Is Genesis Possible in the Framework of Horndeski Gravity?

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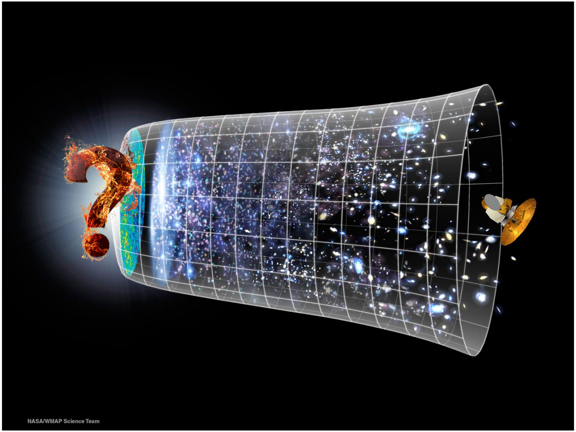
Motivation



Can we replace or complete the inflation model?

All these models are impossible in General Relativity!

Behavior in the asymptotic past (Einstein frame).

model	a	H	$\dot{H}$	singularity
Genesis	$const > 0$	0	$\dot{H} > 0$	no
Bounce	∞	0	$\dot{H} < 0,\ \dot{H}_b > 0$	no
Modified Genesis	0	0	$\dot{H} > 0$	yes
NEC - violating inflation	0	$const > 0$	$\dot{H} > 0$	yes

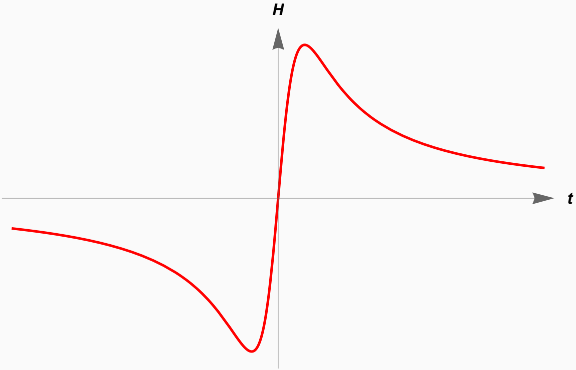


Figure 1: Hubble parameter: bounce

Qui'2011,2013; Easson'2011; Cai'2012;
Osipov'2013; Koehn'2013; Battarra'2014; Ijjas'2016

Genesis

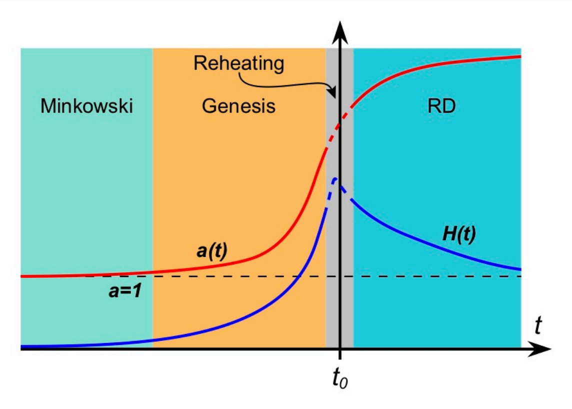


Figure 2: Hubble parameter: Genesis

Creminelli'2010, Creminelli'2012, Hinterbichler'2012, Elder'2013, Pirtskhalava'2014, Nishi'2015, Kobayashi'2015

Null Energy Condition

Realization of non-singular evolution within classical field theory requires the violation of the Null Energy Condition (NEC) $T_{\mu\nu}n^\mu n^\nu > 0$ (or Null Convergence Condition (NCC) $R_{\mu\nu}n^\mu n^\nu > 0$ for modified gravity).

$$T_{00} = \rho, \quad T_{ij} = a^2\gamma_{ij}p,$$

$$\dot{H} = -\frac{1}{2}(\rho + p) + \cancel{\text{curvature term}}.$$

Let us use $n_\mu = (1, a^{-1}\nu^i)$ with $\gamma_{ij}\nu^i\nu^j = 1$ and then NEC leads to

$$T_{\mu\nu}n^\mu n^\nu > 0 \to \rho + p \geq 0 \to \dot{H} \leq 0.$$

Penrose theorem: singularity in the past if:

- The NEC holds
- The Cauchy hypersurface is noncompact.

