Appendix C

Cartan-Maurer and Spinors

The existence of the metric, accompanied by the Levi-Civita connection, defines an invariant notion of the length. This also means that the "rotation" by the structure group of the tangent bundle reduces to $SO(1, d-1)$ or to $SO(d)$. However, this is not manifest in the coordinate basis; for instance, $\mathbb{\Gamma}$ takes values in $GL(d)$. An effective way to make the structure group manifest at the level of the connection is to use an alternate basis of orthonormal frames, also known as the vielbein, in place of the familiar coordinate basis.

This alternate description, the Cartan-Maurer formulation, makes clear that the Riemannian geometry is not different from more general and abstract geometric structure of bundles and connections, on which we have devoted a section in the main text. Here, we will take a more practical approach by adapting the most basic aspects of Yang-Mills connections and the curvatures thereof, to which the Cartan-Maurer fits perfectly. The fundamental distinction for the gravity is the presence of the covariantly constant orthonormal frame, which of course connects back to the metric and makes the dynamics of the gravity qualitatively different from the Yang-Mills gauge theories.

It is worthwhile to warn the readers that, below, we use multiple notations for the covariant derivative, such as ∇ and D, although they are not fundamentally distinct objects. We will use ∇ when we want it to be aware only of the coordinate indices of the object it acts on, while D knows not only the coordinate indices but the orthonormal indices, or the local Lorentz indices, as well. For local computations, i.e., chart by chart, one does have the option of using ∇ only, although this tends to complicate various middle steps. In the main text, we also invoked the symbol D when the covariant derivative includes gauge connections. This is actually congruent to our usage of D here since the new part in the symbol D, relative to ∇, can be also considered an SO connection of the frame bundle.

In the same spirit, the notation $\mathscr{D}$ is introduced when the covariant derivative extends to the spinors as well. In the language of bundles, all these connections originate from the connection as the horizontal lift of the common SO principle bundle, as in Section 5.1.2. These different notations ∇, D, $\mathscr{D}$, merely refers to how we realize this abstract connection incrementally on sections of the tangent bundle, the frame bundle, and the spin bundle.

This practice is also in line with how we must extend the definition of the Lie derivative $\mathfrak{L}_\xi$ to $\mathscr{L}_\xi$ when it acts on the vielbein and on spinors. The latter is known as the Kosmann lift and, after Kosmann's 1970 treatise, has been independently discovered by later scholars as well, sometimes referred to as the Lie-Lorentz derivative. For some reason, this important fact did not receive wide recognition in the physics community. We will see how this extension is entirely unavoidable and devote the middle half of this Appendix to the Kosmann lift.

C.1 Cartan-Maurer Formulation

C.1.1 Orthonormal Frame

The Christoffel connection 1-form, $\mathbb{\Gamma}$, is useful but remains somewhat misleading in the larger context of the modern differential geometry. The parallel transport of Riemannian geometry is designed to preserve the metric, which means that it actually rotates vectors preserving the length. The rotation should be $SO(d)$ or $SO(1, d-1)$ types, depending on the signature, yet in the coordinate basis, the matrix rotates as $GL(d)$. With the Christoffel symbol, therefore, the underlying $SO(d)$ or $SO(1, d-1)$ structure is not manifest. Here we wish to introduce the orthonormal frames, also known as the vielbein with which the structure group of the Riemannian manifold becomes more transparent.

Given a metric, one can always find a basis for vectors that behave like the usual orthonormal basis, in the sense that

$$e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}. \tag{C.1.1}$$

These are d-many $d \times d$ invertible matrices at any given point, and define the orthonormal frame,

$$e_a = e_a{}^\mu \frac{\partial}{\partial x^\mu} \ . \tag{C.1.2}$$

The same relation may be inverted to

$$g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu e^b{}_\nu \tag{C.1.3}$$

where the inverse $e^a{}_\mu$'s obeying

$$e^a{}_\mu e_a{}^\nu = \delta_\mu{}^\nu \ , \qquad e^a{}_\mu e_b{}^\mu = \delta^a{}_b \tag{C.1.4}$$

are introduced.

With these, we find

$$e^a{}_\mu e^b{}_\nu g^{\mu\nu} = \eta^{ab}. \tag{C.1.5}$$

define a set of 1-forms,

$$e^a = e^a{}_\mu \ dx^\mu \ , \tag{C.1.6}$$

with which we may write the metric as

$$g \ = \ \eta_{ab}\ e^a \otimes e^b \ . \tag{C.1.7}$$

These orthonormal basis are also called the vierbein or the tetrad for $d = 4$, the dreibein or the triad for $d = 3$, and the zweibein or the dyad for $d = 2$.

This orthonormality allows us to use the basis 1-forms to build up unit n-forms that measures area, volume, etc. For example, a unit area element spanned by e^a and e^b is

$$e^a \wedge e^b \ , \tag{C.1.8}$$

while unit 3-volume enclosed by $e^{a,b,c}$ is

$$e^a \wedge e^b \wedge e^c \ . \tag{C.1.9}$$

If one is interested in computing d-volume, we merely wedge-product the entire basis 1-forms as

$$e^{\hat{1}} \wedge e^{\hat{2}} \wedge \cdots \wedge e^{\hat{d}} = (\det e)\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^d \ , \tag{C.1.10}$$

where the antisymmetrization induced by the wedge product is responsible for the coefficient $(\det e)$.

On the other hand,

$$g_{\alpha\beta} = e^a{}_\alpha e^b{}_\beta \eta_{ab} \ , \tag{C.1.11}$$

so that

$$\det g = (\det e)^2 \underbrace{(\det \eta)}_{\pm 1} \ , \qquad \det e = \underbrace{\sqrt{|\det g|}}_{\equiv \sqrt{g}} \tag{C.1.12}$$

bringing us back to the volume form we have defined earlier,

$$\mathcal{V} = \sqrt{g}\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^d = e^{\hat{1}} \wedge e^{\hat{2}} \wedge \cdots \wedge e^{\hat{d}} \ . \tag{C.1.13}$$

It is clear that, given the antisymmetric nature of the wedge product, there are two possible choices of ordering involved here with relative sign difference for the d-volume form. This choice is called the Orientation, as we already encountered in the main text as well as in the previous chapter of the appendix.

These unit length, area, and general n-volume built out of e^a's allow us to define physical quantities such as energy densities and charge densities naturally. For instance, given a current j^μ, we interpret its time component as the charge density, but with respect to what 3-volume? In the orthonormal frame, a natural $(d-1)$-volume is

$$dS_a = \frac{1}{(d-1)!} \epsilon_{ab_1 \cdots b_{d-1}}\, e^{b_1} \wedge \cdots \wedge e^{b_{d-1}} \ , \tag{C.1.14}$$

so one naturally integrates

$$j^{\hat{0}} dS_{\hat{0}} = j^{\hat{0}} e^{\hat{1}} \wedge e^{\hat{2}} \wedge \cdots \wedge e^{\widehat{d-1}} \tag{C.1.15}$$

along a spatial hypersurface, and compute the total charge.

C.1.2 Connection 1-Form and Curvature 2-Form

These e's are merely particularly selected sets of vectors and of 1-forms, so the Levi-Civita covariant derivative acts on the tangent index as usual,

$$\begin{aligned} \nabla_\mu e_a{}^\nu &= \partial_\mu e_a{}^\nu + \Gamma^\nu{}_{\mu\lambda} e_a{}^\lambda \;, \\ \nabla_\mu e^a{}_\nu &= \partial_\mu e^a{}_\nu - \Gamma^\lambda{}_{\mu\nu} e^a{}_\lambda \;. \end{aligned} \tag{C.1.16}$$

The properties (C.1.1) and (C.1.5) imply that the change of these under covariant derivative can, at most, rotate these objects among themselves, and since they are each complete basis, the following must happen,

$$\nabla_\mu e_a{}^\nu = w_\mu{}^b{}_a \, e_b{}^\nu \;, \qquad \nabla_\mu e^a{}_\nu = -w_\mu{}^a{}_b e^b{}_\nu \tag{C.1.17}$$

for some set of w's. The basis e^a's and e_a's are of unit lengths each, so the rotation represented by $w_\mu{}^a{}_b$ must preserve this unit length. This means that w is a matrix-valued 1-form with the matrices belonging to the Lie Algebra of $SO(d)$ or $SO(1, d-1)$, depending on the signature of η.

Since the metric component is η_{ab} in this orthonormal basis, the usual condition of the covariant derivative killing the metric translates to

$$\begin{aligned} 0 &= \nabla_\mu \eta_{ab} = \nabla_\mu \left(e_a{}^\alpha \, e_b{}^\beta \, g_{\alpha\beta} \right) \\ &= (\nabla_\mu \, e_a{}^\alpha) \, e_b{}^\beta \, g_{\alpha\beta} + e_a{}^\alpha \, (\nabla_\mu \, e_b{}^\beta) \, g_{\alpha\beta} \\ &= (w_\mu{}^c{}_a \, e_c{}^\alpha) \, e_b{}^\beta \, g_{\alpha\beta} + e_a{}^\alpha \, (w_\mu{}^c{}_b \, e_c{}^\beta) \, g_{\alpha\beta} \;. \end{aligned} \tag{C.1.18}$$

So we find

$$0 = \eta_{bc}\, w^c{}_a + \eta_{ac}\, w^c{}_b \ , \qquad \text{(C.1.19)}$$

where we used (C.1.1).

In particular, (C.1.17) can be used to extract the covariant derivative for tensors in the orthonormal basis. For this, let us consider the usual covariant derivative on $W_\mu = W_a e^a{}_\mu$,

$$\begin{aligned}
\nabla_\mu W_\nu &= \partial_\mu (W_a e^a{}_\nu) - W_a \Gamma^\lambda{}_{\mu\nu} e^a{}_\lambda \\
&= (\partial_\mu W_a) e^a{}_\nu - W_a\, w_\mu{}^a{}_b e^b{}_\nu \\
&= \left(\partial_\mu W_a - W_b\, w_\mu{}^b{}_a\right) e^a{}_\nu \ , \qquad \text{(C.1.20)}
\end{aligned}$$

and similarly,

$$\begin{aligned}
\nabla_\mu V^\nu &= \partial_\mu (V^a e_a{}^\nu) + V^a \Gamma^\nu{}_{\mu\lambda} e_a{}^\lambda \\
&= (\partial_\mu V^a) e_a{}^\nu + V^a\, w_\mu{}^b{}_a e_b{}^\nu \\
&= \left(\partial_\mu V^a + \ w_\mu{}^a{}_b V^b\right) e_a{}^\nu \ . \qquad \text{(C.1.21)}
\end{aligned}$$

These tell us that the role of Γ is now played by w if we try to extend the covariant derivative ∇ to objects with orthonormal basis. We will denote this extended covariant derivative by D.

