Small noise expansion in stochastic inflation

Diego Cruces Mateo

ITP - CAS (Beijing)

D. Cruces, C. Germani, A. Nassiri-Rad and M. Yamaguchi **2410.17987**

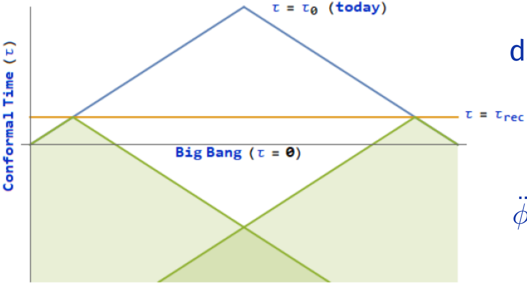
16 01 2025

Outline

Introduction and motivation

- Cosmological inflation provides the most accepted dynamical explanation for the flatness and homogeneity of the universe.

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] .$$



$$ds^2 = -a(t)^2dt^2 + \delta_{ij}dx^idx^j ,$$

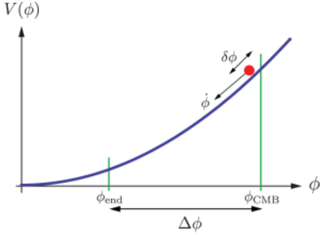
$$H^2 = \rho = \frac{1}{3M_P^2}\left(V + \frac{\dot\phi^2}{2}\right) ,$$

$$\ddot\phi + 3H\dot\phi + V_\phi = 0 .$$

$$\epsilon_1 = -\frac{\dot H}{H^2} = \frac{\dot\phi^2}{2H^2} \ll 1 , \qquad \epsilon_i = \frac{1}{H\epsilon_{i-1}}\frac{d\epsilon_{i-1}}{dt} .$$

Different inflationary regimes

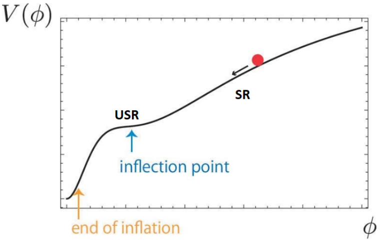
- Slow roll inflation (SR)



$$\cancel{\ddot\phi} + 3H\dot\phi + V_\phi = 0\,.$$

$$\epsilon_i \simeq \text{constant} \ll 1$$

- Ultra slow-roll inflation (USR)



$$\ddot\phi + 3H\dot\phi + \cancel{V_\phi} = 0$$

$$\epsilon_{2i-1} \sim e^{-6Ht} \ll 1\,, \quad \epsilon_{2i} \simeq -6$$

- Constant-roll inflation (CR)

$$\frac{V_\phi}{H\dot\phi} = \kappa\,, \qquad \epsilon_{2i-1} \ll 1\,, \quad \epsilon_{2i} \simeq -6\left(1 + \frac{\kappa}{3}\right)$$

$$\text{SR} \to \kappa \simeq -3 \qquad\qquad \text{USR} \to \kappa = 0$$

- During inflation, small inhomogeneities both in the scalar field and in the metric are usually studied via cosmological perturbation theory.

$$\phi \simeq \bar{\phi} + \delta\phi\,, \qquad g_{\mu\nu} \simeq \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$$

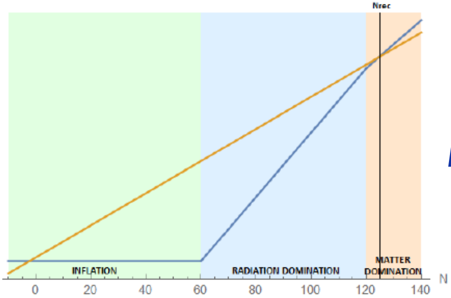
$$ds^2 = -(1+2A)dt^2 + 2a\partial_i B dx^i dt + a^2\left[(1+2D)\delta_{ij} - 2\left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)E\right]dx^i dx^j,$$

We then define the comoving curvature perturbation

$$R = D + \frac{1}{3}\nabla^2 E - \frac{H}{\dot{\phi}}\delta\phi$$

$$\delta G_{\mu\nu} = \frac{1}{M_{PL}^2}\delta T_{\mu\nu} \quad \rightarrow \quad \frac{1}{a^3\epsilon_1}\frac{d}{dt}\left[a^3\epsilon_1\dot{R}\right] - \nabla^2 R = 0$$