Is NEC violation possible

If one considers

$$\mathcal{L} = G_2(\phi, X) + \frac{R}{2},\ X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi,$$

then the NEC violation reads

$$\rho + p = 2XG_{2X} < 0,\ \dot{H} = -\frac{1}{2}(\rho + p)\ .$$

while stability conditions for scalar perturbation are

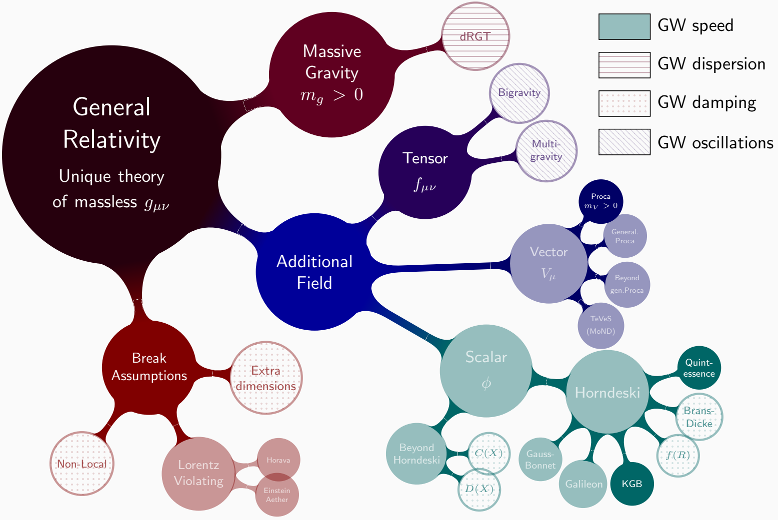
$$\mathcal{L}_{\zeta\zeta} \propto \mathcal{G}_S\dot{\zeta}^2 - \mathcal{F}_S(\nabla\zeta)^2\ ,\ \ \mathcal{G}_S > 0,\ \ \mathcal{F}_S = -\frac{\dot{H}}{H^2} > 0$$

The latter condition leads to the instabilities! Same holds for multi field models.

One must to modify gravity!

Modified gravity theories

Modified gravity roadmap



[from Ezquiaga, Zumalacárregui'18]

1. The most general scalar-tensor theory of gravity!
2. Admits stable NEC - violation.
3. Many modified gravity theories, like: $f(R)$ – gravity, Brans-Dicke theory, Galileons, ect – are subclasses of Horndeski gravity.

Seems like a perfect candidate to build the non-standard cosmology!

Violation of NEC/NCC without obvious pathologies is possible in the class of Horndeski theories [*Horndeski'74*]:

$$\begin{aligned}\mathcal{L}_H &= G_2(\phi, X) - G_3(\phi, X)\Box\phi + \\ &G_4(\phi, X)R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right] \\ &+ G_5(\phi, X)G^{\mu\nu}\nabla_\mu\nabla_\nu\phi \\ &- \frac{1}{6}G_{5,X}\left[(\Box\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2 + 2(\nabla_\mu\nabla_\nu\phi)^3\right],\end{aligned}$$

where $X = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ and $\Box\phi = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi$.

No-Go theorem

- If one works in the framework of Horndeski gravity, then at the some point during the whole evolution ($-\infty < t < +\infty$) of a singularity-free universe: gradient instabilities show up at some moment in the history $\rightarrow$ No-Go theorem. *Rubakov'2016; T. Kobayashi'2016*.
- First way: Abandon Horndeski gravity and move to the more general theories.
- Second way: Reconsider the No-Go theorem once more!
- We want to stay within the framework of Horndeski gravity, so we will choose the second way.

No-Go theorem

Let us consider the following perturbed ADM metric:

$$ds^2 = -N^2 dt^2 + \gamma_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right),$$

$$\gamma_{ij} = a^2 e^{2\zeta}(\delta_{ij} + h_{ij} + \ldots), \quad N = N_0(1+\alpha), \quad N_i = \partial_i \beta.$$

Here α and β are not physical. We work with unitary gauge $\delta\phi = 0$.
The quadratic actions for ζ and h_{ij} are given, respectively:

$$\mathcal{L}_{\zeta\zeta} = a^3\left[\mathcal{G}_S \frac{\dot{\zeta}^2}{N^2} - \frac{\mathcal{F}_S}{a^2}\zeta_{,i}\zeta_{,i}\right], \; \mathcal{L}_{hh} = \frac{a^3}{8}\left[\mathcal{G}_T \frac{\dot{h}_{ij}^2}{N^2} - \frac{\mathcal{F}_T}{a^2} h_{ij,k} h_{ij,k}\right].$$

Remind that bounce solution is $a(t) \to \infty$ as $t \to -\infty$. No-Go works if

$$\int_{-\infty}^{t} a(t)(\mathcal{F}_T + \mathcal{F}_S)dt = \infty \;,$$

$$\int_{t}^{+\infty} a(t)(\mathcal{F}_T + \mathcal{F}_S)dt = \infty \;.$$

No-Go: $\mathcal{F}_{S,T} < 0$ at some moment of time, instability.

No-Go theorem

- One way is to go beyond Horndeski or DHOST Cai' 2016, Creminelli'2016, Kolevatov'2017, Piao'2017
- Another way to avoid No-Go theorem for Horndeski is to obtain such a model/solution that $\mathcal{F}_{S,T}$ coefficients have asymptotics Kobayashi'2016

$$\mathcal{F}_{S,T} \to 0 \text{ as } t \to -\infty, \text{ where } \mathcal{F}_T = 2G_4.$$

- This means that

$$G_4 \to 0 \text{ as } t \to -\infty.$$

- Effective Planck mass goes to zero and it signalizes that we may have strong coupling at $t \to -\infty$.

