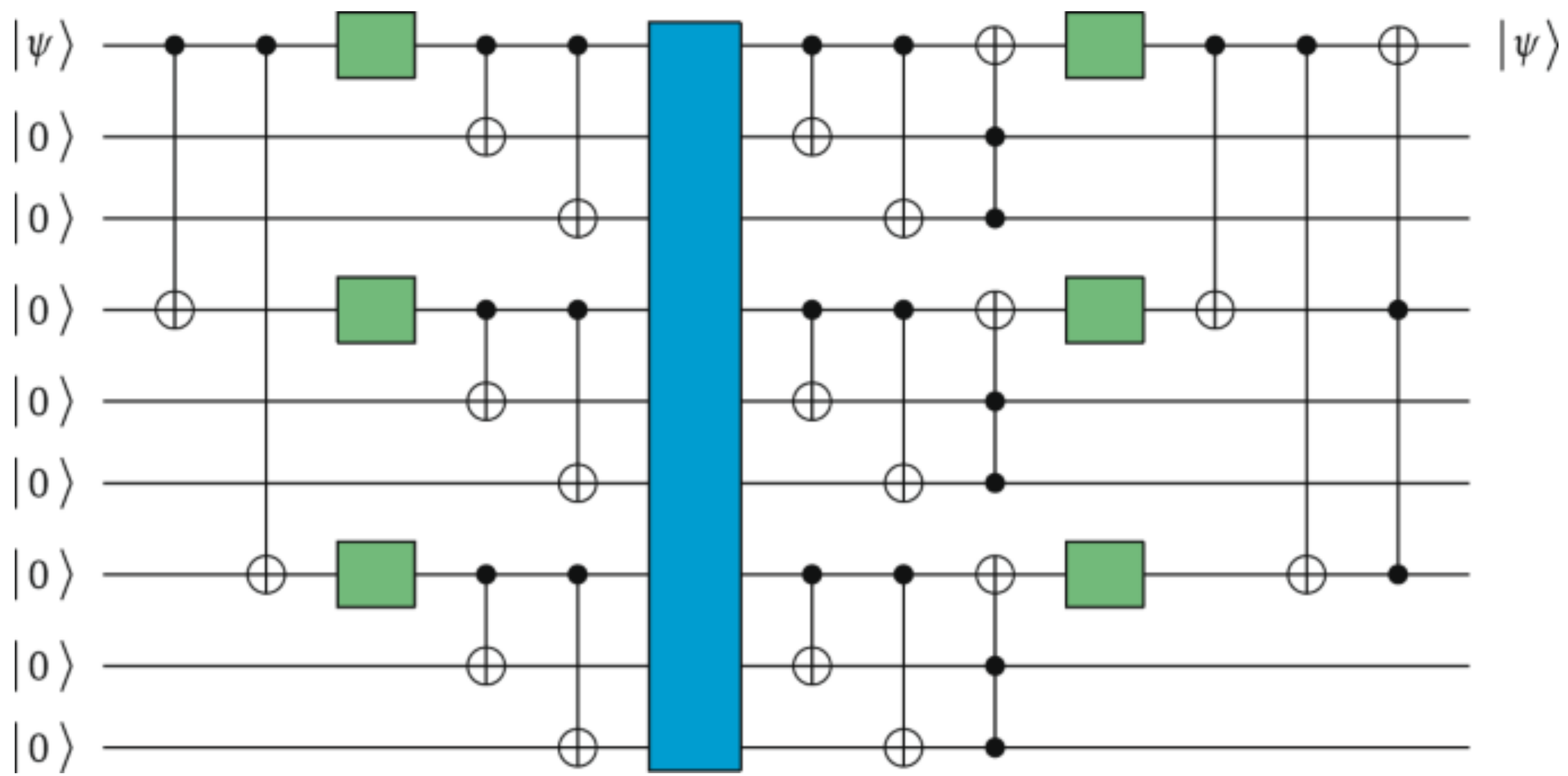
Maximally Multi-Entangled State

Why Multipartite Entanglement?

For Characterizing and Classifying quantum many-body states

Why Multipartite Entanglement?

For quantum error correction?



Why Multipartite Entanglement?

For quantum teleportation?

