Relativistic Quantum Mechanics

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Outline

1. Introduction

Why Relativistic Quantum Mechanics?

- Non-relativistic quantum mechanics does not respect special relativity.
- The Schrödinger equation is built from the non-relativistic energy relation:

$$E = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}).$$

- To describe high–energy particles, we need a framework that combines:
  1. Quantum mechanics
  2. Lorentz invariance
- This naturally points toward **quantum field theory**, but the first step is to study **relativistic single-particle equations**.

2. Natural Units

Definition

- **Natural units** set fundamental constants to 1:

$$\hbar = 1, \qquad c = 1 \quad \text{(often also } k_B = 1\text{)}.$$

- Consequences:
  - Energy, momentum, and mass share the same unit (GeV is standard).
  - Length and time carry inverse energy units:

$$[t] = [\ell] = \frac{1}{\text{mass}}$$

  - Derivatives carry energy dimension:

$$[\partial_\mu] = \text{mass}$$

.

Quick Conversions and Heuristics

- Useful constants:

$$\hbar c \approx 0.1973269804 \text{ GeV fm}.$$

- Core conversions:

$$1 \text{ GeV}^{-1} \approx 0.1973 \text{ fm} \approx 6.582 \times 10^{-25} \text{ s}.$$

- Cross sections:

$$[\sigma] = \mathsf{mass}^{-2} \Rightarrow 1 \text{ GeV}^{-2} \approx 0.3894 \text{ mb}.$$

Action is Dimensionless

- In the path integral,

$$\mathcal{Z} = \int \mathcal{D}\phi \ \exp\left\{ \frac{i}{\hbar} S[\phi] \right\}.$$

- The exponent must be dimensionless. Therefore,

$$[S] = [\hbar].$$

- In natural units $\hbar = 1$, hence S is **dimensionless**.

3. Spacetime Four-Vectors & Minkowski Metric

Spacetime Coordinates

- In relativity, space and time combine into a single **contravariant four-vector**:

$$x^\mu = (t,\, \mathbf{x}) = (t,\, x,\, y,\, z).$$

- We use natural units with $c = 1$.
- Index convention:

$$\mu = 0, 1, 2, 3 \quad \Rightarrow \quad x^0 = t,\ x^1 = x,\ x^2 = y,\ x^3 = z.$$

- Contravariant vector = components with an *upper* index $(\,^\mu)$.
- Later, covariant vectors will be written with a lower index $(\,_\mu)$.

Minkowski Metric (Definition)

- In special relativity, spacetime is described by the **Minkowski metric**:

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1).$$

- This defines the invariant spacetime interval:

$$s^2 = g_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2.$$

- Using the metric, we can **lower indices**:

$$x_\mu = g_{\mu\nu} x^\nu.$$

- For the spacetime coordinate:

$$x^\mu = (t,\ x,\ y,\ z) \quad \longrightarrow \quad x_\mu = (t,\ -x,\ -y,\ -z).$$

- The key: the time component keeps its sign, while spatial components change sign.

Four-Momentum and Invariant Mass

- Define the **contravariant four-momentum**:

$$p^\mu = (E,\, p_x,\, p_y,\, p_z).$$

- Lowering the index with the Minkowski metric:

$$p_\mu = g_{\mu\nu} p^\nu = (E,\ -p_x,\ -p_y,\ -p_z).$$

- Lorentz-invariant scalar $\Rightarrow$ **rest mass squared**

$$p_\mu p^\mu = g_{\mu\nu} p^\mu p^\nu = E^2 - \vec{p}^{\,2} = m^2.$$

- Thus, the relativistic energy–momentum relation:

$$E^2 = \vec{p}^{\,2} + m^2.$$

Covariant and Contravariant Derivatives

- Define the derivative operator:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}.$$

- Explicitly:

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

- Raising the index with the Minkowski metric $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$:

$$\partial^\mu = g^{\mu\nu}\,\partial_\nu = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right).$$

- The **d'Alembertian operator** (wave operator) is defined as

$$\Box \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2.$$

4. Lorentz Transformations

Definition of Lorentz Transformations

- A **Lorentz transformation** is any linear map

$$x^\mu \;\longrightarrow\; x'^\mu = \sum_\nu \Lambda^\mu{}_\nu\, x^\nu = \Lambda^\mu{}_\nu\, x^\nu$$

that preserves the Minkowski metric:

$$g_{\rho\sigma} = g_{\mu\nu}\,\Lambda^\mu{}_\rho\,\Lambda^\nu{}_\sigma.$$

- Invariance of the interval:

$$\begin{aligned} s'^2 &\equiv g_{\alpha\beta}\, x'^\alpha x'^\beta \\ &= g_{\alpha\beta}\,(\Lambda^\alpha{}_\mu x^\mu)(\Lambda^\beta{}_\nu x^\nu) \\ &= \left(g_{\alpha\beta}\Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu\right) x^\mu x^\nu = g_{\mu\nu} x^\mu x^\nu = s^2. \end{aligned}$$

- The Lorentz group therefore contains:
  - **Rotations** in 3D space (leave t unchanged).
  - **Boosts** (mix t with a spatial direction).