Solution: no SC regime at $t \to -\infty$ in some region of Lagrangian parameters. Ageeva'2018

1. The function $G_2(\phi, X)$ is necessary, since this function contains the canonical kinetic term for the field.
2. One must include the function $G_3(\phi, X)$ in order to be able to perform stable NEC violation.
3. At least field-dependent function $G_4(\phi)$ should remain. In order to circumvent the No-Go theorem.
4. $G_5 = 0$.

Thus, the minimal setup is

$$\mathcal{L} = G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi)R \ .$$

ADM

In ADM the Lagrangian has the following form:

$$\mathcal{L} = A_2(t,N) + A_3(t,N)K + A_4(K^2 - K_{ij}^2) + B_4(t,N)R^{(3)}.$$

We remind that we have unitary gauge $\phi = \phi(t)$. ${}^{(3)}R_{ij}$ is the Ricci tensor made of γ_{ij}, $\sqrt{-g} = N\sqrt{\gamma}$, $K = \gamma^{ij}K_{ij}$, ${}^{(3)}R = \gamma^{ij}\ {}^{(3)}R_{ij}$ and K_{ij} is an extrinsic curvature of hypersurfaces $t = \text{const}$:

$$K_{ij} \equiv \frac{1}{2N}\left(\frac{d\gamma_{ij}}{dt} - {}^{(3)}\nabla_i N_j - {}^{(3)}\nabla_j N_i\right) .$$

Early Genesis stage

Let us give our construction in ADM formalism

$$A_2(t,N) = \frac{1}{2}(-ct)^{-2\mu-2-\delta} \cdot a_2(N) \,,$$
$$A_3(t,N) = \frac{1}{2}(-ct)^{-2\mu-1-\delta} \cdot a_3(N) \,,$$
$$A_4(t) = -B_4(t) = -\frac{1}{2}(-ct)^{-2\mu} \,.$$

Circumvent the No-Go theorem:

$$2\mu > 1 + \delta > 1.$$

Unitarity bounds in the asymptotic past

$$\mu + \frac{3}{2}\delta < 1.$$

Leading order solution

$$
\begin{aligned}
H &= \frac{h_0}{(-ct)^{1+\delta}}, \\
a &= a_g\Big(1 + \frac{h_0}{c\delta(-ct)^{\delta}}\Big), \quad t \to -\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
h_0 &\equiv \frac{1}{4}N_0\Big(-\frac{N_0 a_2(N_0)}{c + c\delta + 2c\mu} + a_3(N_0)\Big), \\
a_2(N_0) &+ N_0 \cdot \frac{d}{dN}a_2(N)\Big|_{N=N_0} = 0.
\end{aligned}
$$

The stability requirements reads as follows

$$\mathcal{G}_S > 0, \quad \mathcal{F}_S > 0, \quad \mathcal{G}_T > 0, \quad \mathcal{F}_T > 0.$$

In addition, we also require the absence of superluminal propagation

$$u_S < 1,$$

where

$$\mathcal{G}_S = \frac{4(-ct)^{\delta-2\mu}\left(2a_2'(1)) + a_2''(1)\right)}{\left(4h_0 + a_3'(1)\right)^2},$$

$$\mathcal{F}_S = \frac{4(-ct)^{\delta-2\mu}c(1+\delta-2\mu)}{4h_0 + a_3'(1)},$$

$$\mathcal{G}_T = \mathcal{F}_T = (-ct)^{-2\mu}.$$

$$
\begin{aligned}
g^E_{\mu\nu} &= \Omega \cdot g_{\mu\nu} \ , \\
N^E &= \sqrt{\Omega} \cdot N \ , \\
a^E &= \sqrt{\Omega} \cdot a \ ,
\end{aligned}
$$

where we choose

$$
\Omega(\phi) \equiv \frac{2G_4(\phi)}{M_{Pl}^2} \ .
$$

Therefore, we have the Modified Genesis:

$$
\begin{aligned}
a^E &\propto (-t_c)^{-\frac{\mu}{1-\mu}} \ , \ \mu < 1 \, \text{(No strong-coupling)}, \\
H^E &= \frac{\mu}{1-\mu}(-t_c)^{-1} \ , \\
\dot{H}^E &= \frac{\mu}{1-\mu} \cdot \frac{1}{|t_c|^2} \ .
\end{aligned}
$$