We propose new MEASURE for

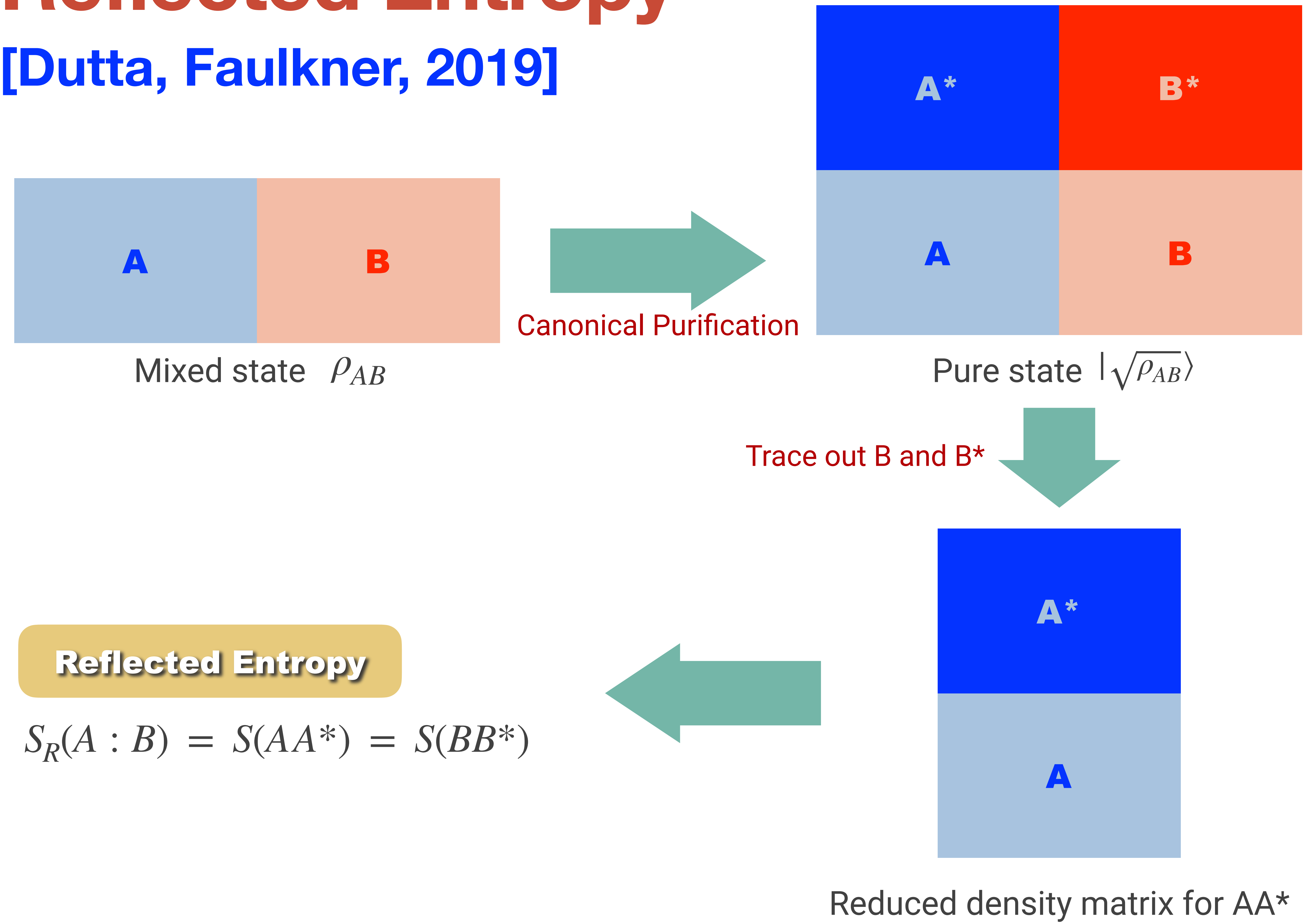
Multi-partite Entanglement.

“L-Entropy” of subsystem A and B

$$\ell_{AB} \equiv 2\min[S(A), S(B)] - S_R(A:B)$$

Reflected Entropy

[Dutta, Faulkner, 2019]



$$S_R(A:B) = S(AA^*) = S(BB^*)$$

Averaged L-Entropy

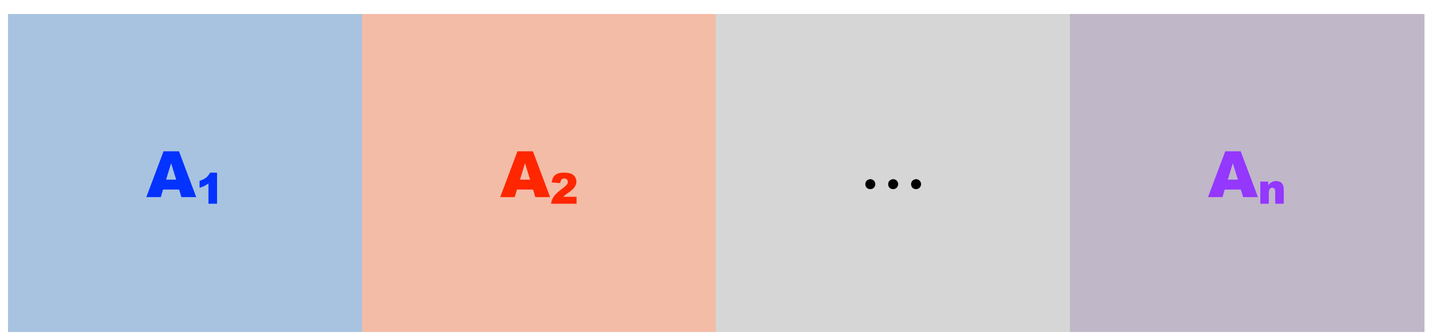
New Measure for Multi-partite Entanglement