If one is to define covariant derivative D in the orthonormal basis, one must ensure

$$(\nabla_\mu V^\nu)\, \partial_\nu = (D_\mu V^a)\, e_a \ , \qquad (\nabla_\mu W_\nu)\, dx^\nu = (D_\mu W_a)\, e^a \qquad \text{(C.1.22)}$$

with $V^\mu\, \partial_\mu = V^a\, e_a$ and $W_\mu\, dx^\mu = W_a\, e^a$, where we extend the covariant derivative ∇ to D which is aware of the orthonormal indices. With this, we find

$$\begin{aligned}
D_\mu W_a &= \partial_\mu W_a - W_b\, w_\mu{}^b{}_a \ , \\
D_\mu V^a &= \partial_\mu V^a + w_\mu{}^a{}_b V^b \ , \qquad \text{(C.1.23)}
\end{aligned}$$

which is the analog of (B.4.14) but for orthonormal basis instead of coordinate basis.

We can repeat this for objects that has both types of indices, e.g.,

$$\begin{aligned} D_\mu W^a_\nu &= \partial_\mu W^a_\nu + w_\mu{}^a{}_b W^b_\nu - \Gamma^\alpha{}_{\mu\nu} W^a_\alpha \ , \\ D_\mu V^\nu_a &= \partial_\mu V^\nu_a - w_\mu{}^b{}_a V^\nu_b + \Gamma^\nu{}_{\mu\alpha} V^\alpha_a \ . \end{aligned} \tag{C.1.24}$$

This guarantees

$$D_\mu e^a{}_\nu = 0 \ , \qquad D_\mu e_a{}^\nu = 0 \ , \tag{C.1.25}$$

which are merely (C.1.17) rewritten by moving the right hand sides to the left hand sides. This in turn tells us that the covariant derivative indeed preserve the metric, which is a symmetric tensor product, $\eta_{ab} e^a \otimes e^b$

$$D_\mu g_{\alpha\beta} = \nabla_\mu g_{\alpha\beta} = 0 \ , \tag{C.1.26}$$

follows automatically.

With this SO structure manifest, we can also introduce spinors on the same manifold. In fact, the new version of the connection w that starts from the orthonormal frame is often called the "spin connection" as it enters the covariant derivative of spinors naturally and is indispensable for understanding spinors in curved backgrounds.

Curvature 2-Form and Vanishing Torsion

With this D, we again find the same condition on w above,

$$0 = D_\mu \eta^{ab} = \partial_\mu \eta^{ab} + w_\mu{}^a{}_c \eta^{cb} + w_\mu{}^b{}_c \eta^{ac} = w_\mu{}^{ab} + w_\mu{}^{ba} \ , \tag{C.1.27}$$

so that w is an SO connection. Antisymmetrizing the coordinate indices of the identity, $D_\mu e^a{}_\nu = 0$, we find a relation between the exterior derivative of the 1-form e^a and the matrix-valued 1-form w,

$$0 = de^a + w^a{}_b \wedge e^b \ , \tag{C.1.28}$$

where the Christoffel symbol drops out due to its symmetric property under the exchange of the lower two indices.

Note that the latter vanishing condition emerged from ${\Gamma^\lambda}_{\mu\nu} = {\Gamma^\lambda}_{\nu\mu}$. From an earlier discussion of torsion, we learned that the antisymmetric part of more general metric-preserving affine connection ${\mathfrak{C}_\mu}^\lambda{}_\nu$ in place of ${\Gamma^\lambda}_{\mu\nu}$ is the torsion ${T^\lambda}_{\mu\nu}$. Imposing $De^a = 0$ for the sake of $\nabla g = 0$, we find that the torsion T with its upper index in the orthonormal basis has the form,

$$2T^a \equiv \; {e^a}_\lambda {T^\lambda}_{\mu\nu} \, dx^\mu \wedge dx^\nu = de^a + {w^a}_b \wedge e^b \; . \tag{C.1.29}$$

Later we will see how $T^a = 0$ condition arises naturally by deriving the General Relativity from an action principle.

(C.1.28) combined with (C.1.19) determines the matrix-valued 1-form w entirely. Once w's are thus determined, the Riemann tensor is recovered in the same manner as the usual gauge field strength, via a matrix-valued 2-form,

$$ {\mathcal{R}^a}_b \equiv {[(d+w)\wedge(d+w)]^a}_b = d{w^a}_b + {w^a}_f \wedge {w^f}_b = \frac{1}{2}\, {R^a}_{bcd}\, e^c \wedge e^d \; , \tag{C.1.30}$$

with

$$ {(d+w)^a}_b = {\delta^a}_b \, d + {w^a}_b \; . \tag{C.1.31}$$

The Bianchi identity $d_{\mathcal{A}}\mathcal{F} = 0$ that holds for general field strength should hold for $\mathcal{R}$ as well,

$$ {(d_w \mathcal{R})^a}_b = d{\mathcal{R}^a}_b + {w^a}_f \wedge {\mathcal{R}^f}_b - {\mathcal{R}^a}_f \wedge {w^f}_b = 0 \; . \tag{C.1.32}$$

The curvature 2-form $\mathcal{R}$ is sometimes called, confusingly, the Ricci 2-form, but to be distinguished from the symmetric Ricci tensor $R_{\alpha\beta} = {R^\mu}_{\alpha\mu\beta}$.

The components of $\mathcal{R}$ here are related to the coordinate basis expression (B.4.17). For instance, with ${R^a}_{b\mu\nu} \equiv {R^a}_{bcd}{e^c}_\mu {e^d}_\nu$ in a mixed basis, we find

$$ {R^a}_{b\mu\nu} = {e^a}_\alpha \, {e_b}^\beta \; {R^\alpha}_{\beta\mu\nu} \; , \tag{C.1.33}$$

via e's. This follows from

$$w_\mu{}^a{}_b = -e_b{}^\nu \nabla_\mu e^a{}_\nu = \Gamma^\lambda{}_{\mu\nu} e^a{}_\lambda e_b{}^\nu - e_b{}^\nu \partial_\mu e^a{}_\nu = e^a{}_\lambda \Gamma^\lambda{}_{\mu\nu} e_b{}^\nu + e^a{}_\nu \partial_\mu e_b{}^\nu \tag{C.1.34}$$

where $e_b{}^\nu$ and its inverse $e^a{}_\nu$ relate Γ and w via a $GL(d)$ "gauge" transformation, on par with (B.4.8) with $e_a{}^\nu$ playing the role of U. Alternatively, we may also write

$$w_{\mu\, b}{}^a = e_b{}^\nu \mathbb{\Gamma}_{\mu\nu}{}^\lambda e^a{}_\lambda + e_b{}^\lambda \partial_\mu e^a{}_\lambda \tag{C.1.35}$$

now with the inverse $e^a{}_\lambda$ acting like U, mapping the Christoffel connection 1-form $\mathbb{\Gamma}$ to w via a "gauge" transformation with $U_\lambda{}^a = e^a{}_\lambda$.

C.2 Spinors and Kosmann Lift

C.2.1 Spinors

Once we understand the matter of orthonormal basis and how to formulate Riemannian geometry in that language, the spinor follows immediately. The connection w that appears in the Cartan-Maurer formulation is also called the spin connection because it is natural and necessary when we couple spinors to gravity minimally. As we have repeatedly in the main text, a spinor Ψ may be thought of as a column vector with $2^{d/2}$-many component, which is a representation of the Clifford algebra. The latter is generated by the Dirac matrices γ's that obey

$$\gamma^a\gamma^b + \gamma^b\gamma^a = 2\,\eta^{ab}\,1_{2^{d/2}\times 2^{d/2}} \ . \tag{C.2.1}$$

It is important to know that the Dirac matrices here are constant matrices, $\partial_\mu\gamma^a = 0$.

With these, the covariant derivative acts on spinors as

$$\mathscr{D}_\mu\Psi = \left(D_\mu + \frac{1}{4} w_{\mu\, ab}\gamma^{ab}\right)\Psi \ , \tag{C.2.2}$$

where SO tensor indices and coordinate indices respond to D in the usual manner, while

$$\gamma^{ab} \equiv \frac{1}{2}\left(\gamma^a\gamma^b - \gamma^b\gamma^a\right) \tag{C.2.3}$$

are the SO rotation generators for spinor indices. The Dirac matrices are not only constant but also covariantly constant since

$$\mathscr{D}_\mu(\gamma^a) = [\mathscr{D}_\mu, \gamma^a] = w_\mu{}^{ab}\gamma_b + \frac{1}{4}w_\mu{}^{fg}[\gamma_{fg}, \gamma^a] = w_\mu{}^{ab}\gamma_b + w_\mu{}^{fa}\gamma_f = 0 \ , \qquad \text{(C.2.4)}$$

which produces, with $\mathscr{D}_a \equiv e_a^{\ \mu}\mathscr{D}_\mu$,

$$(\mathbb{i}\gamma^a\mathscr{D}_a)^2 = -\gamma^a\gamma^b\mathscr{D}_a\mathscr{D}_b = -\mathscr{D}^a\mathscr{D}_a + \frac{1}{4}R \ , \qquad \text{(C.2.5)}$$

the well-known Weitzenböck formula.*

This covariant derivative is consistent with D and thus with ∇ in the following sense. The most general tensors built out of spinors are spanned by

$$V^{a_1\cdots a_k} \equiv \Psi^\dagger\gamma^{a_1\cdots a_k}\Psi \qquad \text{(C.2.6)}$$

with $\gamma^{a_1\cdots a_k}$ an antisymmetric product of γ_a's and tensor products thereof. The covariant derivative D on such V's obey the Leibniz rule,

$$D_\mu V^{a_1\cdots a_k} = (\mathscr{D}_\mu\Psi)^\dagger\,\gamma^{a_1\cdots a_k}\Psi + \Psi^\dagger\gamma^{a_1\cdots a_k}\,(\mathscr{D}_\mu\Psi) \qquad \text{(C.2.7)}$$

which can be verified straightforwardly.

Alternatively, for some purposes, one can also replace

$$\mathscr{D}_\mu \quad \rightarrow \quad \mathbb{i}\pi_\mu \equiv \partial_\mu + w_{\mu\,ab}\gamma^{ab}/4 \qquad \text{(C.2.8)}$$

which are equivalent only if the covariant derivative acts strictly on spinors only. The latter π_μ does not commute with γ's, yet we arrive at an equivalent form of the Weitzenböck formula,

$$(\gamma^a e_a^{\ \mu}\pi_\mu)^2\Psi = \frac{1}{\sqrt{g}}\pi_\lambda\sqrt{g}g^{\lambda\mu}\pi_\mu\Psi + \frac{1}{4}R\Psi \ . \qquad \text{(C.2.9)}$$

The Dirac operator appears naturally in this form as the supercharge of supersymmetric non-linear sigma model, with the target manifold with metric g; π_μ's are the

*In Appendix C, we are using the notation D and $\mathscr{D}$ for the covariant derivatives on tensors and on spinors, respectively, without additional gauge fields. This is in part because the covariant derivative to enter the Kosmann lift in next section should be purely gravitational. Restoring the general gauge fields, e.g. for the general Weiztenböck formulae, should be straightforward.

covariantized conjugate momenta of the bosonic coordinates.