Long wavelenght limit ($k \to 0$)



$$\frac{1}{a^3\epsilon_1}\frac{d}{dt}\left[a^3\epsilon_1\dot{R}_k\right] - \cancel{k^2R_k} = 0$$

$$R_k(k\to 0) = C_1(k) + C_2(k)\int\frac{dt}{a^3\epsilon_1}\,,$$

$$\mathcal{P}_R(k) \equiv \frac{k^3}{2\pi^2}\left|R_k\right|^2$$

- During SR

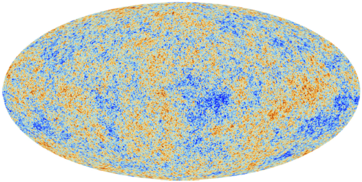
$$\epsilon_1 \simeq \text{constant} \quad \longrightarrow \quad \mathcal{P}_R(k) \sim |C_1(k)|^2$$

- During USR

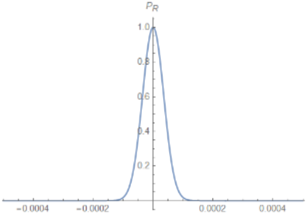
$$\epsilon_1 \sim e^{-6Ht} \quad \longrightarrow \quad \mathcal{P}_R(k) \sim |C_2(k)|^2 e^{6Ht}$$

- Generically, if $\kappa > -\frac{3}{2}$, $\mathcal{P}_R(k)$ at superhorizon scales grows with time

Why SR?



The spectral index $n_s - 1 = \frac{d \log P_R}{d \log k}$ measured at the CMBR is compatible with SR inflation.

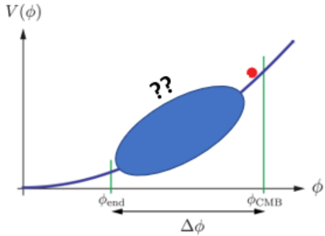


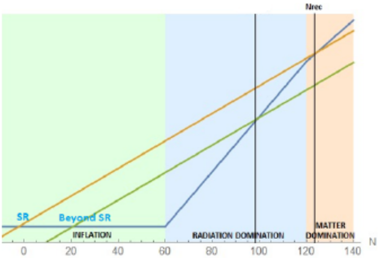
Furthermore:

- $\mathcal{P}_R(k) \sim 2 \cdot 10^{-9}$.
- Small non-gaussianities.

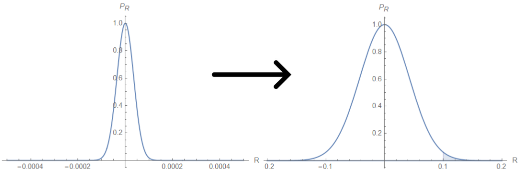
Large fluctuations are almost impossible to generate in this scenario.

Why SR always?





- The power spectrum can grow beyond SR



- Large fluctuations are more probable

Rare large fluctuations

- Rare large fluctuations beyond a given threshold can collapse at horizon re-entry and form Primordial Black Holes (PBHs).
  - Non-negligible fraction of Dark Matter.
  - Seeds of SMBH in the center of some galaxies.
  - ...
- Rare large fluctuations
  1. Are large in amplitude
     - Perturbation theory might not be enough.
  2. Are located at the tail of the probability distribution,
     - Exponentially sensible to inflationary dynamics
  3. Are generated beyond SR
     - SR approximations not valid anymore.

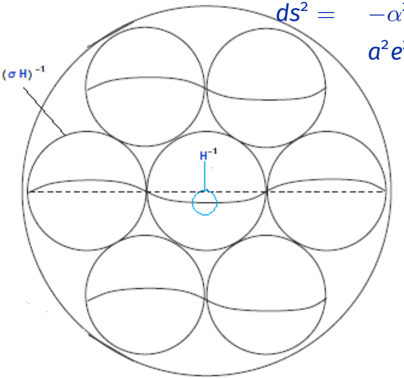
Gradient expansion

Gradient expansion

- The characteristic scale of inhomogeneities is larger that the Hubble horizon scale $L \gg H^{-1}$

