Chapter 9

Gauge Symmetries and BRST

In Chapter 3, we studied how Faddeev and Popov managed to convert an ill-defined path integral with huge gauge redundancies to a well-defined one via their gauge-fixing procedure. It was an elaborate way to mod out the infinite gauge volume. Along the way, we encountered the notion of BRST action, named after Carlo Becchi, Alain Rouet, Raymond Stora and Igor Tyutin, who realized how Faddeev-Popov's action can be recast with ghost fields and an auxiliary field, so that the gauge-fixed action admits a new type of symmetry that inherits the gauge redundancy.

In this brief chapter, we shall go back to the notion of the gauge redundancy for a rudimentary understanding how we handle such ambiguities in characterizing and isolating physical states. We have gone through in Appendix A on how to institute the gauge-invariance of physical states in the Hamiltonian view, but for most of this volume we approach the quantum field theories via path integral where a Lorentz-invariant approach would be more desirable. We are naturally led to the BRST symmetry as their reincarnation after the gauge-fixing procedure. Gauge-fixing is needed one way or another, and BRST offers a covariant way to handle the procedure in a sweeping and manifestly covariant manner.

The BRST machinery we find here has broad generalizations to theories equipped with other gauge principles, and is one of more beautiful story associated with gauged systems. In next chapter, we will again resort to the BRST algebra discovered here for a solution-generating technique known as the anomaly descent.

9.1 Canonical Quantization in Brief

The canonical quantization of quantum mechanics finds immediate generalization to scalar fields. The only new thing is that, instead of finite degrees of freedom q's as functions of time t, we start with scalar fields $\phi(t, \mathbf{x})$ as function of t and $\mathbf{x}$. In a sense we can consider it as an infinite-dimensional quantum mechanics with $q_{\mathbf{x}}(t) = \phi(t, \mathbf{x})$ labeled by the spatial position. The most immediate prerequisite to the canonical quantization is how we single out a time coordinate t along which a unitary evolution via the Hamiltonian dictates the dynamics. How this does not spoil the relativistic invariance has been established for the action which is Lorentz invariant.

We have already gone through much of quantum field theory exercises via the Wick-rotated path integrals, so the canonical quantization here may sounds pretty late. The main purpose of recalling this fundamental aspect of quantum field theory is to glimpse into how classical quantities elevate to quantum ones, among which is how the classical gauge redundancy elevates to quantum level. For this reason, the content of this section will be limited to the most basic dictionaries.

Note how we do not really venture into spinor fields and gauge fields for which the quantization machinaries would be in use if we proceed with the remaining chapters in this canonical language. The rudimentary schemes for scalar already contain the essential part of the story and can be elevated to these higher spin fields, as offered by many standard texts. Our purpose here is merely to convince ourselves that such exists; real computation would be instead via the path integral as we have done.

9.1.1 Canonical Quantization of a Real Scalar

We start with a single bosonic field and quantize a free massive scalar field, with the action,

$$\int d^d x \, \frac{1}{2} \left(-\eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right) \, . \tag{9.1.1}$$

As we have seen earlier numerous times, a quantum field theory needs more than the Lagrangian to define. Issues connected to the renormalization flow need to be dealt with eventually. Before that, a more technical issue of how the continuous label $\mathbf{x}$ comes with unbounded values can be often treated sensibly by replacing the infinite space by a box of finite volume; this way we have at least traded off the continuous label $\mathbf{x}$ in favor of discrete labels, such as the quantized Fourier momenta. We can then take the infinite volume limit, to go back to the original field theory.

For simplicity, we will take the space to be a cubic box of linear size L and impose the periodic boundary condition on ϕ,

$$\phi(t, \mathbf{x}^i) = \phi(t, \mathbf{x}^i + L\delta^i_k) \ , \qquad k = 1, \ldots, d-1 \tag{9.1.2}$$

The field equation,

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi = m^2\phi \ , \tag{9.1.3}$$

is naturally solved by the following plane waves,

$$f^\pm_{\mathbf{p}} = \frac{1}{(2\omega_{\mathbf{p}})^{1/2}L^{(d-1)/2}}e^{\mp \mathbb{i}\omega_{\mathbf{p}}t + \mathbb{i}2\pi\mathbf{p}\cdot\mathbf{x}/L} \ , \qquad \omega_{\mathbf{p}} = \sqrt{m^2 + (2\pi\mathbf{p})^2/L^2} \tag{9.1.4}$$

with $\mathbf{p} \in \mathbb{Z}^{d-1}$. Note that we have $(f^-_{-\mathbf{p}})^* = f^+_{\mathbf{p}}$.

The particular normalization is designed such that they are orthonormal with respect to the following inner product, among $\mathbb{Z}^{d-1}$-many independent Fourier modes,

$$\langle\!\langle f^+_{\mathbf{p}}, f^+_{\mathbf{q}} \rangle\!\rangle \equiv \int d^{d-1}\mathbf{x}\ 2\omega_{\mathbf{p}}\, (f^+_{\mathbf{p}})^* f^+_{\mathbf{q}} = \delta_{\mathbf{p},\mathbf{q}} \ . \tag{9.1.5}$$

Later, when we generalize to curved spacetime, it should become evident why this choice with $\omega_{\mathbf{p}}$ factor in the integrand is inevitable. In turn, the completeness relation with these Fourier modes is

$$\sum_{\mathbf{p}} 2\omega_{\mathbf{p}}\, (f^+_{\mathbf{p}}(t,\mathbf{y}))^* f^+_{\mathbf{p}}(t,\mathbf{x}) = \delta^{d-1}(\mathbf{x}-\mathbf{y}) \ , \tag{9.1.6}$$

or equivalently,

$$\sum_{\mathbf{p}} 2\omega_{\mathbf{p}}\, f^-_{-\mathbf{p}}(t,\mathbf{y})\, f^+_{\mathbf{p}}(t,\mathbf{x}) = \delta^{d-1}(\mathbf{x}-\mathbf{y}) \ . \tag{9.1.7}$$

The most general solution for ϕ takes the form,

$$\phi(t,\mathbf{x}) = \sum_{\mathbf{p}} a_{\mathbf{p}}\, f^+_{\mathbf{p}} + \sum_{\mathbf{p}} a^*_{\mathbf{p}}\, \left(f^+_{\mathbf{p}}\right)^* \tag{9.1.8}$$

with complex coefficients $a_{\mathbf{p}}$ and their conjugates $a^*_{\mathbf{p}}$ to ensure the reality of ϕ.