- The bound for the reflected entropy

$$2\min[S(A), S(B)] \geqq S_R(A:B) \geqq I(A:B)$$

$$\ell_{AB} \equiv 2\min[S(A), S(B)] - S_R(A:B) \geqq 0$$

n-partite system



Averaged "L-Entropy"

$$\ell_{av} \equiv \prod_{i<j} \left[\ell_{A_iA_j}\right]^{\frac{2}{n(n-1)}}$$

Criteria for Multipartite Entanglement

Genuine Multipartite Entanglement Measure $\mathscr{E}$ (GME)

[Ma, Chen, Chen, Spengler, Gabriel, and Huber, 2011] [Xie, Eberly, 2021]

I. $\mathscr{E} = 0$ for fully-seperable or bi-seperable state

$|000\rangle$ $\quad$ $|\mathrm{Bell}\rangle \otimes |0\rangle$

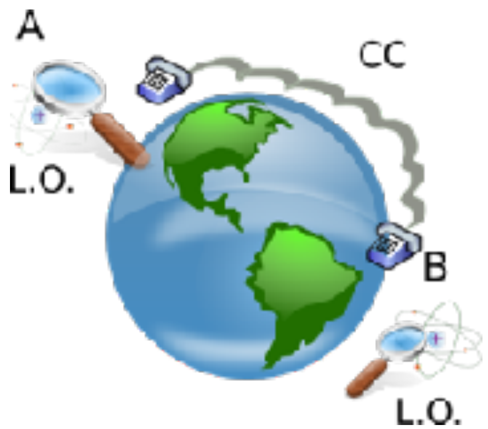
II. $\mathscr{E} > 0$ for non-biseperable state

III. $\mathscr{E}$: invariant under Local Unitary operation.

IV. $\mathscr{E}$: Non-increasing under LOCC [Entanglement Monotone]

Local Operations and Classical Communication

V. $\mathscr{E}(GHZ) > \mathscr{E}(W)$



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Local Operations and Classical Communication

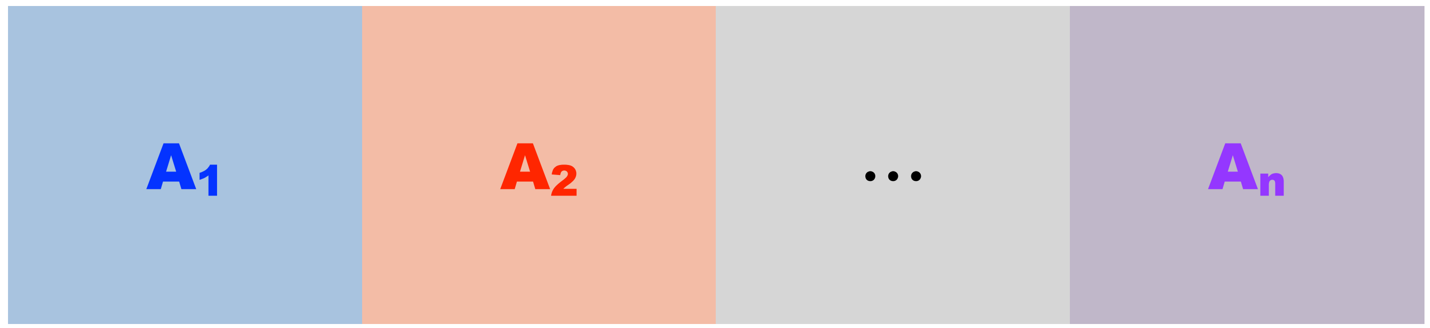
V

Our averaged L-entropy satisfies this criteria.

Maximally Multi-entangled State

The Bound of L-entropy

n-partite system



$d_1 = d_2 = \cdots = d_n = d$

- In general , the L-entropy is bounded by $2\log[d]$

$$\ell_{av} \leq 2\log[d]$$

- Depending on n and d, the bound is not saturated.

For tri-partite system ($n = 3$), the averaged L-entropy is bounded by $\log[d]$

which is the averaged L-entropy of (generalized) GHZ state

$$\ell_{av} \leqq \ell_{GHZ} = \log[d] < 2\log[d] \qquad |\psi\rangle_{GGHZ} = \frac{1}{\sqrt{d}}\sum_{j=1}^{d} |j_A j_B j_C\rangle$$

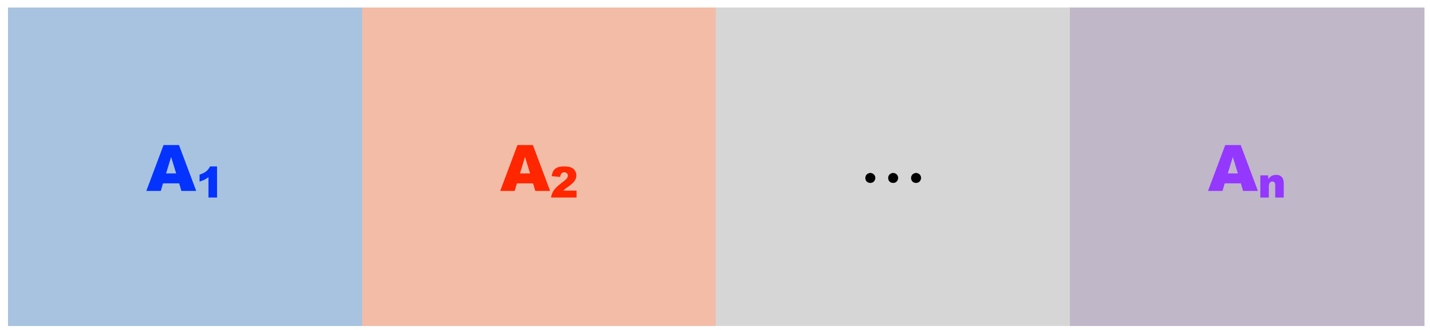
Which states saturate the bound of L-entropy?