- We want to have a kination epoch in the future $t \to +\infty$, so

$$A_2 = \frac{1}{3t^2N^2},\ A_3 = 0,\ A_4 = -\frac{1}{2}.$$

The Lagrangian above is corresponds to the massless scalar field

$$\mathcal{L} = X + \frac{R}{2}.$$

- In the past one has

$$A_2 = \frac{1}{2}|c \cdot t|^{-2\mu-2-\delta}\left(\frac{-g}{N^2} + \frac{g}{3N^4}\right),\ A_3 = 0,\ A_4 = -\frac{1}{2}|c \cdot t|^{-2\mu}\ .$$

- Smoothly connect two stages. The most tricky part is to ensure the stability during transition phase...

$$A_2 = \frac{1}{2} f^{-2\mu-2-\delta}\Big(-\frac{g}{N^2} + \frac{g}{3N^4}\Big) \cdot (1-U) + \frac{U}{3N^2\big(\frac{2f}{c}+t\big)^2} \ ,$$

$$A_3 = 0 \ ,$$

$$A_4 = -\frac{1}{2} f^{-2\mu} \ ,$$

where

$$U(t) = \frac{e^{st}}{1+e^{st}} \ ,$$

$$f(t) = \frac{c}{2}\Big(-t + \frac{\ln[2\cosh(st)]}{s}\Big) + 1 \ .$$

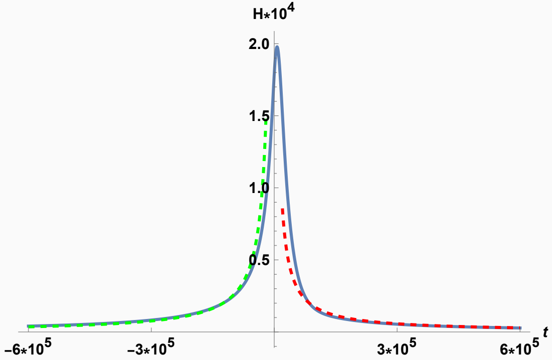
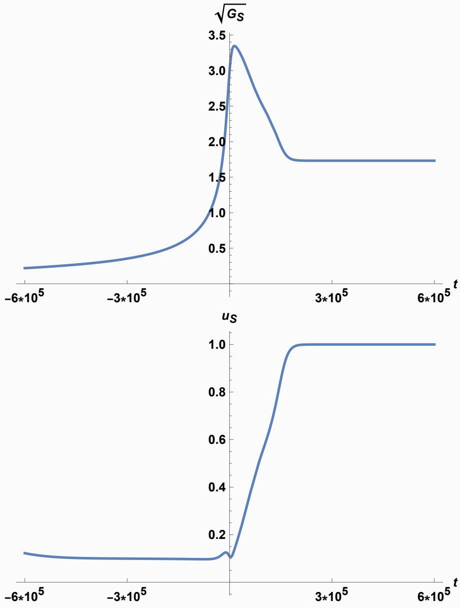


Figure 3: The Hubble parameter. The red dashed line is the Hubble parameter for kination stage $H = (3tN_f)^{-1},\ N_f = N(t_0)$, while the green dashed line is the Hubble parameter for early Genesis stage $H = h_0 \cdot (-ct)^{-1-\delta}$.

Stability



$\sqrt{G_S}$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
$-6*10^5$
$-3*10^5$
$3*10^5$
$6*10^5$
t
u_S
1.0
0.8
0.6
0.4
0.2
$-6*10^5$
$-3*10^5$
$3*10^5$
$6*10^5$
t

Mode equation for scalars

$$\ddot{\zeta} + \dot{\zeta} \cdot \frac{\theta^s}{t} + \vec{k}^2 \cdot \mathcal{B}^s \cdot \zeta = 0,$$

where we introduce

$$\mathcal{A}^s \equiv \frac{\mathcal{G}_S a^3}{N},$$

$$\theta^s \equiv t \cdot \frac{\dot{\mathcal{A}}^s}{\mathcal{A}^s},$$

$$\mathcal{B}^s \equiv \frac{u_S^2 N^2}{a^2}.$$

If mode freezes at the early times

$$
\begin{aligned}
\mathcal{A}_g^s &\equiv \frac{4(-ct)^{\delta-2\mu}\left(2a_2'(1)) + a_2''(1)\right)}{\left(4h_0 + a_3'(1)\right)^2} \cdot a_g^3 \ , \\
\theta_g^s &\equiv t \cdot \frac{\dot{\mathcal{A}}_0^s}{\mathcal{A}_0^s} = \delta - 2\mu < 0 \ , \\
\mathcal{B}_g^s &= \frac{c\left(2\mu - 1 - \delta\right)\left(a_3'(1) + 4h_0\right)}{a_g^2 \cdot \left(a_2''(1) + 2a_2'(1)\right)} > 0 \ .
\end{aligned}
$$

We introduce the canonically normalized field ψ via

$$\zeta \equiv \frac{\psi}{\left(2\mathcal{A}_g^s\right)^{1/2}} \ ,$$

so that the quadratic action is given by

$$\mathcal{S}_{\psi\psi}^{(2)} = \int d^3x dt \left[\frac{1}{2}\dot{\psi}^2 - \frac{\mathcal{B}_g^s}{2}\left(\vec{\nabla}\psi\right)^2 + O\left(t^{-2}\right)\right] \ .$$