The quantization proceeds from this, but with the canonical variables, as usual. The conjugate momentum is

$$\Pi = \frac{\delta}{\delta \dot{\phi}} \int \mathcal{L} = \dot{\phi} = \sum_{\mathbf{p}} -\mathbb{i}\omega_{\mathbf{p}}\, a_{\mathbf{p}}\, f^+_{\mathbf{p}} + \sum_{\mathbf{p}} \mathbb{i}\omega_{\mathbf{p}}\, a^*_{\mathbf{p}} \left(f^+_{\mathbf{p}}\right)^* \ . \tag{9.1.9}$$

The Poisson bracket is

$$[\phi(t,\mathbf{x}), \Pi(t,\mathbf{y})]_{\text{P.B.}} = \delta^{d-1}(\mathbf{x}-\mathbf{y}) \ , \tag{9.1.10}$$

so the quantized version is

$$[\boldsymbol{\phi}(t,\mathbf{x}), \boldsymbol{\Pi}(t,\mathbf{y})] = \mathbb{i}\hbar\, \delta^{d-1}(\mathbf{x}-\mathbf{y}) \ . \tag{9.1.11}$$

now with the boldfaced symbols to emphasize that they are now considered operators.

One sees that the desired canonical commutator is achieved by demanding, after elevating the coefficients a to operators $\mathbf{a}$,

$$[\mathbf{a}_{\mathbf{p}}\, , \mathbf{a}^\dagger_{\mathbf{q}}] = \hbar\, \delta_{\mathbf{p},\mathbf{q}} \ , \qquad [\mathbf{a}_{\mathbf{p}}\, , \mathbf{a}_{\mathbf{q}}] = 0 = [\mathbf{a}^\dagger_{\mathbf{p}}\, , \mathbf{a}^\dagger_{\mathbf{q}}] \ , \tag{9.1.12}$$

where $\delta_{\mathbf{p},\mathbf{q}}$ should be taken with a grain of salt; since the momenta $\mathbf{p}$ would become continuous variables in the limit of $L \to \infty$, $\delta_{\mathbf{p},\mathbf{q}}$ is meant to represent the identity operator in the $\mathbf{p}$ space, eventually. This reduces the above equal time commutator to

$$\begin{aligned} & [\boldsymbol{\phi}(t,\mathbf{x}), \boldsymbol{\Pi}(t,\mathbf{y})] \\ = \;& \sum_{\mathbf{p}} \sum_{\mathbf{q}} \left(\mathbb{i}\omega_{\mathbf{q}} [\mathbf{a}_{\mathbf{p}}, \mathbf{a}^\dagger_{\mathbf{q}}] f^+_{\mathbf{p}}(t,\mathbf{x}) f^-_{-\mathbf{q}}(t,\mathbf{y}) - \mathbb{i}\omega_{\mathbf{q}} [\mathbf{a}^\dagger_{\mathbf{p}}, \mathbf{a}_{\mathbf{q}}] f^-_{-\mathbf{p}}(t,\mathbf{x}) f^+_{\mathbf{q}}(t,\mathbf{y})\right) \\ = \;& \mathbb{i}\hbar\, \delta^{d-1}(\mathbf{x}-\mathbf{y}) \ , \end{aligned} \tag{9.1.13}$$

as desired.

The universal factor of $\hbar$ can be absorbed by normalizing the oscillator $\mathbf{a}$'s as

$$\mathbf{a}_{\mathbf{p}} \to \sqrt{\hbar}\, \mathbf{a}_{\mathbf{p}} \ , \tag{9.1.14}$$

so that

$$[\mathbf{a}_{\mathbf{p}}\,,\mathbf{a}^\dagger_{\mathbf{q}}] = \delta_{\mathbf{p},\mathbf{q}}\ . \tag{9.1.15}$$

With this normalization, the Hamiltonian is

$$\mathbf{H} \;=\; \int d^{d-1}\mathbf{x}\,\frac{1}{2}\left(\mathbf{\Pi}^2 + (\boldsymbol{\partial\phi})^2 + m^2\boldsymbol{\phi}^2\right) \;=\; \frac{1}{2}\sum_{\mathbf{p}}\left(\mathbf{a}_{\mathbf{p}}\,\mathbf{a}^\dagger_{\mathbf{p}} + \mathbf{a}^\dagger_{\mathbf{p}}\,\mathbf{a}_{\mathbf{p}}\right)\hbar\omega_{\mathbf{p}}\ , \tag{9.1.16}$$

reducing the content of ϕ to an infinite number of harmonic oscillators, labeled by $\mathbf{p} \in \mathbb{Z}^{d-1}$, with the respective energy gaps $\omega_{\mathbf{p}}$. The total space constructed from these harmonic oscillators is called the Fock space.

Defining the number operator

$$N_{\mathbf{p}} = \mathbf{a}^\dagger_{\mathbf{p}}\,\mathbf{a}_{\mathbf{p}}\ , \tag{9.1.17}$$

the Hamiltonian can be also written as

$$\mathbf{H} \;=\; \sum_{\mathbf{p}}\left(\mathbf{a}^\dagger_{\mathbf{p}}\,\mathbf{a}_{\mathbf{p}} + \frac{1}{2}\right)\hbar\omega_{\mathbf{p}}\ . \tag{9.1.18}$$

Now that we reviewed the basics of the quantization procedure, we will often drop $\hbar$ by taking the unit where $\hbar = 1$ for simplicity in many middle computations. In this unit, combined with $c = 1$, energy is measured in the unit of inverse length. Reviving it is not too difficult as we merely need to keep track of such units. We will periodically remind ourselves how $\hbar$ enters the story by reviving $\hbar$ at key places, throughout the rest of the note.

As is familiar from quantum harmonic oscillators, the Fock space vacuum is annihilated by all of $\mathbf{a}_{\mathbf{p}}$'s,

$$\mathbf{a}_{\mathbf{p}}|0\rangle = 0 \tag{9.1.19}$$

the Hilbert space constructed via creation operators $\mathbf{a}^\dagger_{\mathbf{p}}$'s acting on this vacuum. Accumulated action of such creation operators add a set of elementary particles of ϕ field carrying such momenta, propagating in the form of a plane-wave. The quantum field theory goes well beyond quantum mechanics in the sense that the number of particles is not fixed, nor are the particle species unchangeable. The Hawking effect we explore in a later chapter is possible fundamentally because particles themselves are derived concept, as the unit excitation of the quantum field in question.