$$\ell_{av} \leq 2\log[d]$$

k-Uniform State

Saturates the bound of the L-entropy

n-partite system



✳ ***k*-uniform state**: In n-partite system, the reduced density matrix of any k numbers of subsystems is maximally mixed.

$$\rho_{A_1 A_2 \cdots A_k} = \frac{1}{d^k} \mathbb{I}_{A_1 \cdots A_k} = \frac{1}{d^k} \mathbb{I}_{A_1} \otimes \cdots \otimes \mathbb{I}_{A_k}$$: **Factorized**

✳ ***k*-uniform state** has maximum L-entropy $2\log[d]$ $(k \geqq 2)$

$$\ell_{av}(k\text{-uniform}) = 2\log[d]$$

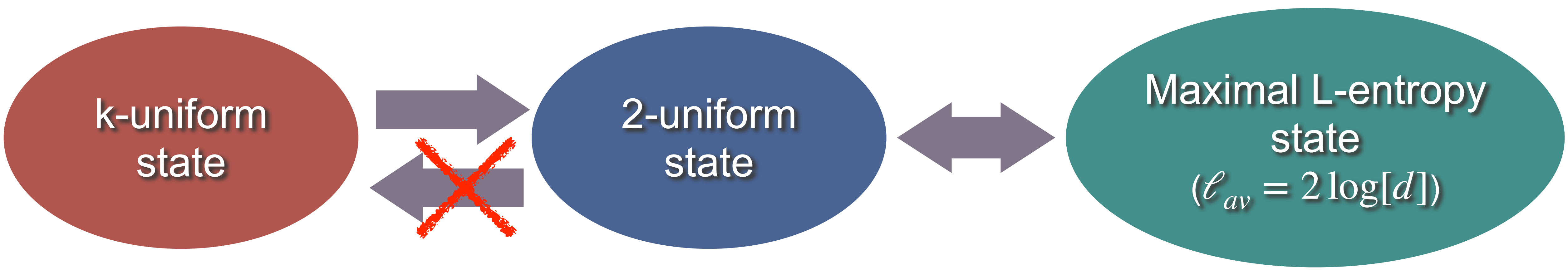
✳ In n-partite system, k-uniform state can exists only if $k \leqq \lfloor \frac{n}{2} \rfloor$ [necessary condition]