The negative-frequency normalized solution is

$$\psi_{-\infty} = \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \cdot \mathrm{e}^{-i\int \omega dt} \ , \omega \equiv \sqrt{\vec{k}^2 \mathcal{B}_g^s}.$$

$$\zeta = \frac{(|\vec{k}|\sqrt{\mathcal{B}_g^s})^{-\nu_s}}{2^3\pi\sqrt{A_g^s}} \cdot \left(-t\sqrt{\mathcal{B}_g^s}|\vec{k}|\right)^{\nu_s} \cdot H_{\nu_s}^{1,2}\left(-t\sqrt{\mathcal{B}_g^s}|\vec{k}|\right) ,$$

$$n_s = 3 + \theta_g^s = 3 - 2\mu + \delta ,$$

$$A_\zeta = \frac{\Gamma^2(\nu_s)(\mathcal{B}_g^s)^{-\nu_s} k_*^{3-2\nu_s}}{2^{4-2\nu_s}\pi^3 A_g^s} .$$

where

$$\nu_s \equiv \frac{1-\theta_g^s}{2} = \mu + \frac{1}{2} - \frac{\delta}{2} .$$

The tensor spectrum

$$\mathcal{A}_g^T = \frac{(-ct)^{-2\mu}}{8} \cdot a_g^3 \ , \ \ \mathcal{B}_g^s = \frac{1}{a_g^2} > 0,$$

$$\theta_g^T = t \cdot \frac{\dot{\mathcal{A}}_g^s}{\mathcal{A}_g^s} = -2\mu < 0 \ .$$

$$n_T = 2 + \theta_g^T = 2 - 2\mu \ ,$$

$$A_T = \frac{2\Gamma^2(\nu_T)(\mathcal{B}_g^T)^{-\nu_T} k_*^{3-2\nu_T}}{2^{4-2\nu_T}\pi^3 A_g^T} \ ,$$

where

$$\nu_T \equiv \frac{1-\theta_g^T}{2} = \mu + \frac{1}{2} \ .$$

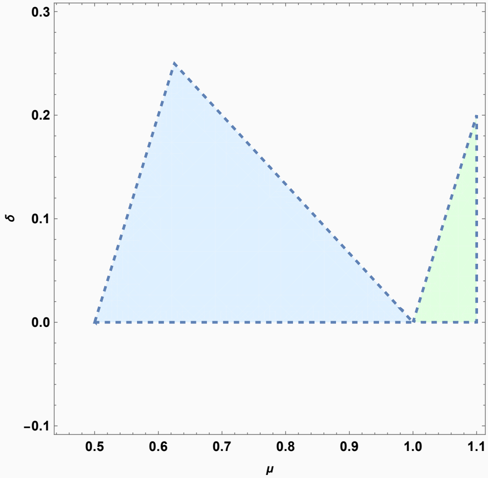


Figure 4: The range of parameters $(\mu,\ \delta)$. The blue area corresponds to unitarity and No-Go constraints, while the green area corresponds to the condition $n_s < 1$.

Do we believe in spectrum calculations?

Not for every model parameters...

$$\mu = 7/10,\ \delta = 1/10,\ c = 10^{-4},\ g = \frac{1}{77}\cdot 10^{-4},\ s = 10^{-4}.$$

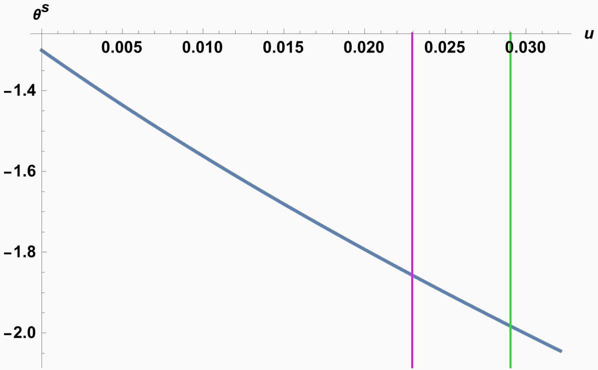


Figure 5: The $\theta^s(u)$ in the vicinity of freeze point. The magenta line is $u(t^{fr})$, while the green line is $u(t_0^{fr})$.

What is going on?

The Hubble parameter is

$$H = \frac{h(u)}{N(u) \cdot (-c \cdot t)^{1+\delta}},\ u = (-c \cdot t)^{-\delta},\ \delta > 0\ ,$$

where

$$h(u) = \frac{g}{6c(\delta + 2\mu + 1)} - \frac{ug^2(5\delta + 8\mu + 4)}{72c^3(\delta + 2\mu + 1)^3(2\delta + 2\mu + 1)} + O(u^2)\ .$$

The corrections by u variable are negligible if

$$|t| \ll t_{nl} = \frac{1}{c}\left(\frac{g}{c^2 M_{Pl}^2}\right)^{1/\delta}\ ,$$

We roughly checked all parameter space ($2 \cdot 10^4$ points) and still no red-tilted spectrum!