$$L \sim \mathcal{O}\left(\frac{1}{\sigma}\right) \rightarrow L \sim \frac{1}{\sigma H}$$

with $\sigma \ll 1$



$$\begin{aligned} ds^2 = \quad & -\alpha^2 d\bar{t}^2 \\ & a^2 e^{2D^{NL}} \tilde{\gamma}_{ij} \left(d\bar{x}^i + \beta^i d\bar{t}\right)\left(d\bar{x}^j + \beta^j d\bar{t}\right) . \end{aligned}$$

$$ds^2 = -dt_p^2 + a(t_p)^2 \delta_{ij} dx_p^i dx_p^j .$$

$$H_p^2 = \rho_p ,$$

$$\frac{d^2}{dt_p^2}\phi_p + 3H_p \frac{d}{dt_p}\phi_p + V_{\phi_p} = 0$$

Stochastic approach to inflation

Stochastic approach to inflation

A. A. Starobinsky, Lect. Notes Phys. 246 (1986).

Evolution of the full quantum field

IR part ($k < \sigma a H$)

Gradient expansion

Equation of motion

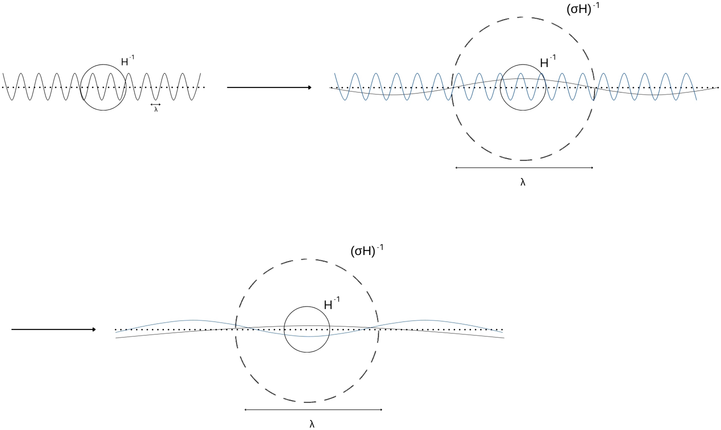
UV part ($k > \sigma a H$)

Linear perturbation theory

Noises

$$X^{IR}(t, \mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \Theta(\sigma a H - k) \mathcal{X}_{\mathbf{k}}(\mathbf{x}, t),$$

$$X^{UV}(t, \mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \Theta(k - \sigma a H) \mathcal{X}_{\mathbf{k}}(\mathbf{x}, t),$$


H-1
λ
(σH)-1
H-1
λ
(σH)-1
H-1
λ

Stochastic approach to inflation

Formulation

As an example we are going to derive the stochastic equation for the Hamiltonian constraint

$$R^{(3)} - \tilde{A}_{ij}\tilde{A}^{ij} + \frac{2}{3}K^2 = \frac{2}{M_{PL}^2} T_{\mu\nu} n^\mu n^\nu,$$

in uniform N gauge ($N = \int H_l dt_l = \int H^b dt^b$)

$$D^{NL} = 0, \quad \beta_i = 0; \qquad D = 0, \quad B = 0$$

$$\alpha \to \alpha^{IR} + \alpha^{UV}$$

$$\phi \to \phi^{IR} + \delta\phi$$

$$\dots$$

$$\left(\frac{H}{\alpha^{IR}}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\left(\frac{\dot{\phi}^{IR}}{\alpha^{IR}}\right)^2 + V\left(\phi^{IR}\right)\right) =$$

$$-\frac{1}{3\left(\alpha^{IR}\right)^3}\left(-6H^2\alpha^{UV} + \left(\dot{\phi}^{IR}\right)^2\alpha^{UV} - \alpha^{IR}\dot{\phi}^{IR}\delta\dot{\phi} - \left(\alpha^{IR}\right)^3 V_\phi \delta\phi\right) + \frac{\nabla^2}{3a^2}\nabla^2 E^{UV}$$

$$\left(\frac{H}{\alpha^{IR}}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\left(\frac{\dot{\phi}^{IR}}{\alpha^{IR}}\right)^2 + V\left(\phi^{IR}\right)\right)$$