Ex) There is no k-uniform state $(k \geqq 2)$ in tri-partite system $(n = 3)$

$$\ell_{av} \leqq \ell_{GHZ} = \log[d] < 2\log[d]$$

2-Uniform State

L-entropy can capture 2-uniform state



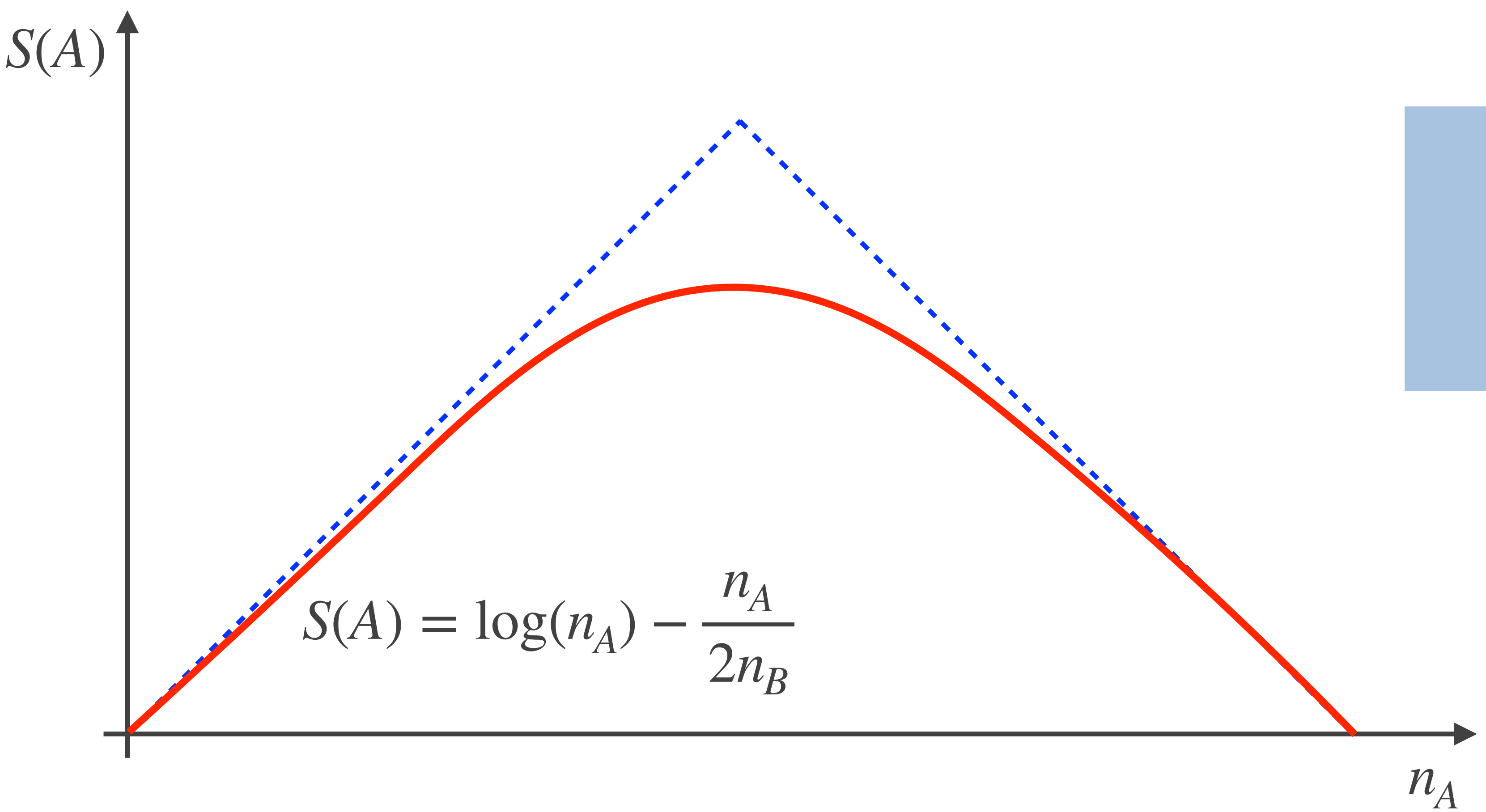
The 2-uniform state is

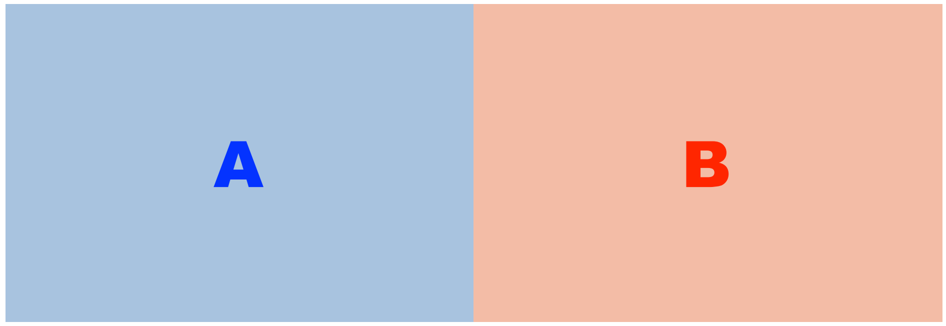
maximally multi-entangled

with respect to averaged L-entropy.

Page Curve

A typical state (random state) is maximally (bi-partite) entangled





Is a typical state

maximally multi-entangled?

Is a typical state

maximally multi-entangled?

Mostly, Yes.

But not always.

n-partite Random State ($n \geqq 5$)

Estimate the reflected entropy by resolvent technique

[Akers, Faulkner, Lin and Rath, 2021]

n-partite system

✳ Reflected Entropy of random state

$$S_R(A:B) = \frac{d^2 + 4d^2\log(d) - 2d^2\log\left(\frac{d^2}{4d_{\overline{AB}}}\right)}{8d_{\overline{AB}}} + O\left(\frac{1}{d_{\overline{AB}}^2}\right)$$

cf. Entanglement entropy of random state [Page, 1993]

$$S_A \approx \log[d_A] - \frac{d_A}{2d_{\overline{A}}}$$

n-partite Random State ($n \geqq 5$)