9.1.2 Canonical Quantization in a Curved spacetime

This section came from another volume by the author on General Relativity with materials on the Hawking radiation and related quantum effect, where we needed to extend the canonical quantization in the curved spacetime as well. We include part of the latter material here as well for the sake of completeness, as we discuss on and off how curved geometry responds to quantum matter. Although the path integral offers a natural language for such interplay between quantum matter and curved spacetime, it is also important to recognize that the canonical quantization can also proceed in such nontrivial background as we will be discussing how the effective action of chiral matter fields would behave in such spacetimes.*

Let us consider how one might repeat the above quantization procedure for a free real scalar field, now in a curved spacetime. With the action

$$\int d^dx\,\sqrt{g}\,\frac{1}{2}\left(-g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right)\ , \tag{9.1.20}$$

the equation of motion is modified by the geometry

$$\nabla^2\phi = m^2\phi\ . \tag{9.1.21}$$

The Hamiltonian, or canonical, formulation needs a reasonably well-defined notion of time, or a foliation. Let us be not too specific but call the time coordinate t and the space-like hypersurfaces of constant t as Σ_t.

Since the field equation is linear, one should be able to construct the most general solution via linear superposition,

$$\phi = \sum_{\mathbf{p}} a_{\mathbf{p}}\, f^+_{\mathbf{p}} + \sum_{\mathbf{p}} a^*_{\mathbf{p}}\, f^-_{\mathbf{p}}\ , \tag{9.1.22}$$

where we kept the same $\mathbf{p}$ as the labels, although these solutions are no longer the

*For more immediate applications of the quantization scheme we outline to black holes and cosmology, we refer the readers to “Gravitation: A Geometric Field Theory” a separate volume by the author, although in a later chapter in this volume we will again visit the Hawking effect from a path integral approach.

simple plane waves on $\mathbb{R}^d$ but rather follow a generalized version of the Fourier analysis. Nor are these solutions be generally countable, if we consider the infinite volume of the typical spacetime we would encounter later; one should not take the discrete labels $\mathbf{p}$ and the summation over them too literally, therefore. In actual applications, the summations would be replaced by integrals over appropriately generalized Fourier modes. Fortunately, many of these detail will not affect our main aim of understanding the quantum effect in the presence of relativistic horizons.

The first thing to ask is whether there is a sensible inner product among these eigenmodes. Inspired by the flat case, we define the following inner product,

$$\langle\!\langle \tilde{f}, f \rangle\!\rangle_{\Sigma_t} = \mathbb{i} \int_{\Sigma_t} dS^\mu \left(\tilde{f}^* (\nabla_\mu f) - (\nabla_\mu \tilde{f}^*) f \right) \;, \tag{9.1.23}$$

over a constant t hypersurface Σ_t. This inner product is invariant under the time evolution since

$$\langle\!\langle \tilde{f}, f \rangle\!\rangle_{\Sigma_{t'}} - \langle\!\langle \tilde{f}, f \rangle\!\rangle_{\Sigma_t} = \mathbb{i} \int_{M_{t't}} d^d x \, \sqrt{g} \, \nabla^\mu \left(\tilde{f}^* (\nabla_\mu f) - (\nabla_\mu \tilde{f}^*) f \right) = 0 \tag{9.1.24}$$

thanks to the field equation, where $\partial M_{t't} = \Sigma_{t'} - \Sigma_t$. Actually this statement is generally false since $M_{t't}$ can have boundary at spatial infinities, in principle, but the idea here is that relevant physical states are wave-packets with localized profiles in the end. These would obey some vanishing condition far away enough to allow us to ignore such asymptotic boundaries at spatial infinities.

More generally, given a pair of hypersurfaces, Σ' and Σ, we have

$$\langle\!\langle \tilde{f}, f \rangle\!\rangle_{\Sigma'} - \langle\!\langle \tilde{f}, f \rangle\!\rangle_{\Sigma} = \mathbb{i} \int_{M_{\Sigma'\Sigma}} d^d x \, \sqrt{g} \, \nabla^\mu \left(\tilde{f}^* (\nabla_\mu f) - (\nabla_\mu \tilde{f}^*) f \right) = 0 \tag{9.1.25}$$

as long as we can find $M_{\Sigma'\Sigma}$ whose boundary is $\Sigma' - \Sigma$. As such, the inner product itself does not depend on the choice of the time-coordinate t. As such, we will often denote the inner product without the labels,

$$\langle\!\langle \cdots , \cdots \rangle\!\rangle \;, \tag{9.1.26}$$

which obeys

$$\langle\!\langle \tilde{f}, f \rangle\!\rangle = \langle\!\langle f, \tilde{f} \rangle\!\rangle^* = -\langle\!\langle f^*, \tilde{f}^* \rangle\!\rangle = -\langle\!\langle \tilde{f}^*, f^* \rangle\!\rangle^* \;. \tag{9.1.27}$$

under complex conjugations.

One can set up examples most analogous to those in the flat spacetime when the metric is time-foliated and ∂_t is a Killing vector field,

$$g = -N(\mathbf{x})^2 dt^2 + \mathbf{h}_{ij}(\mathbf{x}) d\mathbf{x}^i d\mathbf{x}^j \tag{9.1.28}$$

This metric allows a separation of variables to occur for the Fourier modes, with $\omega_{\mathbf{p}} \geq 0$,

$$f_{\mathbf{p}}^+ = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{-\mathbb{i}\omega_{\mathbf{p}} t} \psi_{\mathbf{p}}(\mathbf{x}) , \qquad f_{\mathbf{p}}^- = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} e^{+\mathbb{i}\omega_{\mathbf{p}} t} \psi_{\mathbf{p}}^*(\mathbf{x}) \tag{9.1.29}$$

with

$$-\frac{N}{\sqrt{\mathbf{h}}}\partial_i \left(N\sqrt{\mathbf{h}}\,\mathbf{h}^{ij}\partial_i \psi_{\mathbf{p}} \right) = \omega_{\mathbf{p}}^2 \psi_{\mathbf{p}} \tag{9.1.30}$$

The operator on the left is Hermitian under the natural pairing on each time slices, so that the eigenfunctions obey the orthonormality,

$$\int_{\Sigma_t} \frac{1}{N(\mathbf{x})} (\psi_{\mathbf{q}}(\mathbf{x}))^* \psi_{\mathbf{p}}(\mathbf{x}) = \delta_{\mathbf{q},\mathbf{p}} \tag{9.1.31}$$

with the induced metric $\mathbf{h}$ in Σ_t understood. The label $\mathbf{p}$ need not be discrete so the expression $\delta_{\mathbf{q},\mathbf{p}}$ should taken with a grain of salt, as in the flat spacetime case.