C.2.2 The Kosmann Lift

We need to emphasize here that the Cartan-Maurer formulation and the vielbein we introduced above are indispensable for physics in the curved spacetime if we are to discuss spinor fields. For metric and other tensors, the coordinate basis and the orthonormal basis are two equivalent and interchangeable formulations. For spinors, however, there is no natural starting point if we insist on the coordinate basis, as indicated by the prominent appearance of the spin connection for the covariant derivative $\mathscr{D}_\mu$ on the spinor and how γ^a's labeled by the orthonormal indices are truly constant matrices. This forces us to rethink how the diffeomorphim should acts on spinors and by inference on the vielbein.

The diffeomorphism and the Lie derivative that generates it are most primitive part of the differential geometry, and can be defined as soon as we have the notion of the vector field, with the help of the pull-back and the push-forward. On differential forms Ω, we have

$$\mathfrak{L}_\xi \Omega = d(\xi \lrcorner \Omega) + \xi \lrcorner \, d\Omega \ , \tag{C.2.10}$$

for example. In particular the Lie derivative is something we can define, well ahead of introducing the metric or the connection.

The Clifford algebra itself requires the metric data, on the other hand, which implies that extension of the Lie derivative to the spinor bundle and the frame bundle may be more involved. How should such a Lie derivative be extended when Ω acquires an SO index or spinor indices? The question is how we should view the vector field ξ when we go beyond the tangent bundle and the co-tangent bundle. Is there a natural extension of ξ when we introduce the frame bundle and the spinor bundle? The answer to this geometric question came pretty late in the early 1970, a full century after Riemann's initial proclamation on how Euclid's geometry should be extended. We will devote this section on this relatively obscure feature of the Lie derivative, known as the Kosmann lift.

To explore the Kosmann lift, a convenient starting point is how it acts on spinors, from which the action on the vielbein follows immediately. The answer by Kosmann is to extend the Lie derivative $\mathfrak{L}_\xi$ to spinors as, with purely gravitational $\mathscr{D}_\mu$,

$$\mathscr{L}_\xi \Psi = \xi^\mu \mathscr{D}_\mu \Psi - \frac{1}{4}\hat{\xi}_V^{ab}\gamma_{ab}\Psi \ , \tag{C.2.11}$$

where $\hat{\xi}_V^{ab} \equiv D^{[b}\xi^{a]}$, with $D^a \equiv e^{a\mu}D_\mu$ and SO indices raised and lowered using $\eta^{ab} = e^a{}_\mu e^{b\mu}$ and its inverse η_{ab}. We introduced a new notation $\mathscr{L}$ for the full Lie derivative with the SO local Lorentz indices and the spinor indices taken into account. Below, $\mathfrak{L}$ will continue to denote the same old Lie derivative which is aware of only the coordinate indices. This is similar to how we used ∇ to be aware only of tangent indices, while $\mathscr{D}$ is aware of both the local Lorentz indices and the spinor indices.

Given this, the additional action of $\mathscr{L}_\xi$ on SO indices can be inferred from the fact that $\bar{\Psi}\gamma^a\Psi$ transforms as a vector,

$$\mathscr{L}_\xi\left(\bar{\Psi}\gamma^a\Psi\right) = \xi^\mu D_\mu\left(\bar{\Psi}\gamma^a\Psi\right) - \hat{\xi}_V^{ab}\left(\bar{\Psi}\gamma_b\Psi\right) = \xi^\mu\partial_\mu\left(\bar{\Psi}\gamma^a\Psi\right) - \hat{\xi}_K^{ab}\left(\bar{\Psi}\gamma_b\Psi\right) \tag{C.2.12}$$

where

$$\hat{\xi}_K^{ab} \equiv \hat{\xi}_V^{ab} - \xi^\lambda w_\lambda{}^{ab} \tag{C.2.13}$$

is known as the Kosmann lift.

Note that along the way we used

$$\mathscr{D}_\mu(\gamma^a) = 0 = \mathscr{L}_\xi(\gamma^a) \tag{C.2.14}$$

self-consistently; for the latter, the rotation by $-\hat{\xi}_V$ on the SO local Lorentz index on γ^a is negated by its commutator with $-\hat{\xi}_V^{ab}\gamma_{ab}/4$, via the same type of index-chasing that shows $\mathscr{D}_\mu(\gamma^a) = 0$ from $\partial_\mu\gamma^a = 0$.

For more general SO vector, with $D_\mu e^a = 0$, this leads to

$$\begin{aligned}
\mathscr{L}_\xi v^a &= \xi^b D_b v^a - \hat{\xi}_V^{ab} v_b \\
&= \xi^b D_b v^a - v^b D_b \xi^a + (D^b\xi^a)v_b - \hat{\xi}_V^{ab} v_b \\
&= e^a{}_\mu(\mathfrak{L}_\xi v)^\mu + D^{(b}\xi^{a)}v_b \ .
\end{aligned} \tag{C.2.15}$$

where the last piece,

$$D^{(b}\xi^{a)} = e^b{}_\mu e^a{}_\nu \nabla^{(\mu}\xi^{\nu)} \ . \tag{C.2.16}$$

vanishes, if and only if ξ is Killing.

Since $e^a{}_\mu$'s, needed for the conversion between the coordinate basis and the orthonormal basis, construct the metric, it is hardly surprising that $\mathscr{L}_\xi$ and $\mathfrak{L}_\xi$ do not agree with each other unless ξ is Killing. In turn, this implies that the Kosmann lift $\mathscr{L}$ does not in general obey the familiar commutator algebra that would have related $[\mathscr{L}_\xi, \mathscr{L}_\zeta]$ to $\mathscr{L}_{[\xi,\zeta]}$. Nevertheless, one reason why the Kosmann lift stands out is how such standard algebra among diffeomorphims work at least among Killing vector fields, i.e., among isometry generators.

We can then read off $\mathscr{L}_\xi e^a{}_\mu$ from the Leibniz rules

$$\mathscr{L}_\xi v^a \;\;=\;\; \mathscr{L}_\xi(v^\mu e^a{}_\mu) = (\mathfrak{L}_\xi v^\mu) e^a{}_\mu + v^\mu(\mathscr{L}_\xi e^a{}_\mu) \ , \tag{C.2.17}$$

so that

$$\mathscr{L}_\xi e^a{}_\mu = D^{(b}\xi^{a)} e_{b\mu} = \left(D^b\xi^a - \hat{\xi}_V^{ab}\right) e_{b\mu} \ , \tag{C.2.18}$$

The Lie derivative on the inverse vielbein is

$$\mathscr{L}_\xi e_b{}^\mu = -D_{(b}\xi_{c)} e^{c\mu} = -(D_b\xi^\mu) + (\hat{\xi}_V)_{cb} e^{c\mu} \ , \tag{C.2.19}$$

so that

$$\mathscr{L}_\xi(\delta^a_b) \;=\; \mathscr{L}_\xi(e_b{}^\mu e^a{}_\mu) \;=\; 0 \tag{C.2.20}$$

holds, self-consistently.

The Lie derivative of the vielbein may be written in the following many forms,

$$\begin{aligned}
\mathscr{L}_\xi e^a{}_\mu \;&=\; D_\mu\xi^\lambda e^a{}_\lambda - \hat{\xi}_V^{ab} e_{b\mu} \;=\; \xi^\lambda D_\lambda e^a{}_\mu + D_\mu \xi^\lambda e^a{}_\lambda - \hat{\xi}_V^{ab} e_{b\mu} \\
&=\; \xi^\lambda \nabla_\lambda e^a{}_\mu + \nabla_\mu \xi^\lambda e^a{}_\lambda - \hat{\xi}_K^{ab} e_{b\mu} \\
&=\; \mathfrak{L}_\xi e^a{}_\mu - \hat{\xi}_K^{ab} e_{b\mu} \ ,
\end{aligned} \tag{C.2.21}$$

where we used $De^a = 0$. The last of these again shows how the Kosmann lift augments $\mathfrak{L}_\xi$ by $-\hat{\xi}_K$ rotation of orthonormal indices.

More generally, on a tensor $V^{a_1\,\cdots}_{\mu_1\,\cdots}$ with both SO indices and coordinate indices, $\mathscr{L}_\xi$ acts as

$$\mathscr{L}_\xi(V) = \mathfrak{L}_\xi(V) - \hat{\xi}_K(V) \ , \tag{C.2.22}$$

where $\mathfrak{L}_\xi$ ignores the SO indices of V while $\hat{\xi}_K$ rotates SO indices of V. On an SO-vector-valued differential form Ω^a, this induces

$$\mathscr{L}_\xi \Omega^a = d(\xi\lrcorner\,\Omega^a) + \xi\lrcorner\, d\Omega^a - \hat{\xi}_K^{ab}\Omega_b \ . \tag{C.2.23}$$

for example.

As one can see from above, either $-\hat{\xi}_K$ or $-\hat{\xi}_V$ ends up rotating the SO indices, including the spinor ones, depending on whether one uses ∇ or D ($\mathscr{D}$ if spinor index is present as well). ∇ and $\hat{\xi}_K$ are natural for the coordinate basis, although D (or $\mathscr{D}$) and $\hat{\xi}_V$ are generally more versatile since they can be used on objects with all three types of indices.

More abstractly, we may consider this Kosmann lift as the process of elevating the vector field ξ

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu} \quad \rightarrow \quad \xi^\mu \frac{\partial}{\partial x^\mu} - \hat{\xi}_K^{ab}\mathscr{M}_{ab} \tag{C.2.24}$$

to the relevant SO frame bundle, where $\mathscr{M}$'s are the local Lorentz generators. This Kosmann lift of the vector fields underlies the action of $\mathscr{L}_\xi$ on the frame bundle and by inference on the spinor bundle, which we have backtracked by starting from (C.2.11) instead.