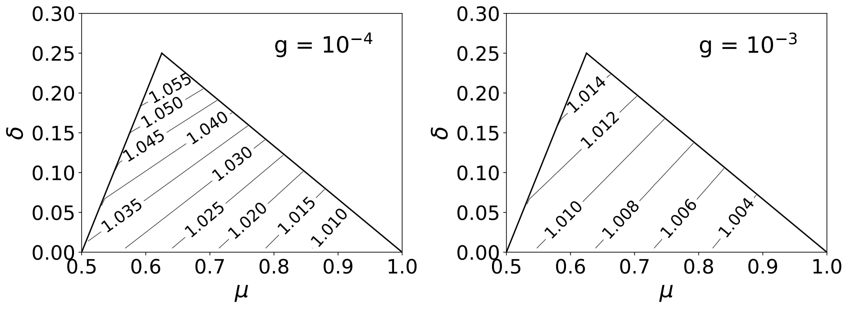


Figure 6: Numerical results of spectral tilt n_s in $\mu - \delta$ plane, which are shown as level curves. Each panels assumes different g values, which are shown in the upper right corners. $c = s = 10^{-4}$ is common for all panels.

$$E_{\text{class}} = \max\left\{H,\ \frac{\dot{H}}{H},\ \frac{\dot{\phi}}{\phi}\right\} \propto \frac{1}{|t|}\ .$$

We consider the regime, when $E_{strong} \gg E_{scatter} \gg E_{classical} > H$.

$$\mathcal{S}^{(2)}_{\zeta\zeta} = \int dt d^3x N a^3 \left[\frac{\mathcal{G}_S}{N^2}\left(\frac{\partial\zeta}{\partial t}\right)^2 - \frac{\mathcal{F}_S}{a^2}\left(\vec{\nabla}\zeta\right)^2\right]$$

Canonical normalizations

$$\zeta_c \propto \sqrt{\mathcal{G}_S}\zeta\ ,\quad \dot{\mathcal{G}}_S/\mathcal{G}_S \ll E_{scatter}\ .$$

The dispersion relation

$$w^2 = u_s^2|\vec{k}|^2\ .$$

The most strict bounds arises from cubic scalar action for perturbations Ageeva'2018, Ageeva'2020.

The Cubic Lagrangian

$$S_1^{(3)} \equiv \int N dt a^3 d^3x \Big[\Lambda_2(\dot{\zeta}^2/N^2)\zeta + (a^{-2})\Lambda_4(\dot{\zeta}/N)\zeta\partial^2\zeta + (a^{-2})\Lambda_5(\dot{\zeta}/N)\left(\partial_i\zeta\right)^2 + \dots\Big] .$$

The most restrictive terms are: $\Lambda_1,\ \Lambda_3,\ \Lambda_7, \Lambda_{10},\ \Lambda_{14},\ \Lambda_{16}$. After the canonical normalization

$$S_0^{(3)} = \int d\tilde{t} d^3\tilde{x} \Big[\Lambda_1 \frac{\zeta_c'^3}{\mathcal{G}_S^{3/2}} + \Lambda_3 \frac{\zeta_c'^2}{\mathcal{G}_S^{3/2}} \tilde{\partial}^2\zeta_c + \Lambda_7 \frac{\zeta_c'}{\mathcal{G}_S^{3/2}} \left(\tilde{\partial}^2\zeta_c\right)^2 + \dots\Big] .$$

We consider two to two scattering. The PWA are

$$a_l \propto \int d(cosx) P_l(cosx) M \, .$$

The optical theorem is follows from unitarity of S - matrix and it is

$$Im\ a_l = |a_l|^2 \, .$$

Then, bounds are

$$\left| Re\ a_l \right| < \frac{1}{2} \, .$$

One defines

$$\mathcal{E} \equiv \frac{E}{N} \ .$$

Omitting all numerical factors, one arrives to

$$|M(\mathcal{E}_{strong})| \propto u_s^3 < 1 \Rightarrow \mathcal{E}_{strong} = \ldots .$$

The u_s^3 factor is arises from non-trivial dispersion relation.

- The unitarity bound is saturated when the absolute the tree matrix element is roughly equal to unity.
- If one wants to obtain exact unitarity bound (at tree level), one needs to calculate the s, u, t – channels for tree level $2 \to 2$ matrix element, then go to PWAs and use the optical theorem.