Example: Lorentz Boost in the z-Direction

- Consider a boost along the z axis with velocity v.
- Define

$$\gamma = \frac{1}{\sqrt{1-v^2}}, \qquad \beta = v.$$

- The Lorentz transformation matrix is

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}.$$

- Transformation of coordinates:

$$\begin{aligned} t' &= \gamma(t - \beta z), \\ x' &= x, \quad y' = y, \\ z' &= \gamma(z - \beta t). \end{aligned}$$

5. Conserved Current and Charge

Conservation Law

- A **conserved quantity** is described by a continuity equation:

$$\partial_t j^0 + \nabla \cdot \mathbf{j} = 0.$$

- In relativistic notation, we combine density and current into a four-vector:

$$j^\mu(x) = (j^0, \mathbf{j}), \qquad \partial_\mu j^\mu = 0.$$

Conserved Charge

- Define the **charge** as the spatial integral of the time component:

$$Q(t) = \int d^3x\, j^0(t, \mathbf{x}).$$

- Take the time derivative:

$$\frac{dQ}{dt} = \int d^3x\, \partial_t j^0(t, \mathbf{x}).$$

- Using $\partial_\mu j^\mu = 0$:

$$\partial_t j^0 = -\nabla \cdot \mathbf{j}.$$

- Thus,

$$\frac{dQ}{dt} = -\int d^3x\, \nabla \cdot \mathbf{j}.$$

Divergence Theorem and Conservation

- Apply Gauss's (divergence) theorem:

$$\int d^3x \, \nabla \cdot \mathbf{j} = \oint_{\partial V} d\mathbf{S} \cdot \mathbf{j}.$$

- If the current $\mathbf{j}$ vanishes sufficiently fast at spatial infinity:

$$\oint_{\infty} d\mathbf{S} \cdot \mathbf{j} = 0.$$

- Therefore:

$$\frac{dQ}{dt} = 0.$$

$\Rightarrow$ The charge Q is a conserved quantity.

Probability Current in Quantum Mechanics

- Start from the Schrödinger equation:

$$i\frac{\partial \psi}{\partial t} = \left(-\frac{\nabla^2}{2m} + V(\mathbf{x})\right)\psi.$$

- Define probability density:

$$\rho(t, \mathbf{x}) = |\psi(t, \mathbf{x})|^2.$$

- Derive the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

$\Rightarrow$ Probability current:

$$\mathbf{j} = \frac{1}{2mi}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right).$$

Interpretation: Probability Conservation

- The conserved "charge" in quantum mechanics is the total probability:

$$Q = \int d^3x \, \rho(t, \mathbf{x}) = \int d^3x \, |\psi|^2.$$

- Conservation law:

$$\frac{dQ}{dt} = 0.$$

- Physical meaning:
  - Probability never disappears or appears spontaneously.
  - A particle is always found *somewhere* in space.
- This is the non-relativistic counterpart of current conservation.

6. From $E = \frac{\mathbf{p}^2}{2m} + V$ to the Schrödinger equation

Four-Momentum as an Operator

- In quantum theory, promote the four-momentum to a differential operator:

$$p^\mu = (E, \mathbf{p}) = i\,\partial^\mu.$$

- With the Minkowski metric $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$:

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t},\ -\nabla\right).$$

- Therefore:

$$p^\mu = \big(i\,\partial_t,\ -i\nabla\big).$$

- Components:

$$p^0 = i\,\partial_t \quad \text{(energy operator)}, \qquad \mathbf{p} = -i\nabla \quad \text{(momentum operator)}.$$

Schrödinger Equation from $E = \frac{\mathbf{p}^2}{2m} + V$

- Start from the non-relativistic energy relation:

$$E = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}).$$

- Substitute the operators:

$$E \to i\,\partial_t, \qquad \mathbf{p} \to -i\nabla.$$

- Acting on a wavefunction $\psi(t, \mathbf{x})$:

$$i\,\partial_t \psi(t, \mathbf{x}) = \left(-\frac{\nabla^2}{2m} + V(\mathbf{x})\right)\psi(t, \mathbf{x}).$$

- This is the time-dependent Schrödinger equation.