Estimate the reflected entropy by resolvent technique

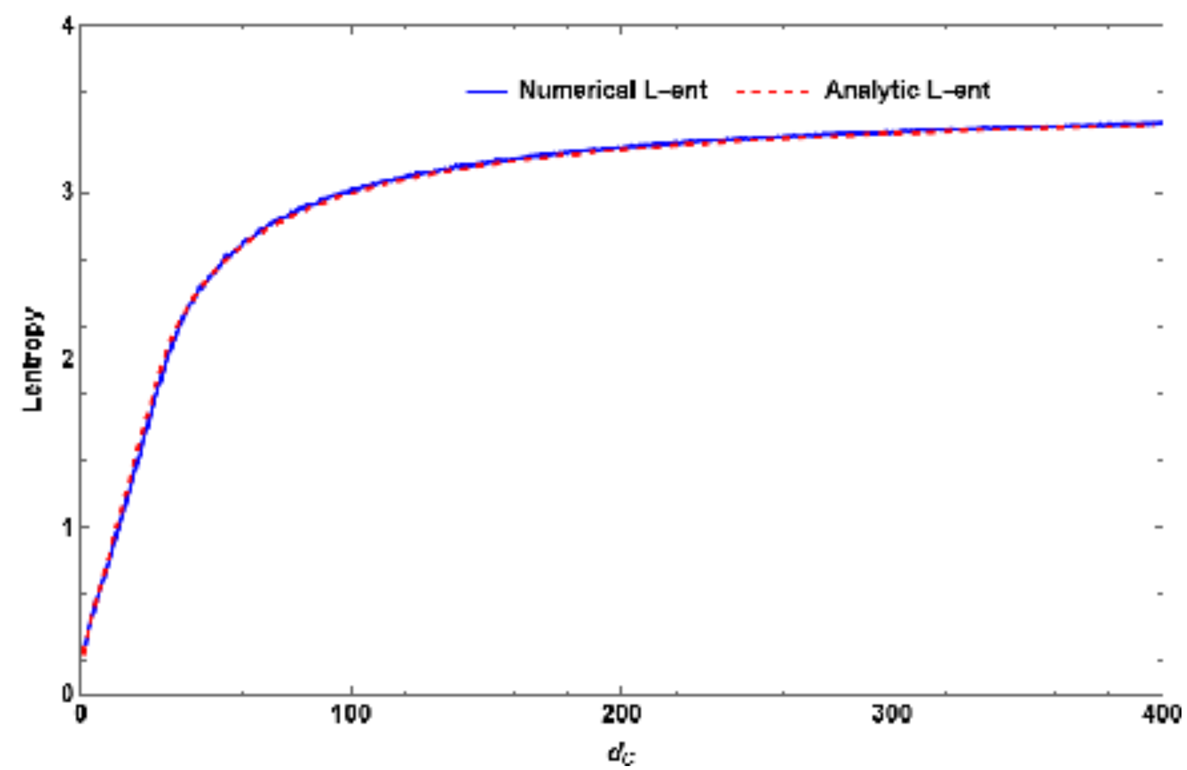
n-partite system

n parties

$$\ell_{AB} \equiv 2\min[S(A), S(B)] - S_R(A:B)$$

$$= 2\log[d] - \frac{8 + d^2 + 4d^2\log(d) - 2d^2\log(\frac{d^2}{4d_{\overline{AB}}})}{8d_{\overline{AB}}}$$

$$+ O\left(\frac{1}{d_{\overline{AB}}^2}\right)$$

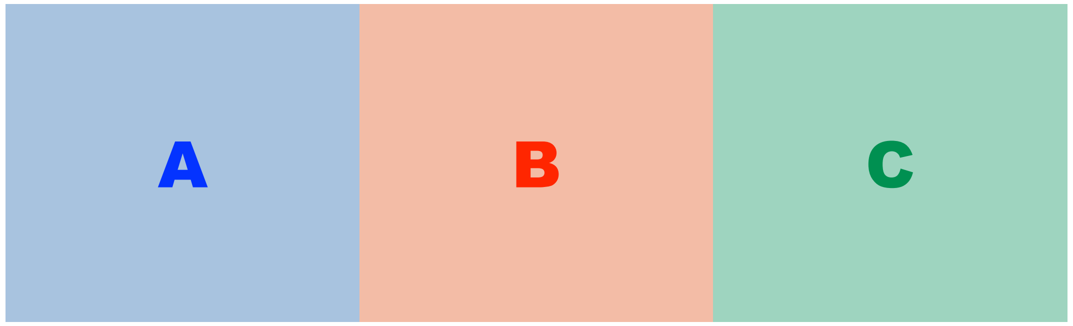


$d_A = d_B = 6\ ,\ \ d_{\overline{AB}} = d_C$

3-partite Random State

It is NOT 2-uniform state

3-partite system



$$d_A = d_B = d_C = d$$

✳ Mutual Information

$= S(C)$ $\because$ Total system ABC is pure

$$I(A:B) = S(A) + S(B) - S(AB) \approx \log[d] \neq 0$$

The subsystem A and B cannot be factorized.

✳ L-entorpy is smaller than the L-entropy of GHZ state

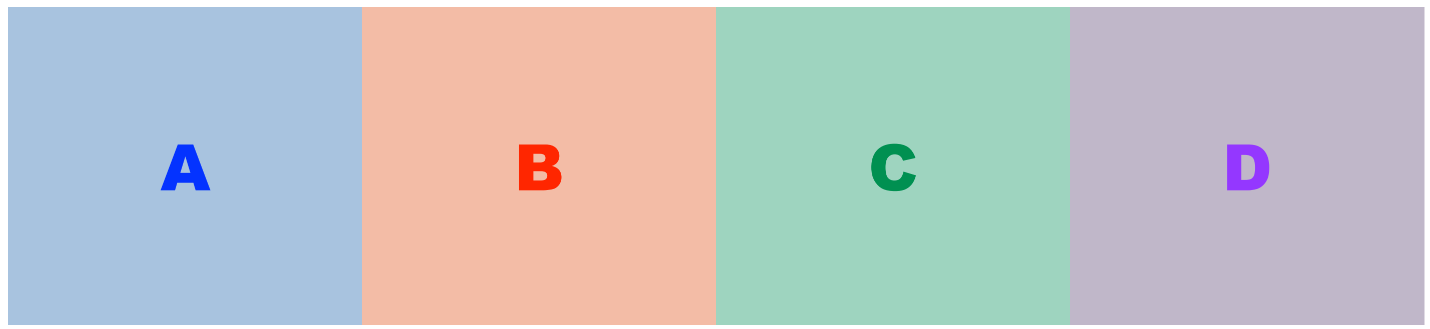
$$\ell_{AB} = \frac{1}{2} + \frac{2\log[d] - 5}{2d} + O(\frac{1}{d^2})$$: The leading contribution is independent of d