The Kosmann Lift on Spin Connection

It remains to understand how diffeomorphism by ξ acts on the spin connection. For this we use

$$\mathscr{L}_\xi(\gamma^a\mathscr{D}_a\Psi) = \boldsymbol{\delta}_\xi(\gamma^a\mathscr{D}_a)\Psi + \gamma^a\mathscr{D}_a(\mathscr{L}_\xi\Psi) \ . \tag{C.2.25}$$

where $\mathscr{D}_\mu = D_\mu + w_\mu{}^{ab}\gamma_{ab}/4$ which equals $\partial_\mu + w_\mu{}^{ab}\gamma_{ab}/4$ on pure spinors. We used the symbol $\boldsymbol{\delta}_\xi$ here in place of $\mathscr{L}_\xi$ to remind ourselves how the connection is not a tensor. The left hand side is

$$\xi^b \mathscr{D}_b(\gamma^a \mathscr{D}_a \Psi) - \frac{1}{4}\hat{\xi}_V^{bc}\gamma_{bc}(\gamma^a \mathscr{D}_a \Psi) \ , \tag{C.2.26}$$

while the right hand side is

$$\left(\gamma^a(\mathscr{L}_\xi e_a{}^\mu)\mathscr{D}_\mu + \frac{1}{4}\gamma^a e_a{}^\mu(\boldsymbol{\delta}_\xi w_\mu{}^{bc})\gamma_{bc}\right)\Psi + \gamma^a \mathscr{D}_a\left((\xi^b \mathscr{D}_b - \frac{1}{4}\hat{\xi}_V^{bc}\gamma_{bc})\Psi\right) \ . \tag{C.2.27}$$

Equating the two sides, we find

$$\begin{aligned}
& \frac{1}{4}\gamma^a e_a{}^\mu(\boldsymbol{\delta}_\xi w_\mu{}^{bc})\gamma_{bc}\Psi \\
= \ & \gamma^a\left((D_a\xi^\mu) + (\hat{\xi}_V)_{ab}e^{b\mu}\right)\mathscr{D}_\mu\Psi - \left[\gamma^a\mathscr{D}_a, \xi^b\mathscr{D}_b - \frac{1}{4}\hat{\xi}_V^{bc}\gamma_{bc}\right]\Psi \\
= \ & \frac{1}{4}\gamma^a e_a{}^\mu(D_\mu\hat{\xi}_V^{bc})\gamma_{bc}\Psi + \gamma^a\xi^b[\mathscr{D}_b, \mathscr{D}_a]\Psi \\
= \ & \frac{1}{4}\gamma^a e_a{}^\mu(D_\mu\hat{\xi}_V^{bc})\gamma_{bc}\Psi + \frac{1}{4}\gamma^a\xi^f R^{bc}{}_{fa}\gamma_{bc}\Psi
\end{aligned} \tag{C.2.28}$$

from which we read off

$$\boldsymbol{\delta}_\xi w_\mu{}^{bc} = D_\mu(\hat{\xi}_V^{bc}) - R^{bc}{}_{\mu f}\xi^f \ , \tag{C.2.29}$$

whose first term represents the usual SO gauge transformation.

Writing the same out more abstractly,

$$\boldsymbol{\delta}_\xi w = d_w(\hat{\xi}_V) + \xi \lrcorner \mathcal{R} \ , \tag{C.2.30}$$

with the matrix-valued curvature 2-form $\mathcal{R} = dw + w \wedge w$ (C.1.30), we find

$$\begin{aligned}
\boldsymbol{\delta}_\xi w \ &= \ d(\hat{\xi}_K + \xi \lrcorner w) + w\,(\hat{\xi}_K + \xi \lrcorner w) - (\hat{\xi}_K + \xi \lrcorner w)\,w + \xi \lrcorner (dw + w \wedge w) \\
&= \ d\hat{\xi}_K + w\,\hat{\xi}_K - \hat{\xi}_K w + \mathfrak{L}_\xi w \\
&= \ d_w\hat{\xi}_K + \mathfrak{L}_\xi w \ ,
\end{aligned} \tag{C.2.31}$$

where the matrix multiplications are understood for w, $\hat{\xi}_V$, and $\hat{\xi}_K$, each carrying a pair of SO indices and, as before, $\mathfrak{L}_\xi$ in the last two lines acts on w as if the latter is an ordinary differential 1-form and ignores SO indices therein. This implies, for the curvature 2-form $\mathcal{R}^a{}_b$,

$$\mathscr{L}_\xi \mathcal{R} = \mathfrak{L}_\xi \mathcal{R} + [\mathcal{R}, \hat{\xi}_K] \tag{C.2.32}$$

as expected from (C.2.22), where we invoked $[\, d, \mathfrak{L}_\xi \,] = 0$.

Equivalently, we could have started from the general variation formula,

$$\gamma_{bc}\, \boldsymbol{\delta}_\xi w_\mu{}^{bc} = -\gamma^{\rho\sigma} \left(e_{f\mu} D_\rho \boldsymbol{\delta}_\xi e^f{}_\sigma + e_{f\sigma} D_\rho \boldsymbol{\delta}_\xi e^f{}_\mu - e_{f\sigma} D_\mu \boldsymbol{\delta}_\xi e^f{}_\rho \right) \tag{C.2.33}$$

with

$$\boldsymbol{\delta}_\xi e^f{}_\mu = \mathscr{L}_\xi e^f{}_\mu = D_\mu \xi^f - \hat{\xi}_V^{fb} e_{b\mu} \;. \tag{C.2.34}$$

The first term $D_\mu \xi^f$ generates the following variation in (C.2.33)

$$\begin{aligned} & -\gamma^{\rho\sigma} \left(e_{f\mu} D_\rho D_\sigma \xi^f + e_{f\sigma} D_\rho D_\mu \xi^f - e_{f\sigma} D_\mu D_\rho \xi^f \right) \\ = \;\; & -\gamma^{bc} \left(\frac{1}{2} [D_b, D_c] \xi_\mu + [D_b, D_\mu] \xi_c \right) \\ = \;\; & -\gamma^{bc} R_{bc\mu f} \xi^f \end{aligned} \tag{C.2.35}$$

with $\gamma^{bc} = -\gamma^{cb}$, reproducing the second piece of (C.2.29). The other $-\hat{\xi}_V^{fb} e_{b\mu}$ further generates from (C.2.33)

$$\begin{aligned} & \gamma^{\rho\sigma} \left(e_{f\mu} D_\rho (\hat{\xi}_V)^f{}_\sigma + e_{f\sigma} D_\rho (\hat{\xi}_V)^f{}_\mu - e_{f\sigma} D_\mu (\hat{\xi}_V)^f{}_\rho \right) \\ = \;\; & \gamma^{\rho\sigma} \left(D_\rho (\hat{\xi}_V)_{\mu\sigma} + D_\rho (\hat{\xi}_V)_{\sigma\mu} - D_\mu (\hat{\xi}_V)_{\sigma\rho} \right) \\ = \;\; & \gamma_{bc} D_\mu (\hat{\xi}_V^{bc}) \end{aligned} \tag{C.2.36}$$

producing the first piece of (C.2.29), completing another derivation of (C.2.29).

Is $\mathfrak{L}_\xi$ Suitable for Local Coordinate Transformation?

The Kosmann lift is the natural extension of vector fields on $\mathcal{M}_d$, which arises from how the SO structure based on the orthonormal frame is embedded into the GL structure relevant for the coordinate basis.

On the other hand, if one is interested in reaching the right Lie derivative structures for tensors with coordinate indices only, the rotation of the orthonormal indices by $\hat{\xi}_K$ appears irrelevant. This rotation will drop out eventually if all such orthonormal indices and spinor indices are contracted away leaving behind the coordinate indices only. As such, we seem to have an option of dropping $\hat{\xi}_K$ in (C.2.24), reverting back to $\mathfrak{L}$,

$$\mathfrak{L}_\xi \Psi = \xi^\mu \partial_\mu \Psi \ , \qquad \mathfrak{L}_\xi e^a = \xi \lrcorner\, de^a + d(\xi^a) \tag{C.2.37}$$

where we effectively view the spinor indices and the orthonormal indices as mere extra labels for these multi-component functions and 1-forms. Needless to say,

$$\mathfrak{L}_\xi \gamma^a = \xi^\mu \partial_\mu \gamma^a = 0 \tag{C.2.38}$$

does hold.

For instance, the resulting action, $\mathfrak{L}_\xi v^a = \xi^\mu \partial_\mu v^a$ on a vector can be understood from how $v^a = e^a_{\ \mu} v^\mu$ are here being treated as a set of d-many functions once the tangent and the co-tangent indices are contracted away. Until we connect the SO structure to that of the GL structure underlying the coordinate basis, this appears to be a perfectly sane thing to do. In a sense we did start with such an attitude when we extended the covariant derivative to the orthonormal frame as

$$D_\mu\, e^a_{\ \nu} \equiv \nabla_\mu\, e^a_{\ \nu} + w_\mu{}^a{}_b\, e^b_{\ \nu} \tag{C.2.39}$$

At this stage, where we are yet to demand $De^a = 0$, w remains independent from the Levi-Civita connection in ∇.

Recall how the Lie derivative appears in the passive transformation, i.e., in the context of local coordinate transformation. Consider an infinitesimal coordinate redefinition $\tilde{x}(x) = x - \epsilon\, \xi(x)$ with $\epsilon \ll 1$. Since this does not involve a map between point, the function value itself should be the same in the end. The variables being used to label the point have changed, on the other hand, so we cannot use the same functional form. Calling the new function $\tilde{f}$ of the coordinate values, the statement that the function did not change translates to $f(x) = \tilde{f}(\tilde{x})$.

We will represent the necessary infinitesimal change as $\tilde{f} = f + \epsilon\, \boldsymbol{\delta}_\xi^{\text{passive}} f$ so that

$$
\begin{aligned}
& f(x) = f(x) + \epsilon \left(-\xi^\mu \partial_\mu f + \boldsymbol{\delta}_\xi^{\text{passive}} f \right) \Big|_x + O(\epsilon^2) \\
\Rightarrow \quad & \boldsymbol{\delta}_\xi^{\text{passive}} f(x) = \xi^\mu \partial_\mu f \Big|_x = \mathfrak{L}_\xi f \Big|_x
\end{aligned}
\tag{C.2.40}
$$

as the "variation" of the function, again bringing us to the Lie derivative. The same works for more general tensors. Upon $x \to \tilde{x} = x - \epsilon\xi$ and following the same procedure as above resulting in (B.2.41), $e^a_{\ \mu}\, dx^\mu$, treated as a set of unrelated 1-forms, would respond to the coordinate shift as $\boldsymbol{\delta}_\xi^{\text{passive}} e^a = \mathfrak{L}_\xi e^a$. The same goes for the spinors.

This seemingly suggests that, as long as are interested in local coordinate transformations, $\mathfrak{L}_\xi$ may do as well. In fact, most physics literature tend to take this attitude. However, the SO structure of the orthonormal frames is eventually tied to the tangent indices with $De^a = 0$, so that the spin connection cannot be independent of the Levi-Civita connection. After all, the two connections compute one and the same curvature tensor in the end, only in different basis. This tells us that although the above vanilla Lie derivative is available locally in a given coordinate patch, it won't generally extend to the entire manifold.

One place where we can see the fatal problem with adopting $\mathfrak{L}_\xi$ on spinors most clearly is (C.2.13). Turning off the Kosmann lift in favor of $\mathfrak{L}_\xi$ means setting $\hat{\xi}_K^{ab} = 0$. But the latter means that we must equate

$$
\xi^\lambda {w_\lambda}^{ab} = -D^{[a}\xi^{b]}
\tag{C.2.41}
$$

with the covariant expression on the right against a non-covariant one on the left. The vanilla Lie derivative $\mathfrak{L}_\xi$ on spinor can be neither extended covariantly beyond a given local chart nor definable in a truly frame-independent manner.

In other words, $\mathfrak{L}_\xi$ does not map sections of the frame bundle and those of the spinor bundle to sections of either. Recall how we motivated the Lie derivative $\mathfrak{L}_\xi$ on tensors as a unique directional derivative that maps tensors to tensors. There is no reason to give up this sacred principle simply because spinors carry different indices that the coordinate indices. In fact the same consideration should be applicable to sections of the gauge bundles, i.e., when the spinor in question also take value in some representation of a Lie algebra. We start the next section with the appropriate generalization of the Kosmann lift for this most general setting.