$$= \boxed{\frac{\dot{\phi}^{IR}}{3\left(\alpha^{IR}\right)^2}\left(-\sigma a H\left(1-\epsilon_1\right)\int \frac{d\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k - \sigma a H\right)\varphi_{\mathbf{k}}\right)} \quad \Rightarrow \frac{\dot{\phi}^{IR}}{3(\alpha^{IR})^2}\xi_1$$

$$\boxed{\begin{aligned}&+\int \frac{d\mathbf{k}}{(2\pi)^{3/2}}\Theta(k-\sigma a H)\left\{\frac{1}{3(\alpha^{IR})^3}\left(6H^2\mathcal{A}_{\mathbf{k}} + \alpha^{IR}\dot{\phi}^{IR}\dot{\varphi}_{\mathbf{k}}\ .\right.\right.\\ &\left.\left.-\left(\dot{\phi}^{IR}\right)^2\mathcal{A}_{\mathbf{k}} + V_\phi\left(\phi^{IR}\right)\varphi_{\mathbf{k}}\right) + \frac{k^4}{2a^2}\mathcal{E}_{\mathbf{k}}\right\}\end{aligned}} \quad \Longrightarrow \quad 0$$

$$\left(\frac{H}{\alpha^{IR}}\right)^2 - \frac{1}{3}\left(\frac{1}{2}\left(\frac{\dot{\phi}^{IR}}{\alpha^{IR}}\right)^2 + V\left(\phi^{IR}\right)\right) = \frac{\dot{\phi}^{IR}}{3\left(\alpha^{IR}\right)^2}\xi_1$$

$$\xi_1 = -\sigma a H\left(1-\epsilon_1\right)\int \frac{d\mathbf{k}}{(2\pi)^{3/2}}\delta\left(k-\sigma a H\right)\varphi_{\mathbf{k}}$$

Two important aspects:

- $\varphi_{\mathbf{k}}$ must be computed in the same gauge that we choose for the *IR* evolution i.e uniform N gauge.
- $\varphi_{\mathbf{k}}$ must be computed over an stochastic background.

$$\frac{1}{3\left(\alpha^{IR}\right)^3}\left\{6H^2\mathcal{A}_{\mathbf{k}} + \alpha^{IR}\dot{\phi}^{IR}\dot{\varphi}_{\mathbf{k}} - \left(\dot{\phi}^{IR}\right)^2\mathcal{A}_{\mathbf{k}} - V_{\phi}\left(\phi^{IR}\right)\varphi_{\mathbf{k}} + \frac{\nabla^2}{3a^2}\nabla^2\mathcal{E}_{\mathbf{k}}\right\} = 0$$

$$\ldots$$

- We must solve

$$\frac{1}{3\left(\alpha^{IR}\right)^3}\left\{6H^2\mathcal{A}_{\mathbf{k}}+\alpha^{IR}\dot{\phi}^{IR}\dot{\varphi}_{\mathbf{k}}-\left(\dot{\phi}^{IR}\right)^2\mathcal{A}_{\mathbf{k}}-V_{\phi}\left(\phi^{IR}\right)\varphi_{\mathbf{k}}+\frac{\nabla^2}{3a^2}\nabla^2\mathcal{E}_{\mathbf{k}}\right\}=0$$

  to characterize the noises
- We need the noises to solve

$$\frac{1}{3\left(\alpha^{IR}\right)^3}\left\{6H^2\mathcal{A}_{\mathbf{k}}+\alpha^{IR}\dot{\phi}^{IR}\dot{\varphi}_{\mathbf{k}}-\left(\dot{\phi}^{IR}\right)^2\mathcal{A}_{\mathbf{k}}-V_{\phi}\left(\phi^{IR}\right)\varphi_{\mathbf{k}}+\frac{\nabla^2}{3a^2}\nabla^2\mathcal{E}_{\mathbf{k}}\right\}=0$$

- Stochastic inflation is in general a non-Markovian process!