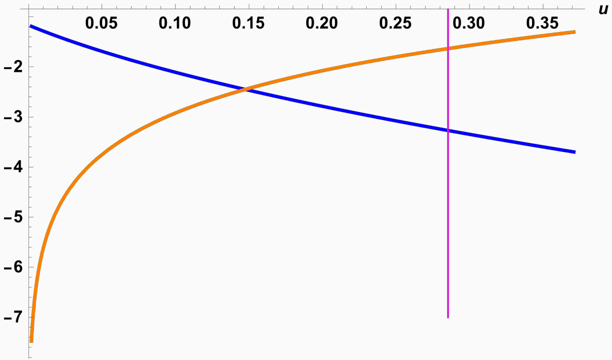


Figure 7: The blue line is $\theta^s(u)$. The orange line is $-\log_{10}\left(\frac{\mathcal{E}_{strong}}{\mathcal{E}_{classical}}\right)$. The magenta line is $u(t^{fr})$.

Additional check

The calculation of two to two scattering is valid only if the coefficients in vertexes Λ_i are changing slowly in comparison with the characteristic time scale $t_{scatter}$ of the scattering ($E_{scatter} \gg E_{class}$):

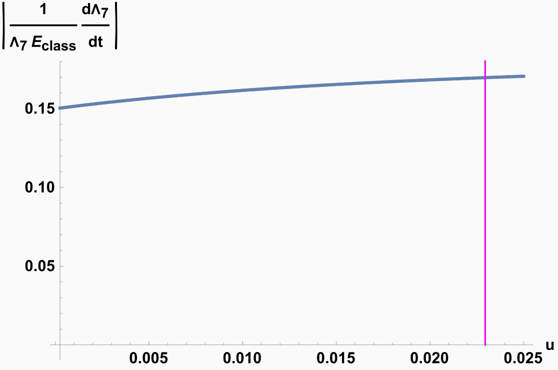


Figure 8: The magenta line is $u(t^{fr})$. The Λ_7 is one of the most restrictive terms!

1. It is nearly impossible to obtain the red-tilted spectrum!
2. The resolution of this problem is quite straightforward – the introduction of the spectator field.
3. Let us introduced the spectator in the spirit of Creminelli'2010,Libanov'2016 Tahara'2020fmn, i.e. in the way when the spectator field is invariant under the scaling transformation.

Scaling transformation

For $\mu = 1$ and $\delta = 0$ the Jordan frame Lagrangian is invariant under the scale symmetry (global conformal symmetry):

$$\tilde{\phi} = \phi - \ln \lambda \ ,$$
$$\tilde{g}_{\mu\nu} = \lambda^2 g_{\mu\nu} \ .$$

Thus, the spectator field is

$$S_\sigma = \int dt d^3x \sqrt{-g} e^{2\phi} \left(-\frac{1}{2} (\partial\sigma)^2 \right) .$$

- The effective scale factor is

$$a_{eff} = e^{\phi} \cdot a.$$

- By adding the potential terms one can tilt the power spectrum in either way!
- The conversion of fluctuations σ into adiabatic modes could happen through the one of the mechanisms from Lyth'2001, Dvali'2003, Dvali'2003.
- The model suggests a natural way to deform, which changes the amplitude of tensor perturbations and gives us an easy opportunity to obtain a subsequent small value for the r – ratio!

- We have found the minimum setup in the framework of Horndeski gravity that could describe non-singular cosmology.
- In this setup, we build the Genesis scenario.
- We show that the background solution is stable during the whole evolution, and the speed of scalar perturbations does not exceed the speed of light.
- While the speed of the tensor perturbation stays equal to unity.

- In our model, there exists two distinct regimes (power-law behavior and non-power law).
- We have implicitly shown that our first solution does not breaks unitarity at early times.
- Despite the highly non power-law behavior, there exist parameters for which the background solution stays out of the strong-coupling regime.

- It is impossible to have a red scalar spectrum and maintain unitarity both for power– and non power– law regimes!
- We came up with the spectator field mechanism $\Rightarrow$ Allows us to produced red-tilted scalar spectrum!
- We suggested a deformation of the model $\Rightarrow$ give an opportunity to achieve small r – ratio!

The end

Thank you for your attention!

No-Go theorem

Coefficients $\mathcal{F}_S, \mathcal{G}_S, \mathcal{F}_T, \mathcal{G}_T$ are given by:

$$\mathcal{F}_T = 2G_4 + ..., \quad \mathcal{G}_T = 2G_4 + ...,$$

and

$$\mathcal{F}_S = \frac{1}{a}\frac{d}{dt}\left(\frac{a}{\Theta}\mathcal{G}_T^2\right) - \mathcal{F}_T, \quad \mathcal{G}_S = \frac{\Sigma}{\Theta^2}\mathcal{G}_T^2 + 3\mathcal{G}_T,$$

where Σ and Θ are some cumbersome expression of G_2, G_3, G_4 and H.
Stability conditions are:

$$\mathcal{G}_T \geq \mathcal{F}_T > 0, \quad \mathcal{G}_S \geq \mathcal{F}_S > 0.$$