C.3 Energy-Momentum of Fermions

C.3.1 Generalized Kosmann

Before we get to physics applications, there is one additional ingredient we need to mull over. Spinors in field theory are often in some representations of gauge groups, carrying additional internal indices. Under the vanilla Lie derivative $\mathfrak{L}_\xi$, the addition of the gauge field does not change the action on spinor; we merely need to remember that the gauge fields $\mathcal{A}$ should transform by $\mathfrak{L}_\xi$ as well, and at least locally this suffices to guarantee the general covariance of the Dirac action, for example. The question is if and how this situation changes once we adopt the Kosmann lift. What we mean by the general covariance of the matter action is itself at stake.

Let us recall our notation for the gauge sector first, with connection $\mathcal{A}$ and the gauge function Θ, both anti-Hermitian,

$$\boldsymbol{\delta}_\Theta^{\text{gauge}}\Psi = -\Theta\Psi \ , \qquad \boldsymbol{\delta}_\Theta^{\text{gauge}}\mathcal{A} = d\Theta + [\mathcal{A},\Theta] \tag{C.3.1}$$

and the covariant derivative,

$$\mathscr{D}_\mu = D_\mu + \mathcal{A}_\mu + \frac{1}{4}w_{\mu\,ab}\gamma^{ab} \tag{C.3.2}$$

now equipped with the gauge-covariant derivative.

Under the Kosmann-lifted diffeomorphism,

$$\begin{aligned} \boldsymbol{\delta}_\xi\Psi &= \xi^\mu(\mathscr{D}_\mu - \mathcal{A}_\mu)\Psi - \frac{1}{4}\hat{\xi}_V^{ab}\gamma_{ab}\Psi \ = \ \mathscr{L}_\xi\Psi \ , \\ \boldsymbol{\delta}_\xi\mathcal{A}_\mu &= \xi^\nu\partial_\nu\mathcal{A}_\mu + (\partial_\mu\xi^\nu)\mathcal{A}_\nu \ = \ \mathfrak{L}_\xi\mathcal{A}_\mu \end{aligned} \tag{C.3.3}$$

the free Dirac action from

$$\mathcal{L}_{\text{Dirac}} \;=\; -\mathbb{i}\bar{\Psi}\gamma^a e_a{}^\mu \mathscr{D}_\mu \Psi - \mathbb{i} m \bar{\Psi}\Psi \;, \tag{C.3.4}$$

is invariant, although in the same restricted sense as how it is preserved by the trasnformation under the vanilla Lie derivative $\mathfrak{L}_\xi$. Being a section of the relevant vector bundle as well as a section of the spinor bundle, on the other hand, the question of how we glue the local sections for Ψ across overlapping patches with regard to the gauge indices enters the Lie derivative also.

This motivates an alternative transformation rule for the diffeomorphism, augmented by gauge transformation by $\Theta = -(\xi \lrcorner \mathcal{A})$,

$$\begin{aligned} \hat{\boldsymbol{\delta}}_\xi \Psi \;&\equiv\; \left(\mathscr{L}_\xi + \boldsymbol{\delta}^{\text{gauge}}_{-(\xi \lrcorner \mathcal{A})}\right)\Psi \;=\; \xi^\mu \mathscr{D}_\mu \Psi - \frac{1}{4}\hat{\xi}_V^{ab}\gamma_{ab}\Psi \\ \hat{\boldsymbol{\delta}}_\xi \mathcal{A}_\mu \;&\equiv\; \left(\mathfrak{L}_\xi + \boldsymbol{\delta}^{\text{gauge}}_{-(\xi \lrcorner \mathcal{A})}\right)\mathcal{A}_\mu \;=\; \xi^\nu \mathcal{F}_{\nu\mu} \end{aligned} \tag{C.3.5}$$

accommodates the latter need on equal footing with the spin indices. This $\hat{\boldsymbol{\delta}}_\xi$ has an obvious advantage over $\boldsymbol{\delta}_\xi$ for being fully covariant under gauge transformation as well.

Since the difference between $\boldsymbol{\delta}_\xi$ and $\hat{\boldsymbol{\delta}}_\xi$ is a gauge transformation, local in the sense of a given coordinate patch, the local covariance of the Dirac action holds equally. In addition, however, the latter action makes sense globally on individual fields as well, which leads to

$$\hat{\boldsymbol{\delta}}_\xi \left(\mathcal{V}\mathcal{L}_{\text{Dirac}}\right) = d\left(\xi \lrcorner \mathcal{V}\mathcal{L}_{\text{Dirac}}\right) \;. \tag{C.3.6}$$

Henceforth, we will refer to the latter transformation rule $\hat{\boldsymbol{\delta}}_\xi$ on spinors and gauge fields as the generalized Kosmann lift.

C.3.2 Symmetric Energy-Momentum T

Let us now turn to the question of the energy-momentum and the conservation law for theories involving spinors, for which the vielbein and the spin connection are indispensable. One central fact is how the Kosmann-lift of the Lie derivative on the vielbein involves symmetric combinations in that

$$\hat{\boldsymbol{\delta}}_\xi e^a{}_\mu = \boldsymbol{\delta}_\xi e^a{}_\mu = D^{(b}\xi^{a)} e_{b\mu} \ , \qquad \hat{\boldsymbol{\delta}}_\xi = \boldsymbol{\delta}_\xi e_b{}^\mu = -D_{(b}\xi_{c)} e^{c\mu} \tag{C.3.7}$$

where the gauge-generalization is moot given how the vielbein has no gauge indices.†

Given this, the variation of the vielbein and the spin connection would yield

$$\sim (\boldsymbol{\delta}_\xi e_b{}^\mu)(-T^b{}_\mu) = D_{(b}\xi_{c)} T^{bc} \tag{C.3.9}$$

in the end, with symmetric tensor T^{bc} emerging naturally for any $\mathcal{L}(\Psi, \mathscr{D}_\mu\Psi; e)$. This is entirely analogous to how $\mathfrak{L}_\xi g^{\mu\nu} = -(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu)$ enters crucially as in

$$\sim (\boldsymbol{\delta}_\xi g^{\mu\nu})(-T_{\mu\nu}/2) = \nabla^{(\mu}\xi^{\nu)} T_{\mu\nu} \tag{C.3.10}$$

for any generally covariant matter Lagrangian involving scalars and tensors. For both, the conservation law involves a symmetric energy-momentum tensor, signaling that the closest analog of $\boldsymbol{\delta}_\xi g^{\mu\nu} = \mathfrak{L}_\xi g^{\mu\nu}$ resides in the Kosmann-lifted Lie derivative on the vielbein, $\boldsymbol{\delta}_\xi e_b{}^\mu$. Also, the same quantity will appear as the source term for the Einstein equation. Eventually we will see that the Kosmann lift naturally brings us to the conservation law of this symmetric energy-momentum tensor.

As a minimal exercise, let us consider free and massless Dirac spinor Ψ, with the Lagrangian,

$$\mathcal{L} \ = \ -\mathbb{i}\bar{\Psi}\gamma^\mu \mathscr{D}_\mu \Psi \ = \ -\mathbb{i}\bar{\Psi}\gamma^a e_a{}^\mu \left(D_\mu + \frac{1}{4} w_{\mu cd}\gamma^{cd} + \mathcal{A}_\mu \right) \Psi \tag{C.3.11}$$

The variation of the action under the arbitrary shift of the vielbein $e \to e + \boldsymbol{\delta} e$ is

$$\begin{aligned} &\boldsymbol{\delta} \int d^d x\, |e|\, \mathcal{L} \bigg|_{\Psi, \mathcal{A}\,\text{fixed}} \\ &= \int d^d x\, |e| \left[-\mathbb{i}\bar{\Psi}(\gamma^a \mathscr{D}_\nu - e^a{}_\nu \gamma^\mu \mathscr{D}_\mu)\Psi \cdot \delta e_a{}^\nu - \frac{\mathbb{i}}{4} \bar{\Psi}\gamma^\mu\gamma^{ab}\Psi \cdot \delta w_{\mu ab} \right] \end{aligned} \tag{C.3.12}$$

which should compute the symmetric energy-momentum tensor in the end. Using

†This should be compared to the vanilla Lie derivative of the vielbein, say,

$$\boldsymbol{\delta}'_\xi e_b{}^\mu \equiv \mathfrak{L}_\xi e_b{}^\mu = \xi^\nu \partial_\nu e_b{}^\mu - (\partial_\nu \xi^\mu) e_b{}^\nu \tag{C.3.8}$$

which exhibits no obvious combinatoric property.

how the spin connection responds to such a general variation,

$$\gamma^\mu \gamma^{ab} \delta w_{\mu ab} \;=\; -(\gamma^c \gamma^{ab} + 2\gamma^{(a}\gamma^{b)c}) e_{b\nu} D_c \delta e_a{}^\nu \tag{C.3.13}$$

this can be further manipulated

$$\begin{aligned}
&\boldsymbol{\delta} \int d^d x\, |e|\, \mathcal{L} \Big|_{\Psi, \mathcal{A}\,\text{fixed}} \\
= \;& \int d^d x\, |e| \Big[-\,\mathbb{i} \left(\bar\Psi (\gamma^a \mathscr{D}^b - \eta^{ab} \gamma^\mu \mathscr{D}_\mu) \Psi + \frac{1}{4} \mathscr{D}_c \left(\bar\Psi (\gamma^c \gamma^{ab} + 2\gamma^{(a}\gamma^{b)c}) \Psi \right) \right) \cdot e_{b\nu} \delta e_a{}^\nu \\
& \qquad + \frac{\mathbb{i}}{4} \mathscr{D}_c \left(\bar\Psi (\gamma^c \gamma^{ab} + 2\gamma^{(a}\gamma^{b)c}) \Psi \cdot e_{b\nu} \delta e_a{}^\nu \right) \Big]
\end{aligned} \tag{C.3.14}$$

where we kept all terms including total derivative terms.

Now invoking the Kosmann-lifted diffeomorphism, as advertised,

$$\boldsymbol{\delta} e_a{}^\nu \quad \to \quad \boldsymbol{\delta}_\xi e_a{}^\nu \;=\; \mathscr{L}_\xi e_a{}^\nu \;=\; -D_{(a}\xi_{b)} e^{b\nu} \tag{C.3.15}$$

we find

$$\begin{aligned}
&\boldsymbol{\delta}_\xi \int d^d x\, |e|\, \mathcal{L} \Big|_{\Psi, \mathcal{A}\,\text{fixed}} \\
= \;& \int d^d x\, |e| \Big[(D_a \xi_b)\, \mathbb{i} \left(\bar\Psi (\gamma^{(a} \mathscr{D}^{b)} - \eta^{ab} \gamma^\mu \mathscr{D}_\mu) \Psi + \frac{1}{2} D_c \left(\bar\Psi (\eta^{ab}\gamma^c - \gamma^{(a}\eta^{b)c}) \Psi \right) \right) \\
& \qquad - \frac{\mathbb{i}}{2} D_c \left[(D_a \xi_b) \bar\Psi \gamma^{(a}\gamma^{b)c} \Psi \right] \Big] \\
= \;& \int d^d x\, |e| \left[(D_a \xi_b)\, T^{ab} - \frac{\mathbb{i}}{2} D_c \left[(D_a \xi_b) \bar\Psi \gamma^{(a}\gamma^{b)c} \Psi \right] \right]
\end{aligned} \tag{C.3.16}$$

from which the symmetric energy-momentum tensor

$$T^{ab} \;=\; \frac{\mathbb{i}}{2}\, \bar\Psi \left[(\gamma^{(a} \mathscr{D}^{b)} - \overleftarrow{\mathscr{D}}^{(b} \gamma^{a)}) - \eta^{ab} (\gamma^\mu \mathscr{D}_\mu - \overleftarrow{\mathscr{D}}_\mu \gamma^\mu) \right] \Psi \tag{C.3.17}$$

is easily identified.