Given that the operator for the spatial eigenvalue problem is real, we may as well choose $\psi_{\mathbf{p}}(\mathbf{x})$ themselves real, by taking linear combinations among eigenmodes with the common eigenvalue $\omega_{\mathbf{p}}^2$. This immediately allow us to say that

$$\begin{aligned} &\langle\!\langle f_{\mathbf{q}}^+, f_{\mathbf{p}}^+ \rangle\!\rangle = \delta_{\mathbf{q},\mathbf{p}} , \\ &\langle\!\langle f_{\mathbf{q}}^-, f_{\mathbf{p}}^- \rangle\!\rangle = -\delta_{\mathbf{q},\mathbf{p}} , \\ &\langle\!\langle f_{\mathbf{q}}^-, f_{\mathbf{p}}^+ \rangle\!\rangle = 0 = \langle\!\langle f_{\mathbf{q}}^+, f_{\mathbf{p}}^- \rangle\!\rangle . \end{aligned} \tag{9.1.32}$$

and

$$\mathbf{a}_{\mathbf{q}} = \langle\!\langle f_{\mathbf{q}}^+, \boldsymbol{\phi} \rangle\!\rangle , \qquad \mathbf{a}_{\mathbf{q}}^\dagger = -\langle\!\langle f_{\mathbf{q}}^-, \boldsymbol{\phi} \rangle\!\rangle . \tag{9.1.33}$$

The negative definite nature of this pairing between f^-'s does not imply unphysical states. It merely signals that $f^-_{\mathbf{p}} \sim e^{\mathbb{i}\,\omega_{\mathbf{p}} t}$'s represent anti-particles.

The completeness relation that we need for the equal-time commutator is built from the basis $\{\psi_{\mathbf{p}}(\mathbf{x}^i)\}$ which obeys the usual orthonormality among these spatial eigenmodes, (9.1.31) so is not affected by this extraneous sign. We recover the same canonical commutator

$$[\mathbf{a}_{\mathbf{p}}\,, \mathbf{a}^\dagger_{\mathbf{q}}] = \delta_{\mathbf{p},\mathbf{q}}\ , \qquad [\mathbf{a}_{\mathbf{p}}\,, \mathbf{a}_{\mathbf{q}}] = 0 = [\mathbf{a}^\dagger_{\mathbf{p}}\,, \mathbf{a}^\dagger_{\mathbf{q}}]\ , \tag{9.1.34}$$

for the quantized operators, and the Hamiltonian,

$$\mathbf{H} \;=\; \frac{1}{2}\sum_{\mathbf{p}} \left(\mathbf{a}_{\mathbf{p}}\,\mathbf{a}^\dagger_{\mathbf{p}} + \mathbf{a}^\dagger_{\mathbf{p}}\,\mathbf{a}_{\mathbf{p}}\right)\hbar\omega_{\mathbf{p}} \;=\; \sum_{\mathbf{p}} \left(\mathbf{a}^\dagger_{\mathbf{p}}\,\mathbf{a}_{\mathbf{p}} + \frac{1}{2}\right)\hbar\omega_{\mathbf{p}}\ , \tag{9.1.35}$$

is conserved, as in the flat spacetime, thanks to the time translation invariance assumed.

9.2 Gauss Constraints and Physical States

The treatment we gave for the scalar quantization would apply to free Maxwell theory as well, although the important caveats due to the gauge redundancy would enter nontrivially. For the latter, we need to keep in mind both how to perform the gauge fixing but at the same time how to elevate the gauge redundancy to quantum level. For the latter, a useful thing to do is to recall how the gauged system is handled in the canonical formulation, which is the starting point of the canonical quantization.

Appendix A gave a quick overview of the constrained system, which ended with the example of the classical Maxwell theory. Recall that the free theory, say, with

$$S_{\text{Maxwell}} \;=\; -\frac{1}{4}\int d^d x\; \tilde{F}^2 \tag{9.2.1}$$

is equipped with a pair of first-class constraints,

$$\pi^0 \approx 0\ , \qquad \partial_i \pi^i \approx 0 \tag{9.2.2}$$

where π^μ are the conjugate momenta of $\tilde{A}_\mu$,

$$\pi^\mu \equiv \frac{\delta S_{\text{Maxwell}}}{\delta(\partial_0 \tilde{A}_\mu)} \tag{9.2.3}$$

As explained there, $\approx$ means that these must hold on physical states and one should not rush into using the constraints too early.

Some of related subtleties may be addressed most conveniently by working with the total Hamiltonian

$$\mathbb{H} = \int d^{d-1}\mathbf{x} \left[\frac{1}{2}\left(\pi^i\pi^i + B^iB^i\right) + \vartheta_0\pi_0 + (\tilde{A}_0 + \vartheta_1)(-\partial_i\pi^i)\right] \tag{9.2.4}$$

with the Lagrange multiplier ϑ_0 and ϑ_1. The Lagrange multipliers of the first-class constraints are not determined on-shell, while those of the second-class constraints are, and play the role of the gauge functions. Time derivative of $\tilde{A}_0$ acquires an arbitrary term $\sim \vartheta_0$ while that of $\tilde{A}_i$ is shifted by $\partial_i\vartheta_1$, due to these additional arbitrary pieces. Regardless of suh a shift, which is a matter of convenience, A_0 acts like a Lagrange multiplier, imposing a secondary constraint,

$$-\partial_i\pi^i \approx 0 \tag{9.2.5}$$

as we saw above. This is called the Gauss constraint.

After a harmless redefinition of $\vartheta_0 = -\partial_0\vartheta$, we may as well shift $\vartheta_1 \to -\vartheta$, given how $\tilde{A}_0$ is also entirely arbitrary, so that,

$$\mathbb{H} = \int d^{d-1}\mathbf{x} \left[\frac{1}{2}\left(\pi^2 + B^2\right) + (-\partial_0\vartheta)\pi_0 + (\tilde{A}_0 - \vartheta)(-\partial_i\pi^i)\right] \tag{9.2.6}$$

The time evolution under $\mathbb{H}$ accumulates, in addition to the physical evolution, a gauge transformation of $\tilde{A}$ under the gauge function,

$$\theta = \int^t dt'\, \vartheta(t', \mathbf{x}) \tag{9.2.7}$$

given the Poisson bracket,

$$[\pi^\mu(t,\mathbf{x}), \tilde{A}_\nu(t,\mathbf{x}')]_{\text{P.B.}} = -\delta^\mu{}_\nu\, \delta^{d-1}(\mathbf{x}-\mathbf{x}') \tag{9.2.8}$$

In particular, the Gauss constraint is the generator of the gauge transformation and its vanishing implies that the physical states are gauge-invariant.