$$\ll \log[d]$$: L-entropy of GHZ state

4-partite Random State

NOT 2-uniform state

4-partite system



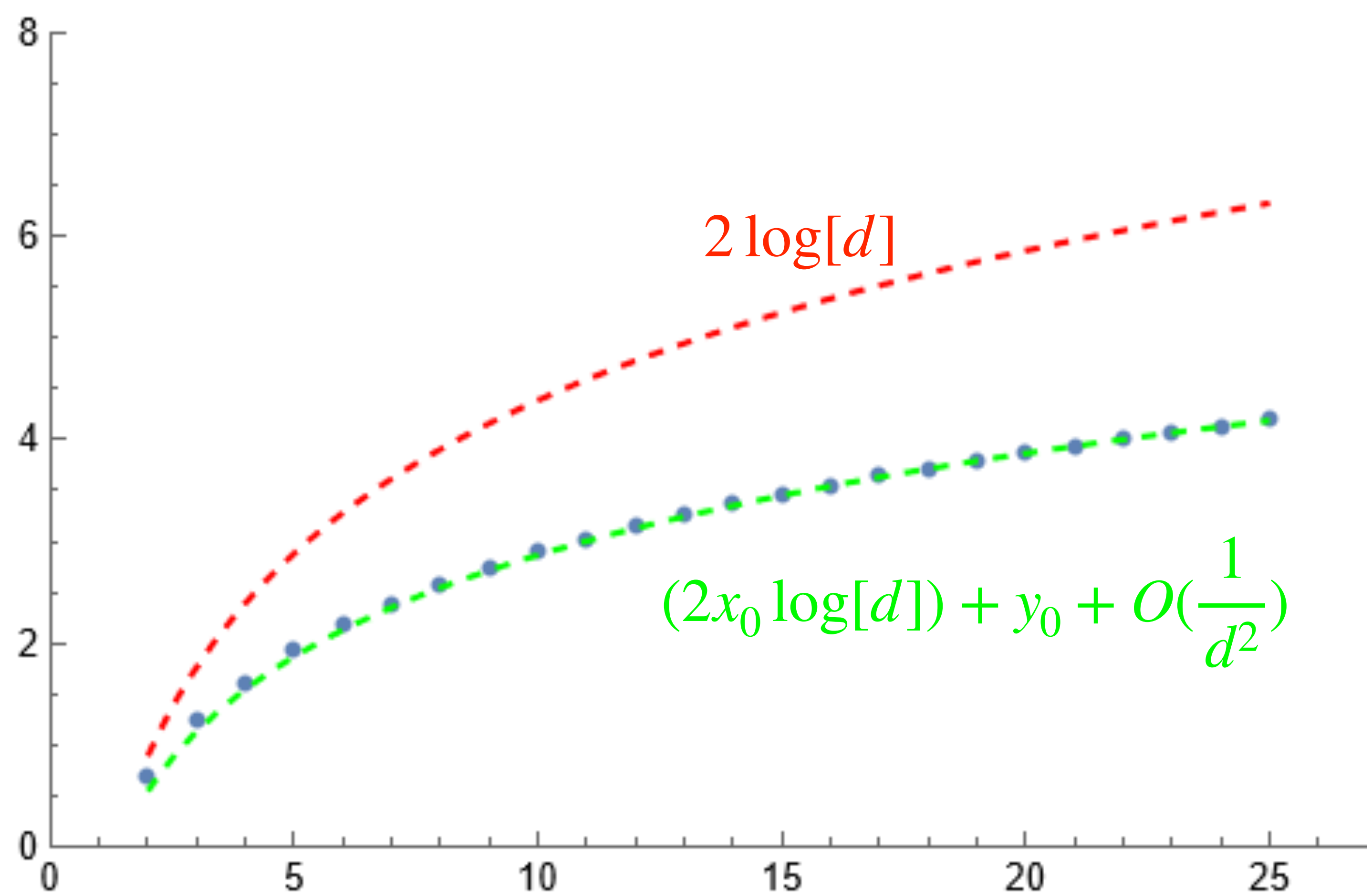
✳ L-entropy of 4-partite random state by resolvent technique

$$\ell_{AB} = (2x_0 \log[d]) + y_0 + O(\frac{1}{d^2})$$

$$x_0 \approx 0.720$$

$$y_0 \approx -0.453$$

: smaller than Maximum value



Maximally Mixed State

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_n |E_n\rangle \otimes |E_n\rangle$$