C.3.3 Noether Energy-Momentum $\hat{\mathbb{T}}$

The fact that the Kosmann-lift of the Lie derivative gave the symmetric energy-momentum T naturally without further manipulation tells us that the Noether procedure must also employ the Kosmann lift when we vary the matter sector. For a free massless Dirac Ψ, again, we now perform the variation of the matter fields using the generalized Kosmann $\hat{\boldsymbol{\delta}}_\xi$,

$$\hat{\boldsymbol{\delta}}_\xi \mathcal{L}\Big|_{e\,\text{fixed}} = -\mathbb{i}\left(\bar{\Psi}\gamma^\mu \mathscr{D}_\mu(\boldsymbol{\delta}_\xi \Psi) + (\boldsymbol{\delta}_\xi \bar{\Psi})\gamma^\mu \mathscr{D}_\mu \Psi + \bar{\Psi}\gamma^\mu(\boldsymbol{\delta}_\xi \mathcal{A}_\mu)\Psi\right) \tag{C.3.18}$$

The expression inside the parenthesis may be manipulated as

$$\begin{aligned}
\mathbb{i}\hat{\boldsymbol{\delta}}_\xi \mathcal{L}\Big|_{e\,\text{fixed}} &= \xi^\nu \partial_\nu(\bar{\Psi}\gamma^\mu \mathscr{D}_\mu \Psi) + (D_a \xi_b)\bar{\Psi}\left(\gamma^a \mathscr{D}^b + \frac{1}{4}[\gamma^c, \gamma^{ab}]\mathscr{D}_c\right)\Psi \\
&\quad + \bar{\Psi}\left(\frac{1}{4}(D_c D_a \xi_b)\gamma^c \gamma^{ab} + \xi^\nu \gamma^\mu([\mathscr{D}_\mu, \mathscr{D}_\nu] - \mathcal{F}_{\mu\nu})\right)\Psi \\
&= \xi^\nu \partial_\nu(\bar{\Psi}\gamma^\mu \mathscr{D}_\mu \Psi) + (D_a \xi_b)\bar{\Psi}\left(\gamma^a \mathscr{D}^b - \gamma^{[a}\mathscr{D}^{b]}\right)\Psi \\
&\quad + \frac{1}{4}\bar{\Psi}\left((D_c D_a \xi_b)\gamma^c \gamma^{ab} + (\mathcal{R}_{ab\mu\nu}\xi^\nu)\gamma^\mu \gamma^{ab}\right)\Psi \\
&= \xi^\nu \partial_\nu(\bar{\Psi}\gamma^\mu \mathscr{D}_\mu \Psi) + (D_a \xi_b)\bar{\Psi}\gamma^{(a}\mathscr{D}^{b)}\Psi \\
&\quad + \frac{1}{4}(D_c D_a \xi_b + [D_a, D_b]\xi_c)\bar{\Psi}\gamma^c \gamma^{ab}\Psi
\end{aligned} \tag{C.3.19}$$

with the definition of the curavture as the commutator of the covariant derivatives.

After some further massaging of this expression, we find

$$\begin{aligned}
\boldsymbol{\delta}_\xi \mathcal{L}\Big|_{e\,\text{fixed}} &= \xi^\nu \partial_\nu(-\mathbb{i}\bar{\Psi}\gamma^\mu \mathscr{D}_\mu \Psi) - (D_a \xi_b)\,\mathbb{i}\bar{\Psi}\gamma^{(a}\mathscr{D}^{b)}\Psi + +\frac{\mathbb{i}}{2}(D_c D_a \xi_b)\bar{\Psi}\gamma^{(a}\gamma^{b)c}\Psi \\
&= \nabla_\mu(\xi^\mu \mathcal{L}) - (D_a \xi_b)\,\mathbb{i}\bar{\Psi}(\gamma^{(a}\mathscr{D}^{b)} - \eta^{ab}\gamma^\mu \mathscr{D}_\mu)\Psi + \frac{\mathbb{i}}{2}(D_c D_a \xi_b)\bar{\Psi}\gamma^{(a}\gamma^{b)c}\Psi \\
&= \nabla_\mu(\xi^\mu \mathcal{L}) - (D_a \xi_b)\,\hat{\mathbb{T}}^{ab} + \frac{\mathbb{i}}{2}\mathscr{D}_c\left[(D_a \xi_b)\bar{\Psi}\gamma^{(a}\gamma^{b)c}\Psi\right]
\end{aligned} \tag{C.3.20}$$

from which we indentify the Noether energy-momentum

$$\hat{\mathbb{T}}^{ab} \;=\; \frac{\mathbb{i}}{2}\,\bar{\Psi}\left[\left(\gamma^{(a}\mathscr{D}^{b)} - \overleftarrow{\mathscr{D}}^{(b}\gamma^{a)}\right) - \eta^{ab}\left(\gamma^{\mu}\mathscr{D}_{\mu} - \overleftarrow{\mathscr{D}}_{\mu}\gamma^{\mu}\right)\right]\Psi \tag{C.3.21}$$

under the generalized Kosmann lift.

One should not be misled to think that the Kosmann lift above can be evaded by starting with coordinate indices everywhere on account of how the rotation by $\hat{\xi}_{K,V}$'s would cancel out by the time all local Lorentz indices are contracted away. One can see clearly that this naive expectation will not work by taking the example of $\gamma^{\mu} = \gamma^{a}e_{a}{}^{\mu}$, given how $\mathscr{L}_{\xi}\gamma^{a} = 0$. In order for γ^{μ} transformation to reduce to that of a vector under $\mathfrak{L}_{\xi}$, after the local Lorentz indices contracted away, one must allow γ^{a} to rotate under $\mathscr{L}_{\xi}$. Otherwise, γ^{μ} would transform in the same manner as $e_{a}{}^{\mu}$ but of course this does not make sense either given the additional index on the latter relative to γ^{μ}.

Although we have performed all computations using the generalized Kosmann $\hat{\boldsymbol{\delta}}_{\xi}$ for the diffeomorphim, exactly the same expressions for the key quantities result under the regular Kosmann lift $\boldsymbol{\delta}_{\xi}$. The difference between the two is the internal gauge transformation by $-\xi \lrcorner \mathcal{A}$, yet the Dirac action is manifestly invariant under this gauge transformation. Furthermore, when we split the diffeomorphism action on the Lagrangian into that of the vielbein and that of the rest, this gauge transformation enters the latter procedure exclusively and undivided so that, again, this additional transformation affects none of the final expressions.

C.3.4 $T = \hat{\mathbb{T}}$ and How Kosmann Preserves Dirac Action

From the above explicit and mutually independent computations, we find

$$\hat{\mathbb{T}}^{ab} \;=\; \frac{\mathbb{i}}{2}\,\bar{\Psi}\left[\left(\gamma^{(a}\mathscr{D}^{b)} - \overleftarrow{\mathscr{D}}^{(b}\gamma^{a)}\right) - \eta^{ab}\left(\gamma^{\mu}\mathscr{D}_{\mu} - \overleftarrow{\mathscr{D}}_{\mu}\gamma^{\mu}\right)\right]\Psi = T^{ab} \tag{C.3.22}$$

and this leads to

$$\begin{aligned}
\hat{\boldsymbol{\delta}}_{\xi}\int \mathcal{V}\mathcal{L} \;&=\; \hat{\boldsymbol{\delta}}_{\xi}\int \mathcal{V}\mathcal{L}\,\Big|_{e\,\text{fixed}} + \hat{\boldsymbol{\delta}}_{\xi}\int \mathcal{V}\mathcal{L}\,\Big|_{\Psi,\mathcal{A}\,\text{fixed}} \\
&=\; \int d^{d}x\,|e|\left[\nabla_{\mu}(\xi^{\mu}\mathcal{L}) - (D_{a}\xi_{b})\,\hat{\mathbb{T}}^{ab} \cancel{+\frac{\mathbb{i}}{2}D_{c}\left((D_{a}\xi_{b})\bar{\Psi}\gamma^{(a}\gamma^{b)c}\Psi\right)}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\int d^dx\,|e|\left[(D_a\xi_b)\,T^{ab}\cancel{-\frac{\mathbb{i}}{2}D_c\left((D_a\xi_b)\bar\Psi\gamma^{(a}\gamma^{b)c}\Psi\right)}\right]\\
=&\int d^dx\,|e|\left[\nabla_\mu(\xi^\mu\mathcal{L})+(D_a\xi_b)\,(T^{ab}-\hat{\mathbb{T}}^{ab})\right]\\
=&\int d(\xi\lrcorner\mathcal{V}\mathcal{L})
\end{aligned}
\tag{C.3.23}
$$

and vice versa.

The above may be considered as a demonstration of how the Dirac action is invariant under the Kosmann lift and also under the generalized Kosmann lift. In the opposite view with the latter fact given, the inevitable equality

$$
T^{ab}=\hat{\mathbb{T}}^{ab}
\tag{C.3.24}
$$

follows. Adding a mass term is a relatively trivial exercise.

If we had started with the democratic $\mathcal{L}'_{\text{Dirac}}$,

$$
\begin{aligned}
\mathcal{L}'_{\text{Dirac}} &= \mathcal{L}_{\text{Dirac}}+\frac{\mathbb{i}}{2}D_\mu(\bar\Psi\gamma^\mu\Psi)\\
&= -\frac{\mathbb{i}}{2}\left(\bar\Psi\gamma^\mu\mathscr{D}_\mu\Psi-\mathscr{D}_\mu\bar\Psi\gamma^\mu\Psi\right)\ ,
\end{aligned}
\tag{C.3.25}
$$

the middle steps would have differed. Yet, remarkably, they eventually lead to the same energy-momentum tensors,

$$
(T')^{ab}=T^{ab}=\hat{\mathbb{T}}^{ab}=(\hat{\mathbb{T}}')^{ab}\ ,
\tag{C.3.26}
$$

despite $\mathcal{L}'_{\text{Dirac}}\neq\mathcal{L}_{\text{Dirac}}$. Surprisingly, the form of the energy-momentum tensor is even more robust than the Lagrangian in that the addition of total derivative terms to the latter is automatically screened out by the procedure we offer, rather than by some after-thought tweaking.