All of these extend to the case with charged matter field, say,

$$S_{\text{Maxwell+matter}} = \int d^dx \left(-\frac{1}{4}\tilde{F}^2 + \mathcal{L}_{\text{matter}}(\phi^\dagger, \phi, \tilde{A})\right) \tag{9.2.9}$$

with

$$\mathcal{L}_{\text{matter}} = -\eta^{\mu\nu}(\partial_\mu + \mathbb{i}q\tilde{A}_\mu)\phi^\dagger(\partial_\nu - \mathbb{i}q\tilde{A}_\nu)\phi - V(\phi^\dagger\phi) \ . \tag{9.2.10}$$

The conjugate momenta are

$$\begin{aligned}
\Pi &= \frac{\delta}{\delta(\partial_0\phi)}\int \mathcal{L}_{\text{matter}} = (\partial_0 + \mathbb{i}q\tilde{A}_0)\phi^\dagger \ , \\
\Pi^\dagger &= \frac{\delta}{\delta(\partial_0\phi^\dagger)}\int \mathcal{L}_{\text{matter}} = (\partial_0 - \mathbb{i}q\tilde{A}_0)\phi \ ,
\end{aligned} \tag{9.2.11}$$

and the total Hamiltonian is

$$\begin{aligned}
\mathbb{H} &= \int d^{d-1}\mathbf{x}\ \left[\frac{1}{2}(\pi^2 + B^2) + \Pi\Pi^\dagger + V(\phi^\dagger\phi)\right] \\
&\quad + \int d^{d-1}\mathbf{x}\ \left[(-\partial_0\vartheta)\pi_0 + (\tilde{A}_0 - \vartheta)\left(-\partial_i\pi^i + \mathbb{i}q(\Pi\phi - \Pi^\dagger\phi^\dagger)\right)\right] \ .
\end{aligned} \tag{9.2.12}$$

Again from π^0's null equation of motion, we find the Gauss constraint now equipped with a charge density,

$$\partial_i\pi^i + \rho \approx 0 \ , \qquad \rho \equiv -\mathbb{i}q(\Pi\phi - \Pi^\dagger\phi^\dagger) \ . \tag{9.2.13}$$

The matter charge part in $\mathbb{H}$,

$$\int d^{d-1}\mathbf{x}\,\vartheta\left[-\mathbb{i}q(\Pi\phi - \Pi^\dagger\phi^\dagger)\right] \tag{9.2.14}$$

generates phase rotation of the matter field $\phi \to e^{\mathbb{i}q\theta}\phi$ and $\phi^\dagger \to e^{-\mathbb{i}q\theta}\phi^\dagger$, again via the Poisson bracket,

$$[\Pi(t,\mathbf{x}), \phi(t,\mathbf{x}')]_{\text{P.B.}} = -\,\delta^{d-1}(\mathbf{x} - \mathbf{x}') \tag{9.2.15}$$

which completes the demonstration that the Gauss constraint is the generator of the gauge transformation.

The ambiguous total Hamiltonian $\mathbb{H}$ allows arbitrary gauge transformation to occur along the time evolution. The only sensible attitude one can take with this undetermined nature of the evolution is to say that the resulting arbitrariness reflects the redundancy in the description. In other words, the time evolution may look ambiguous but all those different paths should be considered equivalent physically, which leads to the gauge principle we have used and relied on. One can see how the gauge redundancy is generated by the Gauss constraint, more or less.

When we quantize the system, we acquire a new option in dealing with such ambiguity. Quantum states are functional on the phase space, so one can now demand invariance of such functional under gauge transformations of its argument. That is, we merely ask that the quantum amplitude remains identical across configurations related by gauge transformations. Calling the quantized Gauss constraint,

$$\mathbf{G}_{\text{gauge}} = \left[\partial_i \boldsymbol{\pi}^i - \mathbb{i} q \left(\boldsymbol{\phi}\boldsymbol{\Pi} - \boldsymbol{\phi}^\dagger \boldsymbol{\Pi}^\dagger\right)\right] \tag{9.2.16}$$

normal-ordered appropriately, with the boldfaced symbols for the quantized quantities, the natural condition to impose on physically sensible states is to demand the gauge-invariance,

$$\mathbf{G}_{\text{gauge}} \,|\text{phys}\rangle = 0 \tag{9.2.17}$$

The procedure outlined here represents the most intuitive and physical treatment of the quantization in the face of gauge redundancy, but is also limited in that we had to separate $\tilde{A}_0$ from the rest, inevitable from how we select out the time direction. Whether this is truly a limitation is a matter of taste, especially for Abelian gauge theories with their relative simple form.

For non-Abelian theories, on the other hand, the Faddeev-Popov procedure offers a fully Lorentz-covariant and systematic gauge-fixing mechanism. The covariant but properly gauge-fixed Lagrangian no longer respects the original gauge symmetry, so the above physical picture of how the gauge redundancy should be handled becomes murky. Some analog of the Gauss constraint, we would like to think, should be respected since the so-called gauge-fixing is a mathematical rewriting of the same path integral divided by the gauge volume. How does one addresses the gauge-invariance of physical quantum states, in the face of the inevitable gauge-fixing procedure? Next we turn to this question, albeit at a formal level.

9.3 BRST Symmetry of the Gauge-Fixed Action

Let us recall the infinitesimal gauge transformations for the gauge field and the matter fields ϕ

$$\boldsymbol{\delta}_\Theta \mathcal{A} = d_\mathcal{A}\Theta = d\Theta + \mathcal{A}\Theta - \Theta\mathcal{A} \ , \qquad \boldsymbol{\delta}_\Theta \phi = -\Theta\phi \ , \tag{9.3.1}$$

For the latter, ϕ stands for both scalars and spinors while the appropriate representation understood implicitly.

With Faddeev-Popov's gauge-fixed action, this transformation no longer preserve the Yang-Mills-Matter path integral

$$\begin{aligned} & \int [D\mathcal{A}][D\phi]\; e^{-S^E(\phi;\mathcal{A})+\zeta^2/2\int \mathrm{tr}\,\mathbb{K}(\mathcal{A})^2} \mathrm{Det}\,(\mathcal{Q}_{\mathrm{ghost}}) \\ = \ & \int [D\mathcal{A}][D\phi][D\mathbb{b}D\mathbb{v}]\; e^{-S^E(\phi;\mathcal{A})+\zeta^2/2\int \mathrm{tr}\,\mathbb{K}(\mathcal{A})^2+\mathrm{tr}(\mathbb{b}\mathcal{Q}_{\mathrm{ghost}}\mathbb{v})} \end{aligned} \tag{9.3.2}$$

since neither the gauge fixing condition $\mathbb{K}(\mathcal{A})$ nor $\mathcal{Q}_{\mathrm{ghost}} = \delta\mathbb{K}/\delta\Theta$ are gauge-invariant. For the structure below, we may keep the gauge-fixing condition $\mathbb{K}$ and the number $\zeta^2 > 0$ arbitrary, although we have employed specific expressions, such as $\mathbb{K} = \bar{D}_\mu \mathcal{A}^\mu$ and $\zeta^2 = 1/\mathbb{g}^2$ in Chapter 3. We left "tr" somewhat ambiguous, although for classical Lie algebras, one can write the ghost fields in the $N \times N$ matrix form and the trace may be performed in the defining representation. For exceptional Lie algebra, these depend pretty much follow how we write the kinetic term of $\mathcal{A}$.