- Following the same procedure with every ADM equation:
  DC, C. Germani, Phys.Rev.D 105 (2022) 2, 023533
  DC, Universe 8 (2022) 6, 334

$$\pi^{IR} = \frac{\partial \phi^{IR}}{\partial N} + \xi_1 \,,$$

$$\frac{\partial \pi^{IR}}{\partial N} + \left(3 - \frac{\left(\pi^{IR}\right)^2}{2M_{PL}^2}\right)\pi^{IR} + M_{PL}^2\left(3 - \frac{\left(\pi^{IR}\right)^2}{2M_{PL}^2}\right)\frac{V_\phi\left(\phi^{IR}\right)}{V\left(\phi^{IR}\right)} = -\xi_2 \,,$$

$$\partial_i\left(\frac{\partial}{\partial N}\left(\frac{1}{3}\nabla^2 C^{IR}\right)\right) - \frac{\partial_i \alpha^{IR}}{\alpha^{IR}} + \frac{\partial \phi^{IR}}{\partial N}\frac{\partial_i \phi}{2M_{PL}^2} = -\partial_i \xi_4 \,,$$

- Because stochastic inflation is in general a non-Markovian process, we cannot generically use these Langevin equations to write the Fokker-Planck equation.

Stochastic approach to inflation

Solution

Approximations: Decoupling limit

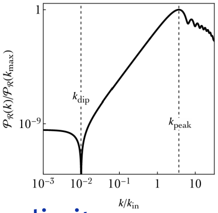
- The evolution of the Hubble rate is not governed by the scalar field.
  - We recover the Markovian behaviour.
  - Momentum constraint absent.
- If, furthermore, we assume Slow-Roll (SR):

$$\left.\begin{array}{c} \pi^{IR} = \dfrac{\partial \phi^{IR}}{\partial N} - \dfrac{H}{2\pi}\xi \\ 3\pi^{IR} + \dfrac{V_\phi\left(\phi^{IR}\right)}{H^2} = 0 \\ \cancel{\mathit{Momentum}} \end{array}\right\} \Rightarrow \frac{\partial \phi^{IR}}{\partial N} + \frac{V_\phi\left(\phi^{IR}\right)}{3H^2} = \frac{H}{2\pi}\xi$$

where $\langle \xi_1(N_1)\xi(N_2)\rangle = \delta(N_1 - N_2)\,.$

Approximations for the noises

- For PBH, we usually need a non-trivial dependence of the Hubble parameter with the scalar field



- We need to go beyond the decoupling limit
- We have to solve the full non-Markovian process
- Numerically? $\rightarrow$ computationally very expensive!
- Any other approximation?

- Small noise approximation:
  We assume that, for $\lambda \ll 1$ the solution for the stochastic equation
  $$dx = a(x)dt + \lambda b(x)dW(t)\,,$$
  can be approximated as
  $$x(t) = x_0(t) + \lambda x_1(t) + \lambda^2 x_2(t) + \dots$$
  We can write the stochastic equation at different orders in λ
  $$\begin{aligned} dx_0 &= a(x_0)dt \\ dx_1 &= \frac{da(x_0)}{dx_0}x_1 dt + b(x_0)dW(t) \\ dx_2 &= \left(\frac{da(x_0)}{dx_0}x_2 + \frac{1}{2}\frac{d^2a(x_0)}{dx_0^2}x_1^2\right)dt + \frac{db(x_0)}{dx_0}x_1 dW(t) \\ &\dots \end{aligned}$$
  Noises computed over a deterministic solution!

In the case of stochastic inflation:

- $\mathcal{O}\left(\lambda^0\right)$

$$\bar{\pi} = \frac{\partial \bar{\phi}}{\partial N}\,,$$
$$\frac{\partial \bar{\pi}}{\partial N} + \left(3 - \frac{\bar{\pi}^2}{2M_{PL}^2}\right)\bar{\pi} + \left(3M_{PL}^2 - \frac{\bar{\pi}^2}{2}\right)\frac{V_{\bar{\phi}}\left(\bar{\phi}\right)}{V\left(\bar{\phi}\right)} = 0\,,$$

- $\mathcal{O}\left(\lambda^1\right)$
  - Exact solution!
  - Gaussian PDF for $Q_1 \equiv \phi_1 - \bar{\pi}\left(\frac{1}{3}\nabla^2 C_1\right)$ with mean $\bar{\phi}$ and variance

$$\sigma_Q^2 \equiv \langle Q^{IR}(N)^2\rangle = \int_{k=\sigma a(0)H(0)}^{k=\sigma a(N)H(N)} \frac{k^3}{2\pi^2}\left|Q(k,N)\right|^2 \frac{dk}{k} = \int_{k=\sigma a(0)H(0)}^{k=\sigma a(N)H(N)} \mathcal{P}_Q(k,N)\frac{dk}{k}$$

which can be easily computed using only perturbation theory (expected).