C.4 Clifford Algebra and Spinor Classification

Now that we have studied a little bit about spinors and their property under diffeomorphisms, it is high time to take a deeper look, especially at how the spinor bundle structure depends on dimensions and the signature. The starting point is again the

Clifford algebra spanned by the Dirac matrices.

In Euclidean signature, we have Cl_d algebra spanned by

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab} \tag{C.4.1}$$

Given these generators, more independent matrices may be constructed from completely antisymmetric products, i.e., sums

$$\gamma^{a_1 \cdots a_p} \equiv \frac{1}{p!} \sum_{\sigma} (-1)^{|\sigma|} \gamma^{a_{\sigma_1}} \cdots \gamma^{a_{\sigma_p}} \tag{C.4.2}$$

over all possible permutations σ with the parity $(-1)^{|\sigma|}$. In particular, when $d = 2n$, there exists a special generator $\gamma^{1 \cdots d}$ which, as we saw earlier, is related to the chirality operator,

$$\Gamma = (-\mathbb{i})^n \gamma^1 \cdots \gamma^{2n} \tag{C.4.3}$$

For $d = 2n + 1$, the same set of γ's for $d = 2n$ may be used for the first $2n$ Dirac matrices, while for the last one, γ^{2n+1}, we would use either Γ or $-\Gamma$.

In the Lorentzian signature, the Clifford algebra takes the form

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \tag{C.4.4}$$

and is denoted $Cl_{1,d-1}$. More generally, we may also consider more negative signs on the right hand side, say,

$$\mathrm{diag}(\underbrace{-, \cdots, -}_{p}, \underbrace{+, \cdots, +}_{d-p}) \tag{C.4.5}$$

denoted as $Cl_{p,d-p}$. The familiar spinors are acted on by multiplication by γ's and Γ's on the left, so form a representation. In this final section, we will explore these spinor representations, with emphasis on how the Lorentz group enter the discussion along the way. We do not really need a separate construction for these alternate signatures since we may as well start from γ^a's and construct $\Gamma^{a \le p} = -\mathbb{i}\gamma^{a \le p}$'s and identify the rest intact, $\Gamma^{a>p} = \gamma^{a>p}$.

The questions here are how these spinor representations of Clifford algebras in various dimensions and signatures can be characterized and sometimes decomposed further when we dote on $\mathfrak{so}(p, d-p)$ subalgebra of $Cl_{p,d-p}$, generated by antisymmetric products of two distinct Γ's. For instance, we have seen how even d implies that the spinor splits into two distinct irreducible representations under $\mathfrak{so}(p, d-p)$, due to the existence of the chirality operator. Equally important are the charge conjugation operations, which can also halve the spinor by imposing reality conditions. Much of these detail played crucial roles for path integrals of fermions, as we have encountered numerously in the main text. One well-known phenomenon we will rediscover here is the so-called Bott periodicity, whereby the structure repeats itself under a shift of d by 8.

C.4.1 A Canonical Representation

We start with discussion for the first where all γ's are Hermitian, and come to the other signatures later. As we saw in earlier discussion of spinors, γ^{ab}'s play a special role as the rotation generators, on spinors, of the underlying Lorentz group. As such, we have a sequence of algebras

$$\mathfrak{so}(d) \quad \subset \quad Cl_d^{\text{even}} \quad \subset \quad Cl_d \tag{C.4.6}$$

where the middle is a subalgebra spanned by even products of γ's. The primary objects of interest are the first two, or more precisely the spin group $Spin(d)$, which is related to $SO(d)$ group by a $\mathbb{Z}_2$ division and can be constructed from Cl_d^{even}, and the representations thereof, but we will start with a canonical representation of Cl_d which is useful for the rest of the discussions.

To construct the representation explicitly, it is convenient to define

$$\alpha_S \equiv \frac{1}{2}\left(\gamma^{n+S} + \mathbb{i}\gamma^S\right) \ , \qquad \alpha_S^\dagger = \frac{1}{2}\left(\gamma^{n+S} - \mathbb{i}\gamma^S\right) \tag{C.4.7}$$

such that the fermionic oscillators,

$$\{\alpha_S, \alpha_T^\dagger\} = \delta_{ST} \tag{C.4.8}$$

may be used to construct a 2^n dimensional Fock space,

$$|0\rangle \ , \qquad \alpha_S^\dagger|0\rangle \ , \quad \cdots \quad , \quad \alpha_1^\dagger\alpha_2^\dagger\cdots\alpha_n^\dagger|0\rangle \tag{C.4.9}$$

starting from the vacuum state, $\alpha_S|0\rangle = 0$, which serve as a basis that span the Dirac spinor. The oscillators merely shuffle a basis state to another, with ± 1 coefficients, so $\gamma^S = -\mathbb{i}(\alpha_S - \alpha_S^\dagger)$ are pure imaginary and antisymmetric while $\gamma^{n+S} = \alpha_S + \alpha_S^\dagger$'s are real symmetric. Finally, since the Hermitian $\gamma^{2n+1} = \pm\Gamma = \pm(-\mathbb{i})^n\gamma^1\cdots\gamma^{2n}$ obeys

$$(\gamma^{2n+1})^* = (-1)^n(-1)^n\gamma^{2n+1} = \gamma^{2n+1} \tag{C.4.10}$$

γ^{n+S}'s are real symmetric all the way for $S = 1, \ldots, n+1$ in this representation.

Note that we can repeat the construction for other signatures; the only new element here is to replace some γ by $-\mathbb{i}\gamma$ as the new Dirac matrices. For a more streamlined notation, let us introduce a different notation for these anti-Hermitian Dirac matrices as

$$\begin{aligned} \Gamma^a &\equiv -\mathbb{i}\gamma^a \quad , \qquad a = 1, \cdots, p \leq n \\ \Gamma^b &\equiv \gamma^b \qquad , \qquad b = p+1, \cdots, d \end{aligned} \tag{C.4.11}$$

where we restricted the number of such anti-Hermitian Γ^a to be no more than $n = \lfloor d/2 \rfloor$, the integer part of $d/2$.

The Fock space construction proceeds the same way as in the Euclidean case, since we may as well use γ^a's and multiply $-\mathbb{i}$ for the first p of them in the end.

$$-\mathbb{i}\gamma^1, \cdots, -\mathbb{i}\gamma^p; \gamma^{p+1}, \cdots, \gamma^n; \gamma^{n+1}, \cdots, \gamma^{2n} \tag{C.4.12}$$

The first p are real antisymmetric, the middle $(n-p)$ are imaginary antisymmetric, and the last n are real symmetric. In particular, when $p = n$, we see that all Dirac matrices are real. The chirality operator, or γ^{2n+1} modulo sign if $d = 2n+1$,

$$\Gamma^{2n+1} \equiv \Gamma = (-\mathbb{i})^n\gamma^1\cdots\gamma^{2n} = (-\mathbb{i})^{n-p}\Gamma^1\cdots\Gamma^{2n} \tag{C.4.13}$$

is real and symmetric, regardless of p and n, in this representation.

C.4.2 Complex Conjugations and Majorana Spinors

One may then construct

$$\mathbb{C} \equiv \Gamma^{p+1} \cdots \Gamma^{n} = \gamma^{p+1} \cdots \gamma^{n} \ , \qquad \mathbb{C}^{-1} = \mathbb{C}^{\dagger} \tag{C.4.14}$$

where one should note that we take product of pure imaginary ones among Γ's in the above Fock space representation of the Clifford algebra. With our choice $p \leq n$, and $d = 2n$ or $d = 2n+1$, these constitute no more than half of all Dirac matrices. Later we will come to the complimentary choice, $\mathcal{C}$, available for $d = 2n$ as the product of $(n+p)$ real Γ's down to Γ^{2n}, playing a similar role. For $d = 2n+1$, no such independent $\mathcal{C}$ exists since product of all Γ's is proportional to 1.

This $\mathbb{C}$ obey

$$\mathbb{C}^{-1}\gamma^{a\leq p}\mathbb{C} = -(-1)^{n-p}(\gamma^{a\leq p})^* \ , \qquad \mathbb{C}^{-1}\gamma^{a>p}\mathbb{C} = (-1)^{n-p}(\gamma^{a>p})^* \tag{C.4.15}$$

which can be used as a charge conjugation for Γ^a's

$$\mathbb{C}^{-1}\Gamma^{a}\mathbb{C} = (-1)^{n-p}(\Gamma^{a})^* \tag{C.4.16}$$

up to $a = 2n+1$, from which we find

$$(\Gamma^{ab})^* = \mathbb{C}^{-1}\Gamma^{ab}\mathbb{C} \tag{C.4.17}$$

on $\mathfrak{so}(p, d-p)$ rotation generators for $d = 2n, 2n+1$. Therefore, the Dirac spinor Ψ and its complex conjugate

$$\Psi_{\mathbb{C}} \equiv \mathbb{C}\Psi^* \tag{C.4.18}$$

transform the same way under $\mathfrak{so}(p, d-p)$.

If we perform this conjugation operation twice, the spinors come back to itself, generally modulo a sign,

$$(\Psi_{\mathbb{C}})_{\mathbb{C}} = \mathbb{C}(\mathbb{C}\Psi^*)^* = \begin{cases} \Psi & n-p = 0, 3 \bmod 4 \\ -\Psi & n-p = 1, 2 \bmod 4 \end{cases} \tag{C.4.19}$$

since

$$\mathbb{C}\mathbb{C}^* = \gamma^{p+1}\cdots\gamma^n(-1)^{n-p}\gamma^{p+1}\cdots\gamma^n = (-1)^{(n-p)(n-p+1)/2} \tag{C.4.20}$$

Since Ψ and Ψ_c transform the same way under $\mathfrak{so}$, one may use this conjugation operation to project the Dirac spinor to real and imaginary halves. With $n-p = 0,3$ mod 4, for which $\mathbb{C}\mathbb{C}^* = 1$,

$$\left[\frac{1}{2}(\Psi + \Psi_{\mathbb{C}})\right]_{\mathbb{C}} = \frac{1}{2}(\Psi + \Psi_{\mathbb{C}}) \ , \qquad \left[\frac{\mathbb{i}}{2}(\Psi - \Psi_{\mathbb{C}})\right]_{\mathbb{C}} = \frac{\mathbb{i}}{2}(\Psi - \Psi_{\mathbb{C}}) \tag{C.4.21}$$

which split the Dirac spinor into real and imaginary part.