Although we have done this for non-Abelian Yang-Mills theories in Chapter 3, the same is needed for Abelian theories as well. We have computed there how the matter field quantization affect the gauge fields in QED, but of course quantized Maxwell field would also affect the matter fields as well via the minimal coupling of the latter to the former. The necessary path integral over the gauge field requires such a gauge-fixing procedure, regardless of Abelian or non-Abelian. One simplification for Abelian gauge theories is that the typical covariant gauge choice $\mathbb{K}(\mathcal{A}) = D^\mu \mathcal{A}_\mu = \partial^\mu \mathcal{A}_\mu$ does not lead to new vertices, so that this gauge-fixing affects only the gauge-field propagators.

Other than that, the basic ingredients to the Faddeev-Popov procedure are identical, so everything we offer in this chapter is equally applicable.

It turns out that this gauge-fixed path integral can be trivially extended to admit a new type of redundancy, by rewriting the $\mathrm{tr}\,\mathbb{K}(\mathcal{A})^2$ in the exponent as

$$e^{\zeta^2/2\int \mathrm{tr}\,\mathbb{K}(\mathcal{A})^2} = \int [D\mathbb{B}]\; e^{1/(2\zeta^2)\int \mathrm{tr}\,\mathbb{B}^2 - \mathbb{i}\int \mathrm{tr}(\mathbb{B}\,\mathbb{K}(\mathcal{A}))} \tag{9.3.3}$$

so that we end up with

$$\int [D\mathcal{A}][D\phi][D\mathbb{b}D\mathbb{v}][D\mathbb{B}]\; e^{-S^E(\phi;\mathcal{A})-S^E_{\text{g.f.}}(\mathcal{A};\mathbb{b},\mathbb{v};\mathbb{B})} \tag{9.3.4}$$

where

$$S^E_{\text{g.f.}}(\mathcal{A};\mathbb{b},\mathbb{v};\mathbb{B}) = \int \left[-\frac{1}{2\zeta^2}\mathrm{tr}\,\mathbb{B}^2 + \mathbb{i}\,\mathrm{tr}\,(\mathbb{B}\,\mathbb{K}(\mathcal{A})) - \mathrm{tr}\left(\mathbb{b}\frac{\delta\mathbb{K}}{\delta\Theta}\mathbb{v}\right)\right] \tag{9.3.5}$$

We use the same trace convention for $\mathbb{B}$ as we do for the ghosts.

As was originally found by Carlo Becchi, Alain Rouet, Raymond Stora and Igor Tyutin, this new action $S^E(\phi;\mathcal{A}) + S^E_{\text{g.f.}}$ remembers the gauge redundancy with a twist. We shall find transformation rules where Θ is replaced by $\mathbb{v}$, including

$$\boldsymbol{\delta}_{\mathbb{v}}\mathcal{A} \equiv -d_{\mathcal{A}}\mathbb{v} = -d\mathbb{v} - \mathcal{A}\mathbb{v} - \mathbb{v}\mathcal{A}\;, \qquad \boldsymbol{\delta}_{\mathbb{v}}\phi = -\mathbb{v}\,\phi\;, \tag{9.3.6}$$

that preserves $S^E(\phi;\mathcal{A})$ and $S^E_{\text{g.f.}}$ separately. For the former, the transformation above is a verbatim translation of $\Theta \to \mathbb{v}$ for the gauge transformation, except one important aspect of the BRST "grading", as we will see below, so

$$\boldsymbol{\delta}_{\mathbb{v}}S^E(\phi;\mathcal{A}) = 0 \tag{9.3.7}$$

should be immediate.

Let us show this more carefully and then move on to $S^E_{\text{g.f.}}$. Recall from the differential calculus how we assign the notion of "degree" to differential forms such that ordinary p-form and p'-form can exchange the relative position modulo a sign,

$$\Omega^{(p)} \wedge \tilde{\Omega}^{(p')} = (-1)^{p\cdot p'}\,\tilde{\Omega}^{(p')} \wedge \Omega^{(p)} \tag{9.3.8}$$

Something similar happens with Grassmann numbers

$$\mathbb{z}'\mathbb{z} = -\mathbb{z}\,\mathbb{z}' \tag{9.3.9}$$

We encode this rule by assigning the "ghost number" $n_{\rm gh}$ such that a sign $(-1)^{n_{\rm gh}\cdot n'_{\rm gh}}$ accompanies all exchanges. All Grassmann quantities are equipped with odd $n_{\rm gh}$. The assignments for the field content are

$$n_{\rm gh}(\mathcal{A}) = 0\ , \qquad n_{\rm gh}(\mathbb{v}) = 1\ , \qquad n_{\rm gh}(\mathbb{b}) = -1\ , \qquad n_{\rm gh}(\mathbb{B}) = 0 \tag{9.3.10}$$

We see that $S^E_{\rm g.f.}$ carries zero ghost number naturally.

Finally, we define the BRST grading $n_{\rm BRST}$ as

$$n_{\rm BRST} = p + n_{\rm gh} \tag{9.3.11}$$

and use it for the rule of signs when various quantities exchange positions, with both the degree p and the ghost number $n_{\rm gh}$ taken into account. For example, there is a potential sign flip, e.g.,

$$\Omega^{(p)} \wedge \left(\mathbb{z}\,\tilde{\Omega}^{(p')}\right) = (-1)^{(p+0)\cdot(p'+1)}\ \left(\mathbb{z}\,\tilde{\Omega}^{(p')}\right) \wedge \Omega^{(p)} \tag{9.3.12}$$

for a Grassmann number $\mathbb{z}$, say, with $n_{\rm gh} = 1$. Under this, with the most general form of $\mathbb{v} = \sum \mathbb{z}^i\,\Theta_i$, we find

$$-d\mathbb{v} = \sum \mathbb{z}^i\,\partial_\mu\Theta_i\ dx^\mu\ , \qquad -\mathcal{A}\ \mathbb{v} - \mathbb{v}\ \mathcal{A} = \sum \mathbb{z}^i\,[\mathcal{A}_\mu, \Theta_i]\ dx^\mu \tag{9.3.13}$$

which results in the usual gauge transformation rule

$$\boldsymbol{\delta}_{\mathbb{v}}\mathcal{A}_\mu = \partial_\mu\mathbb{v} + \mathcal{A}_\mu\mathbb{v} - \mathbb{v}\mathcal{A}_\mu = D_\mu\mathbb{v}\ , \tag{9.3.14}$$

component-wise.