- We have an exact solution for the stochastic approach to inflation at leading order in small noise approximation!
- The PDF described at $\mathcal{O}\left(\lambda^1\right)$ for Q_1 is Gaussian with mean $\bar{\phi}$ and variance

$$\sigma_Q^2 \equiv \langle Q^{IR}(N)^2 \rangle = \int_{k=\sigma a(0)H(0)}^{k=\sigma a(N)H(N)} \frac{k^3}{2\pi^2} \left|Q(k,N)\right|^2 \frac{dk}{k} = \int_{k=\sigma a(0)H(0)}^{k=\sigma a(N)H(N)} \mathcal{P}_Q(k,N) \frac{dk}{k}$$

which can be easily computed using only perturbation theory (expected).

In Slow-Roll inflation

- λ^2

$$\begin{aligned}
\frac{\partial\phi_2}{\partial N} =& \pi_2 + \left(\mathcal{O}(\epsilon^2)\right)\frac{\phi_1}{\bar{\pi}}\xi\,,\\
\frac{\partial\pi_1}{\partial N} +& \left(3-\epsilon_1+\mathcal{O}\left(\epsilon^2\right)\right)\pi_2 + \left(-\frac{3}{2}\epsilon_2+\mathcal{O}\left(\epsilon^2\right)\right)\phi_2 + \left(\mathcal{O}(\epsilon^2)\right)\frac{\phi_1^2}{\bar{\pi}} = -\left(\mathcal{O}(\epsilon^2)\right)\frac{\phi_1}{\bar{\pi}}\xi\,,\\
\frac{\partial}{\partial N}&\left(\frac{1}{3}\nabla^2 E_2\right) + \left(\mathcal{O}(\epsilon^2)\right)\phi_2 + \mathcal{O}(\epsilon^2)\frac{\phi_1^2}{\bar{\pi}} = \left(\mathcal{O}(\epsilon^2)\right)\frac{\phi_1}{\bar{\pi}}\xi\,,
\end{aligned}$$

- The stochastic corrections to the noise are actually of the same order as the non-linearities of the IR equation!
- The convenient approximation of computing the noises over a deterministic background linearizes the system.

- Neglecting $\mathcal{O}(\epsilon^2)$ at all orders in small noise expansion we can write the stochastic system as

$$\frac{\partial \phi^{IR}}{\partial N} = \pi^{IR} + \frac{\bar{H}_0}{2\pi}\left[1 + (\alpha - \log(\sigma))\left(\epsilon_1 + \frac{\epsilon_2}{2}\right) - \frac{3}{2}\epsilon_1 - \epsilon_1 N\right]\xi\,,$$
$$\frac{\partial \pi^{IR}}{\partial N} + (3 - \epsilon_1)\,\pi^{IR} - \frac{3}{2}\epsilon_2\phi^{IR} = -\frac{\bar{H}_0}{2\pi}\frac{\epsilon_2}{2}\xi\,.$$

whose result is, again

$$P\left(\phi^{IR}\right) = \mathcal{N}\left(\bar{\phi}, \sigma_\phi^2\right)$$

where $\sigma_\phi^2 = \int_{k=\sigma a(0)H(0)}^{k=\sigma a(N)H(N)} P_Q(k,N)\frac{dk}{k}$ is computed at leading order in *SR* parameters.

Conclusions.

Conclusions

- A growth of the power spectrum beyond SR makes rare large inhomogeneities, which can produce PBH, more probable. These large inhomogeneities must be described beyond perturbation theory using, for example, the stochastic approach.
- Stochastic inflation is generically a non-Markovian process.
- Small noise expansion is a tool that allows us to study the full non-Markovian problem in terms of an infinite set of Markovian processes.
  - It consistently recovers the tree-level result; we expect to do the same also at loop level.
  - Stochastic corrections of the noises are generically of the same order as non-linearities in the potential.

Thank you!