If we restrict our attention to $d = 2n$, there is one more choice of the charge conjugation operator,

$$\mathcal{C} = \mathbb{C}\Gamma = \mathbb{C}\Gamma^{2n+1} \tag{C.4.22}$$

again with $\mathcal{C}\mathcal{C}^\dagger = 1$, and

$$\mathcal{C}^{-1}\Gamma^a\mathcal{C} = (-1)^{n-p+1}(\Gamma^a)^* \ , \qquad (\Gamma^{ab})^* = \mathcal{C}^{-1}\Gamma^{ab}\mathcal{C} \tag{C.4.23}$$

The charge conjugation under $\mathcal{C}$,

$$\Psi_{\mathcal{C}} \equiv \mathcal{C}\Psi^* \tag{C.4.24}$$

has the property,

$$(\Psi_{\mathcal{C}})_{\mathcal{C}} = \mathcal{C}(\mathcal{C}\Psi^*)^* = \begin{cases} \Psi & n-p = 0,1 \text{ mod } 4 \\ & \\ -\Psi & n-p = 2,3 \text{ mod } 4 \end{cases} \tag{C.4.25}$$

since

$$\mathcal{C}\mathcal{C}^* = (-1)^{(n-p)(n-p+1)/2+(n-p)} = (-1)^{(n-p)(n-p+3)/2} \tag{C.4.26}$$

which equals 1 for $n-p = 0,1$ mod 4 and allows the split

$$\left[\frac{1}{2}(\Psi + \Psi_{\mathcal{C}})\right]_{\mathcal{C}} = \frac{1}{2}(\Psi + \Psi_{\mathcal{C}}) \ , \qquad \left[\frac{\mathbb{i}}{2}(\Psi - \Psi_{\mathcal{C}})\right]_{\mathcal{C}} = \frac{\mathbb{i}}{2}(\Psi - \Psi_{\mathcal{C}}) \tag{C.4.27}$$

in the same manner as above.

Although the nomenclatures on this varies, we will call the projected spinors, possible with the help of $\mathbb{C}\mathbb{C}^* = 1$ or $\mathcal{C}\mathcal{C}^* = 1$, Majorana. The other case, with $\mathbb{C}\mathbb{C}^* = -1$ for odd dimensions or $\mathcal{C}\mathcal{C}^* = \mathbb{C}\mathbb{C}^* = -1$ for even dimensions, are called symplectic Majorana. For such symplectic Majorana spinors, the charge conjugation extends the global symmetry algebra $\mathfrak{u}(1)$ that act on a single Dirac spinor, to the $\mathfrak{sp}(1) = \mathfrak{usp}(2)$, even though the above split of Dirac spinor into "real" and "imaginary" parts is not possible

Majorana spinors, with truly half the degree of freedom relative to Dirac spinors, are possible in odd dimensions under $\mathbb{C}$ if $n - p = 0, 3 \mod 4$. These are $d = 2n + 1 = 7, 9$ etc for $\mathfrak{so}(2n + 1)$'s and $d = 2n + 1 = 3, 9, 11$ etc for $\mathfrak{so}(1, 2n)$'s. In even dimensions, on the other hand, we can use either of $\mathbb{C}$ or $\mathcal{C}$, so the Majorana spinor is possible provided that $n - p = 0, 1, 3$. For $\mathfrak{so}(2n)$, these are $d = 2, 6, 8$ etc while for $\mathfrak{so}(1, 2n - 1)$ these are $d = 2, 4, 8, 10$ etc.

Although we worked with mostly plus sign of the signature, it would be immediately clear that these classifications is symmetric under $\mathfrak{so}(p, d - p) \to \mathfrak{so}(d - p, p)$ since all we need to do is to map $\Gamma^a \to \mathbb{i}\Gamma^a$, under which the rotation generator change signs at most. In even dimensions, this flip exchanges $\mathbb{C}$ and $\mathcal{C}$, for example. Note that the above discussion of reality and pseudo-reality refers to the representation under $\mathfrak{so}(p, d - p)$, rather than those of the Clifford algebra. The (pseudo-)reality of $Cl_{p,d-p}$ representations are another matter, since the much-smaller algebra $\mathfrak{so}(p, d-p)$ resides in $Cl^{\text{even}}_{p,d-p}$ consisting of even number of antisymmetrized product of Γ^a's, inside $Cl_{p,d-p}$. For this reason, the (pseudo-)reality classification of the Clifford algebra $Cl_{p,d-p}$ itself, often found in mathematics literature, looks different from that of $\mathfrak{so}$ spinors above.

(Symplectic) Majorana-Weyl

On the other hand, the Dirac spinor is not irreducible under $\mathfrak{so}(p, 2n - p)$ given how $\Gamma^a\Gamma^{2n+1}$ are no longer rotation generators. As we have seen earlier, $\Gamma^{2n+1} = \Gamma$ plays the role of a chirality operator instead, and split the Dirac spinor into a pair of Weyl spinors

$$\Psi_\pm = \frac{1}{2}(1 \pm \Gamma)\Psi \tag{C.4.28}$$

With this, we need to check whether the two types of the above projections can be simultaneously imposed. The rotation generators for these are respectively,

$$\Sigma^{ab}_{\pm} = \Gamma^{ab}\frac{(1 \pm \Gamma)}{2} \tag{C.4.29}$$

and their properties under the charge conjugation are

$$\mathbb{C}^{-1}(\Sigma^{ab}_{\pm})^*\mathbb{C} = \mathbb{C}^{-1}(\Gamma^{ab})^*\frac{(1 \pm \Gamma^*)}{2}\mathbb{C} = \Gamma^{ab}\frac{(1 \pm (-1)^{n-p}\Gamma)}{2} \tag{C.4.30}$$

and the same with $\mathcal{C}$,

$$\mathcal{C}^{-1}(\Sigma^{ab}_{\pm})^*\mathcal{C} = \mathcal{C}^{-1}(\Gamma^{ab})^*\frac{(1 \pm \Gamma^*)}{2}\mathcal{C} = \Gamma^{ab}\frac{(1 \pm (-1)^{n-p}\Gamma)}{2} \tag{C.4.31}$$

Thus Weyl projection is compatible with either of the charge conjugation if and only if $n-p$ is even.

Let us first concentrate on $\mathfrak{so}(1,d-1)$. Recall that, with $p=1$, the Majorana projection was possible for $d = 2,3,4,8,9,10,11$, respectively with $n = 1,1,2,4,4,5,5$. Among even dimensions, therefore, we find the Majorana projection and the Weyl projections are compatible only in $d = 2,10$ dimensions. Spinors projected twice this way is called Majorana-Weyl. On the other hand, in $d = 6$ dimensions, we have $n-p = 2$ so that the charge conjugation associated with the symplectic Majorana property there does preserve the Weyl projection. We call the Weyl spinor in such cases symplectic Majorana-Weyl whose net effect is merely enlargement of the global symmetry associated with the Weyl spinor.

With the Euclidean signature $\mathfrak{so}(d = 2n)$, the Majorana projection is available for $d = 6,8$ with $n = 3,4$, respectively, so the Majorana-Weyl spinor is possible only for $d = 8$.

C.4.3 Minimal Spinor Representations for $\mathfrak{so}(1,d-1)$

Recall how the two relevant sign factors that entered the above discussion are determined by the combination, $n-p$, as

$$(-1)^{n-p}\ , \qquad (-1)^{(n-p)(n-p+1)/2} \tag{C.4.32}$$

The former repeat itself in $n-p$ mod 2 while we have seen that the latter repeat itself in $n-p$ mod 4. With $\mathfrak{so}(p,d-p)=\mathfrak{so}(p,q)$, on the other hand, $n-p=(q-p)/2+\cdots$, so the first and the second repeat themselves in $q-p$ mod 4 and mod 8 respectively.

Combined, this implies that the pattern repeats itself in d mod 8 and is invariant under the shift $(p,q)\to(p+1,q+1)$. With this understood, it suffices to list the minimal representation for a particular signature, say, $p=1$. In the table, we list the smallest spinor representations for $\mathfrak{so}(1,d-1)$ for $d\le 11$, with the resulting minimal number of components displayed in the second column. The last column represents the largest possible global symmetry when N such spinors are simultaneously present; $N_\pm$ refers to the numbers of chiral and anti-chiral spinors, respectively, when applicable.

	# of components	minimal spinor	global symmetry
$\mathfrak{so}(1,1)$	1 real	Majorana-Weyl	$\mathfrak{so}(N_+)\oplus\mathfrak{so}(N_-)$
$\mathfrak{so}(1,2)$	2 real	Majorana	$\mathfrak{so}(N)$
$\mathfrak{so}(1,3)$	2 complex	Weyl (or Majorana)	$\mathfrak{su}(N)\oplus\mathfrak{u}(1)$
$\mathfrak{so}(1,4)$	4 complex	symplectic Majorana	$\mathfrak{sp}(N)$
$\mathfrak{so}(1,5)$	4 complex	symplectic Majorana-Weyl	$\mathfrak{sp}(N_+)\oplus\mathfrak{sp}(N_-)$
$\mathfrak{so}(1,6)$	8 complex	symplectic Majorana	$\mathfrak{sp}(N)$
$\mathfrak{so}(1,7)$	8 complex	Weyl (or Majorana)	$\mathfrak{su}(N)\oplus\mathfrak{u}(1)$
$\mathfrak{so}(1,8)$	16 real	Majorana	$\mathfrak{so}(N)$
$\mathfrak{so}(1,9)$	16 real	Majorana-Weyl	$\mathfrak{so}(N_+)\oplus\mathfrak{so}(N_-)$
$\mathfrak{so}(1,10)$	32 real	Majorana	$\mathfrak{so}(N)$

For $d=4,8$, where one can choose either Weyl or Majorana, we displayed the Weyl spinor; Weyl is more versatile than Majorana in that it can more naturally accommodate more diverse gauge representations. We should emphasize again that "symplectic Majorana" has the same content as a Dirac; the difference is how "symplectic" case admit maximal global symmetry algebra $\mathfrak{sp}(N)=\mathfrak{usp}(2N)$ instead of $\mathfrak{u}(N)$ for a collection of N such spinors. With $N_\pm$ referring to the number of chiral and anti-chiral Weyl spinors as above, "symplectic Majorana-Weyl" may have the symmetry $\mathfrak{sp}(N_+)\oplus\mathfrak{sp}(N_-)$.

Note how these spinors decompose under the reduction $\mathfrak{so}(1,d-1) \to \mathfrak{so}(1,1) \oplus \mathfrak{so}(d-2)$. Starting from the former's spinor Ψ, there is a universal decomposition,

$$\Psi \qquad \to \qquad \Psi_{1/2} + \Psi_{-1/2} \tag{C.4.33}$$

where $\pm 1/2$ refers to the charge under $\mathfrak{so}(1,1)$. Since the latter's smallest spinor is a single real component and since the type of the minimal spinor representation is common between $\mathfrak{so}(1,d-1)$ and $\mathfrak{so}(d-2)$, the reality property of Ψ is inherited by $\Psi_{1/2}$ and $\Psi_{-1/2}$, each carrying exactly half the component of Ψ. For instance, with $d = 10$, a Majorana-Weyl Ψ carries 16 real components while $\Psi_{\pm 1/2}$ is again Majorana-Weyl with 8 real components each, whose chiralities are determined as ± 1 times that of Ψ. This exercise is closely tied to how the dynamical contents of fields transforming covariantly under $\mathfrak{so}(1,d-1)$ are classified by the little group, which for massless cases is effectively $\mathfrak{so}(d-2)$.