Furthermore, we also demand that

$$d\,\boldsymbol{\delta}_{\mathbb{v}} + \boldsymbol{\delta}_{\mathbb{v}}\,d = 0 \tag{9.3.15}$$

as part of the definition of $\boldsymbol{\delta}_{\mathbb{v}}$. This is needed for how $\boldsymbol{\delta}_{\mathbb{v}}$ should be also a (Grassmann-valued) gauge-transformation on $\mathcal{F}$, i.e.,

$$\boldsymbol{\delta}_{\mathbb{v}}\mathcal{F} = \mathcal{F}\,\mathbb{v} - \mathbb{v}\,\mathcal{F} \tag{9.3.16}$$

With (9.3.15), the left hand side may be expressed as

$$\begin{aligned} \boldsymbol{\delta}_{\mathbb{v}}\mathcal{F} &= -d(\boldsymbol{\delta}_{\mathbb{v}}\mathcal{A}) + \boldsymbol{\delta}_{\mathbb{v}}\mathcal{A}\wedge\mathcal{A} - \mathcal{A}\wedge\boldsymbol{\delta}_{\mathbb{v}}\mathcal{A} \\ &\to \;\; \mathbb{z}\,\left(d(\boldsymbol{\delta}_{\Theta}\mathcal{A}) + \mathcal{A}\wedge\boldsymbol{\delta}_{\Theta}\mathcal{A} + \boldsymbol{\delta}_{\Theta}\mathcal{A}\wedge\mathcal{A}\right) \;=\; \mathbb{z}\,\boldsymbol{\delta}_{\Theta}\mathcal{F} \end{aligned} \tag{9.3.17}$$

upon $\mathbb{v} = \mathbb{z}\,\Theta$, which is also on par with the usual gauge transformation $\boldsymbol{\delta}_{\Theta}\mathcal{F} = \mathcal{F}\,\Theta - \Theta\,\mathcal{F}$. Thus we arrived at $\boldsymbol{\delta}_{\mathbb{v}}S^E(\phi;\mathcal{A}) = 0$, as promised, following from $\boldsymbol{\delta}_{\Theta}S^E(\phi;\mathcal{A}) = 0$.

All we have done here is to translate bosonic Θ to Grassmann $\mathbb{v}$ without losing any part of the infinitesimal gauge transformations. The claim is that this $\boldsymbol{\delta}_{\mathbb{v}}$ can be extended to $\mathbb{v}$, $\mathbb{b}$, and $\mathbb{B}$ such that

$$\boldsymbol{\delta}_{\mathbb{v}}S^E_{\text{g.f.}} = 0 \tag{9.3.18}$$

as well, so that the total action from the Faddeev-Popov gauge fixing retains the essence of the gauge redundancy despite the gauge-fixing. The difference here is that because $\mathbb{v}$ is now Grassmann, the would-be-infinite volume associated with the gauge redundancy is no longer there. The extended rules for $\boldsymbol{\delta}_{\mathbb{v}}$ are fairly simple,

$$\boldsymbol{\delta}_{\mathbb{v}}\mathbb{v} = -\mathbb{v}^2\ , \qquad \boldsymbol{\delta}_{\mathbb{v}}\mathbb{b} = \mathbb{i}\,\mathbb{B}\ , \qquad \boldsymbol{\delta}_{\mathbb{v}}\mathbb{B} = 0 \tag{9.3.19}$$

Note that $\boldsymbol{\delta}_{\mathbb{v}}$ increases the ghost number n_{gh} by unit just as d increases the degree p by unit.

To show $\boldsymbol{\delta}_{\mathbb{v}}S^E_{\text{g.f.}} = 0$, it is useful to show first the nilpotency,

$$\boldsymbol{\delta}_{\mathbb{v}}\boldsymbol{\delta}_{\mathbb{v}} = 0 \tag{9.3.20}$$

analogous to $dd = 0$. On $\mathbb{b}$ and $\mathbb{B}$, this nilpotency follows from $\boldsymbol{\delta}_{\mathbb{v}}\mathbb{B} = 0$. On $\mathbb{v}$ it follows immediately from

$$\boldsymbol{\delta}_{\mathbb{v}}\boldsymbol{\delta}_{\mathbb{v}}\mathbb{v} = -\boldsymbol{\delta}_{\mathbb{v}}\mathbb{v}^2 = -(\boldsymbol{\delta}_{\mathbb{v}}\mathbb{v})\mathbb{v} + \mathbb{v}(\boldsymbol{\delta}_{\mathbb{v}}\mathbb{v}) = \mathbb{v}^3 - \mathbb{v}^3 = 0 \tag{9.3.21}$$

while, on $\mathcal{A}_\mu$, it takes a bit more steps,

$$\begin{aligned}\boldsymbol{\delta}_{\mathbb{v}}\boldsymbol{\delta}_{\mathbb{v}}\mathcal{A}_\mu \;=\; \boldsymbol{\delta}_{\mathbb{v}}\left(D_\mu \mathbb{v}\right) \;&=\; \left(\boldsymbol{\delta}_{\mathbb{v}}\mathcal{A}_\mu\right)\mathbb{v} + \mathbb{v}\left(\boldsymbol{\delta}_{\mathbb{v}}\mathcal{A}_\mu\right) + D_\mu\left(-\mathbb{v}^2\right) \\ &=\; \left(D_\mu \mathbb{v}\right)\mathbb{v} + \mathbb{v}\left(D_\mu \mathbb{v}\right) + D_\mu\left(-\mathbb{v}^2\right) \;=\; 0 \qquad (9.3.22)\end{aligned}$$

For the latter, it is important to recall that, although $\mathcal{A}$ as 1-form is BRST-odd, the components $\mathcal{A}_\mu$ are BRST-even.

Combining all, one can go one more step and find

$$(d+\boldsymbol{\delta}_{\mathbb{v}})^2 \;=\; d\,d + (d\,\boldsymbol{\delta}_{\mathbb{v}} + \boldsymbol{\delta}_{\mathbb{v}}\,d) + \boldsymbol{\delta}_{\mathbb{v}}\boldsymbol{\delta}_{\mathbb{v}} \;=\; 0 \qquad (9.3.23)$$

from each of three pieces in the middle vanishing identically, such that $d+\boldsymbol{\delta}_{\mathbb{v}}$ is also a nilpotent operator. Together with the other nilpotencies, $dd = 0 = \boldsymbol{\delta}_{\mathbb{v}}\boldsymbol{\delta}_{\mathbb{v}}$, we find formal similarities among d, $\boldsymbol{\delta}_{\mathbb{v}}$, and $d+\boldsymbol{\delta}_{\mathbb{v}}$. Each of these nilpotent operators raises p, $n_{\rm gh}$, and $n_{\rm BRST}$ by unit, respectively, and will be of immense use down the road.