Denote $\xi = a\mathcal{G}_T^2/\Theta$, we rewrite $\mathcal{F}_S$ as

$$\mathcal{F}_S = \frac{1}{a}\frac{d\xi}{dt} - \mathcal{F}_T \to \frac{d\xi}{dt} > a\mathcal{F}_T > 0$$

$$\tilde{A}_2 = \frac{g_1}{2} f^{-2\mu-2-\delta}\Big(- \frac{g}{N^2} + \frac{g}{3N^4}\Big) \cdot (1-U) + \frac{U}{3N^2\big(\frac{2f}{c}+t\big)^2} \; ,$$

$$\tilde{A}_3 = 0 \; ,$$

$$\tilde{A}_4 = -\frac{g_1}{2}\left(f - 1 + g_1^{\frac{1}{2\mu}}\right)^{-2\mu} , \; f - 1 + g_1^{\frac{1}{2\mu}} > 0 \; , g_1 > 0 \; .$$

$$\frac{d\xi}{dt} > a\mathcal{F}_T > 0, \quad \xi = a\mathcal{G}_T^2/\Theta,$$

Here $|\Theta| < \infty$ everywhere and it is smooth function of time (as it is function of ϕ and H), so ξ can never vanish (except $a = 0$) $\rightarrow$ thus we demand non-singular model. Integrating from some t_i to t_f, we obtain:

$$\xi(t_f) - \xi(t_i) > \int_{t_i}^{t_f} a(t)\mathcal{F}_T dt,$$

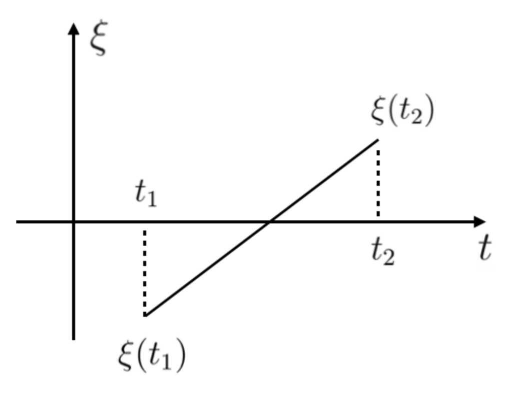
where $a > const > 0$ for $t \rightarrow -\infty$ and it is increasing with $t \rightarrow +\infty$.

$$\xi(t_f) - \xi(t_i) > \int_{t_i}^{t_f} a(t)\mathcal{F}_T dt,$$

- Let $\xi_i < 0$, so
$$-\xi_f < |\xi_i| - \int_{t_i}^{t_f} a\mathcal{F}_T dt,$$
where RHS $\rightarrow$ negative with $t_f \rightarrow +\infty$. So therefore $\xi_f > 0$. And it means that $\xi = 0$ at some moment of time - singularity! So we should demand $\xi > 0$ for all times.
- But on the other had, again just rewritting:
$$-\xi_i > -\xi_f + \int_{t_i}^{t_f} a\mathcal{F}_T dt,$$
and now RHS $\rightarrow$ positive with $t_i \rightarrow -\infty$ and ξ_i must be negative. Again contradiction...

Thus we have two important features here:

$$1.\xi \neq 0,$$
$$2.d\xi/dt > a\mathcal{F}_T > 0.$$



$$\xi(t_f) - \xi(t_i) > \int_{t_i}^{t_f} a(t)\mathcal{F}_T dt,$$

$$G_2 = A_2 - 2XF_\phi,$$
$$G_3 = -2XF_X - F,$$
$$G_4 = B_4,$$

where $F(\phi, X)$ is an auxiliary function, such that

$$F_X = -\frac{A_3}{(2X)^{3/2}} - \frac{B_{4\phi}}{X},$$

with

$$N^{-1}d\phi/dt = \sqrt{2X}.$$

EoMs are

$$(NA_2)_N + 3NA_{3N}H + 6N^2(N^{-1}A_4)_N H^2 = 0,$$
$$A_2 - 6A_4H^2 - \frac{1}{N}\frac{d}{d\hat{t}}(A_3 + 4A_4H) = 0\ .$$

$$G_2 = \frac{gX\left(-3c^2 e^{2\phi} + 2X\right) e^{\phi(\delta+2\mu-2)}}{3c^4} + 4\mu^2 X e^{2\mu\phi} \ln\left(\frac{X}{X_0}\right) \ ,$$

$$G_3 = \mu e^{2\mu\phi}\left(\ln\left(\frac{X}{X_0}\right) + 2\right) \ ,$$

$$G_4 = \frac{1}{2} e^{2\mu\phi} \ .$$

- Geodesic incompleteness for gravitons. Creminelli'16.
- This can be understood by moving to the Einstein frame.

Does we solve the initial singularity problem?!

- The transformation between frames is singular at minus infinity!
- The geodesic (in)completeness should be generalizes Rubakov'22!
- The physical time based on counting the oscillations of photons wave functions Wetterich:2024.