Maximally Mixed State

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Introduce **Temperature**

Thermofield Double(TFD) State
(Canonical Purification of Thermal State)

$$|TFD(\beta)\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{\beta}{2}E_n} |E_n\rangle \otimes |E_n\rangle$$

Black Hole in Gravity

Maximally Mixed State

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_n |E_n\rangle \otimes |E_n\rangle$$

Introduce **Temperature**

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$$|TFD(\beta)\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{\beta}{2}E_n} |E_n\rangle \otimes |E_n\rangle$$

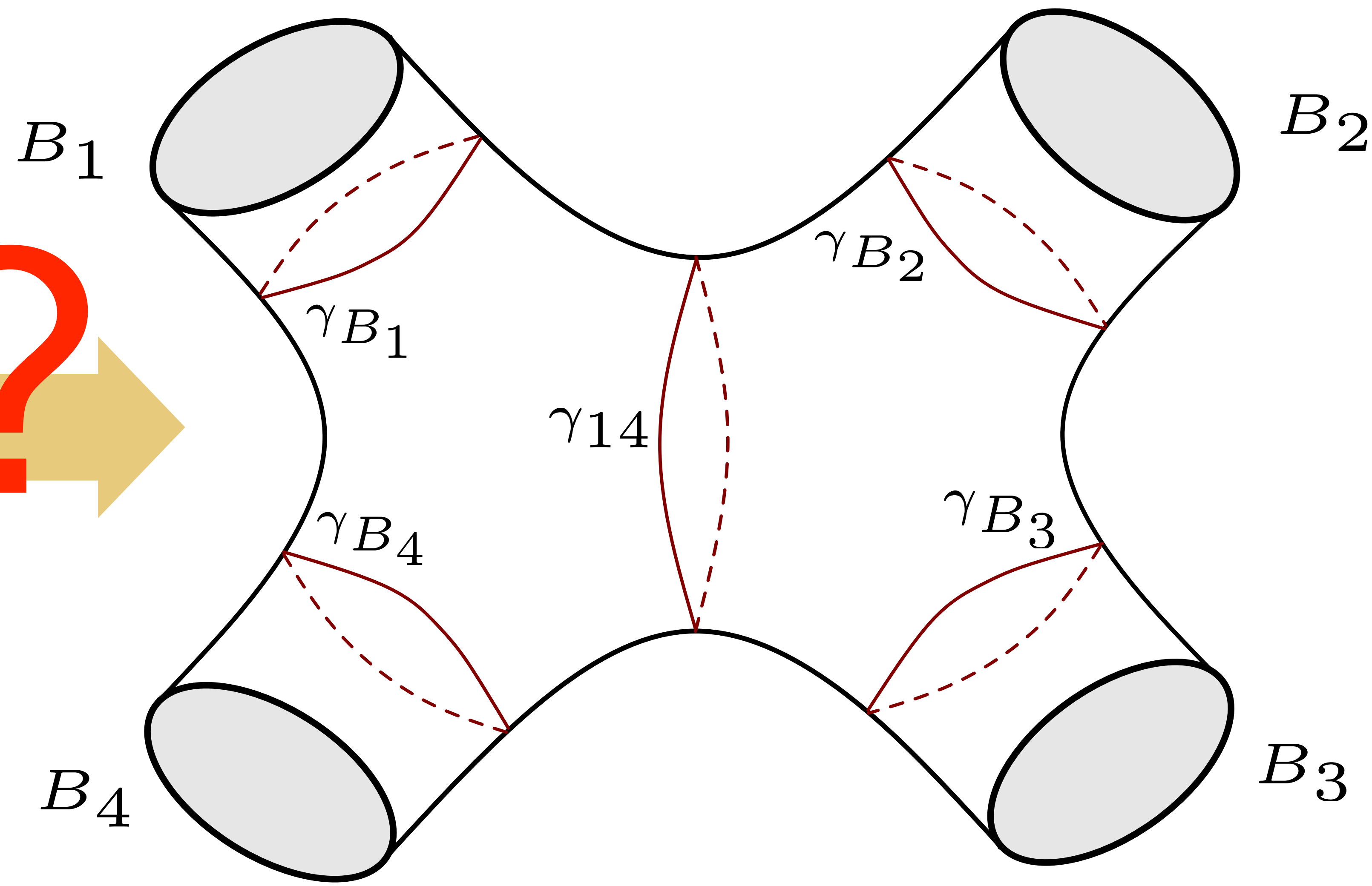
Black Hole in Gravity

What is Finite Temperature version of Multi-entangled State?

Due to the limitation of time

Please read our paper or ask me in person if you are interested in how to introduce the temperature.

$$|\Psi_\alpha\rangle \equiv \prod_{i=1}^{n} e^{-\frac{\alpha}{2}H^{(k)}}|\psi\rangle$$



Summary

- ✓ We introduced "L-entropy" as a measure of the multi-partite entanglement.
- ✓ The k-uniform state is a maximally multipartite entangled state.
- ✓ A random state is a good approximation of the k-uniform state.
- ✓ We discuss the holographic dual between the multipartite entangled state and multiboundary wormhole.

Thank You