The final step for showing how $\boldsymbol{\delta}_{\mathbb{v}}$ is the new symmetry is to notice

$$S^E_{\rm g.f.} = \boldsymbol{\delta}_{\mathbb{v}}\mathbb{P}\;, \qquad \mathbb{P} \;=\; \int \mathrm{tr}\left[\mathbb{b}\left(\frac{\mathbb{i}}{2\zeta^2}\mathbb{B} + \mathbb{K}(\mathcal{A})\right)\right] \qquad (9.3.24)$$

$\mathbb{P}$ is a Grassmann-valued functional with $n_{\rm gh} = -1$. $\boldsymbol{\delta}_{\mathbb{v}}$ acting on $\mathbb{b}$ produces

$$\int \mathrm{tr}\left[\mathbb{i}\,\mathbb{B}\left(\frac{\mathbb{i}}{2\zeta^2}\mathbb{B} + \mathbb{K}(\mathcal{A})\right)\right] \;=\; \int\left[-\frac{1}{2\zeta^2}\mathrm{tr}\,\mathbb{B}^2 + \mathbb{i}\,\mathrm{tr}\left(\mathbb{B}\,\mathbb{K}(\mathcal{A})\right)\right] \qquad (9.3.25)$$

while on the latter factor gives

$$\int \mathrm{tr}\left[-\mathbb{b}\;\boldsymbol{\delta}_{\mathbb{v}}\mathbb{K}(\mathcal{A})\right] = \int -\mathrm{tr}\left[\mathbb{b}\,\frac{\delta\mathbb{K}}{\delta\Theta}\,\mathbb{v}\right] \qquad (9.3.26)$$

respectively producing the two blocks present in $S^E_{\rm g.f.}$. The former generates the gauge-fixing term upon $\mathbb{B}$ integration while the latter reduces to the ghost determinant. Then, we find

$$\boldsymbol{\delta}_{\mathbb{v}}\left[S^E(\phi;\mathcal{A}) + S^E_{\rm g.f.}\right] \;=\; 0 + \boldsymbol{\delta}_{\mathbb{v}}\boldsymbol{\delta}_{\mathbb{v}}\mathbb{P} \;=\; 0 \qquad (9.3.27)$$

from the nilpotency $\boldsymbol{\delta}_{\mathbb{v}}^2 = 0$, and arrive at the claimed BRST invariance of the gauge-fixed action.

9.4 BRST Quantization

This emergent BRST symmetry is useful both technically and conceptually. The gauge-fixing procedure is unavoidable in gauge theories with the infinite gauge-volume, yet the notion of gauge invariance must survive the quantization. The gauge invariance means that the physical states should be annihilated by the Gauss constraint, at quantum level; however, if the path integral lose the gauge redundancy due to the gauge-fixing procedure, it would not have been clear why and how one could impose such an constraint. The BRST symmetry inherits the essence of the gauge symmetry, even after the gauge-fixing is performed, so we can honestly ask about the consequences of and sometimes the failure of the gauge redundancies in the fully quantum language.

Instead, what we have found above implies that the gauge invariance is replaced by the BRST invariance,

$$\mathbf{Q}_{\mathrm{BRST}}\,|\mathrm{phys}\rangle = 0 \tag{9.4.1}$$

where $\mathbf{Q}_{\mathrm{BRST}}$ is the quantum BRST charge operator generated by $\boldsymbol{\delta}_{\mathrm{v}}$. In other words, after the gauge-fixing, $\mathbf{Q}_{\mathrm{BRST}}$ replaces $\mathbf{G}_{\mathrm{gauge}}$ that we used prior to the gauge-fixing description. Physical quantum states should thus belong to

$$\mathrm{kernel}(\mathbf{Q}_{\mathrm{BRST}}) \tag{9.4.2}$$

On the other hand, a physical state should be considered unchanged under a shift

$$|\mathrm{phys}\rangle \quad \rightarrow \quad |\mathrm{phys}\rangle + \mathbf{Q}_{\mathrm{BRST}}\,|\cdots\rangle \tag{9.4.3}$$

since the shift represents a gauge transformation. States mutually related by shifts induced by $\mathbf{Q}_{\mathrm{BRST}}$ acting on some other state should be considered one and the same physical state.

The nilpotency of $\boldsymbol{\delta}_{\mathrm{v}}$ translates to the nilpotency

$$(\mathbf{Q}_{\mathrm{BRST}})^2 = 0 \tag{9.4.4}$$

at quantum level. As such, the image of $\mathbf{Q}_{\mathrm{BRST}}$, representing gauge transformation, is a subset of the kernel of $\mathbf{Q}_{\mathrm{BRST}}$. All these mean that physical states belong to the quotient,

$$\frac{\mathrm{kernel}(\mathbf{Q}_{\mathrm{BRST}})}{\mathrm{image}(\mathbf{Q}_{\mathrm{BRST}})} \tag{9.4.5}$$

constructed starting from the Fock space of $\mathcal{A}$, $\mathbb{v}$, $\mathbb{b}$, and $\mathbb{B}$. Since the field content now includes ghosts as well, the states are graded by n_{gh}, with $\mathbf{Q}_{\mathrm{BRST}}$ increasing the ghost number by unit. This decomposition

$$\frac{\mathrm{kernel}(\mathbf{Q}_{\mathrm{BRST}})}{\mathrm{image}(\mathbf{Q}_{\mathrm{BRST}})} = \oplus_{n_{\mathrm{gh}}} \frac{\mathrm{kernel}(\mathbf{Q}_{\mathrm{BRST}})}{\mathrm{image}(\mathbf{Q}_{\mathrm{BRST}})}\bigg|_{n_{\mathrm{gh}}} \tag{9.4.6}$$

then defines

$$\oplus_{n_{\mathrm{gh}}} \mathcal{H}^{(n_{\mathrm{gh}})}_{\mathrm{BRST}} \tag{9.4.7}$$

a cohomology associated with the complex of $\mathbf{Q}_{\mathrm{BRST}}$. The true statement is that the physical Hilbert space is given by the zero-th such that

$$\mathcal{H}^{(0)}_{\mathrm{BRST}} \;=\; \{|\mathrm{phys}\rangle\} \;. \tag{9.4.8}$$

This offers a manifestly Lorentz-covariant quantization scheme, known as the BRST quantization.

This BRST quantization has proven to be particularly convenient and practical for lower dimensional theories, with the most prominent example being the worldsheet dynamics of fundamental strings. For our purpose in the remainder of this volume, the observation that these BRST symmetries and the resulting BRST quantization sit behind the gauge symmetry, despite the inevitable gauge-fixing, suffices as we wish to explore the consequences and the unexpected failure of the gauge redundancies in the fully quantized framework.