7. Klein–Gordon Equation

Motivation: Relativistic Wave Equation

- Start from the **relativistic energy–momentum relation**:

$$E^2 = \mathbf{p}^2 + m^2.$$

- Historically, the Klein–Gordon (KG) equation was proposed **independently in 1926** by **Oskar Klein** and **Walter Gordon** as the relativistic analogue of the Schrödinger equation.
- Goal: promote E and $\mathbf{p}$ to operators and derive a Lorentz-invariant wave equation for a scalar field $\phi(x)$.

Derivation from $E^2 = \mathbf{p}^2 + m^2$

- Use the four-momentum operator $p^\mu = i\,\partial^\mu$, with

$$g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1) \Rightarrow \partial^\mu = (\partial_t, -\nabla).$$

$\Rightarrow$

$$E \to i\partial_t, \quad \mathbf{p} \to -i\nabla.$$

- Act on a scalar field $\phi(t, \mathbf{x})$:

$$(i\partial_t)^2\phi \;=\; \big[(-i\nabla)^2 + m^2\big]\phi \;\Longrightarrow\; -\partial_t^2\phi \;=\; \big(-\nabla^2 + m^2\big)\phi.$$

- Rearranging:

$$\left(\partial_t^2 - \nabla^2 + m^2\right)\phi(t, \mathbf{x}) = 0 \Rightarrow \boxed{\left(\partial_\mu\partial^\mu + m^2\right)\phi = 0}$$

Covariant Form and Solutions

- Let's interpret the KG equation

$$(\partial_\mu \partial^\mu + m^2)\,\phi = 0$$

- Plane-wave ansatz $\phi(x) = e^{-ip\cdot x}$ with

$$p \cdot x = p_\mu x^\mu = Et - \mathbf{p}\cdot\mathbf{x}$$

- The KG equation gives

$$(-p_\mu p^\mu + m^2)\,\phi = 0 \implies p_\mu p^\mu = m^2 \implies E^2 = \mathbf{p}^2 + m^2.$$

- OOPS! Solutions come with $E = \pm\sqrt{\mathbf{p}^2 + m^2}$. (Negative-frequency solutions are physical; in QFT they correspond to antiparticles.)

8. Klein–Gordon Current and Continuity Equation

Step 1: Klein–Gordon Equation and Its Conjugate

- Klein–Gordon equation for a (complex) scalar field:

$$(\partial_\mu \partial^\mu + m^2)\, \phi(x) = 0.$$

- Its complex conjugate:

$$(\partial_\mu \partial^\mu + m^2)\, \phi^*(x) = 0.$$

- Multiply the first by ϕ^* and the second by ϕ:

$$\phi^*\, \partial_\mu \partial^\mu \phi + m^2 \phi^* \phi = 0, \qquad \phi\, \partial_\mu \partial^\mu \phi^* + m^2 \phi \phi^* = 0.$$

- Subtract:

$$\phi^*\, \partial_\mu \partial^\mu \phi \;-\; \phi\, \partial_\mu \partial^\mu \phi^* \;=\; 0 \quad (m^2 \text{ terms cancel}).$$

Step 2: Turn It Into a Divergence

- Product rule identity:

$$\partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) = \phi^*\,\partial_\mu\partial^\mu\phi \;-\; \phi\,\partial_\mu\partial^\mu\phi^*.$$

- We get the **continuity equation**:

$$\partial_\mu j^\mu = 0, \qquad \boxed{j^\mu \;\equiv\; i\big(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\big)}.$$

- Components (with $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ so $\partial^\mu = (\partial_t, -\nabla)$):

$$j^0 = i\big(\phi^*\partial_t\phi - \phi\partial_t\phi^*\big), \qquad \mathbf{j} = -\,i\big(\phi^*\nabla\phi - \phi\nabla\phi^*\big).$$

Step 3: Conserved Charge

- Define the charge (time component integrated over space):

$$Q(t) \equiv \int d^3x\, j^0(t, \mathbf{x}).$$

- Since $\partial_\mu j^\mu = 0$, we have

$$\frac{dQ}{dt} = 0 \quad \Rightarrow \quad Q \text{ is conserved.}$$

- $j^0(t, \mathbf{x})$ is the probability density?

KG Current: j^0 is not positive definite

- Take a plane-wave solution:

$$\phi(x) = A\, e^{-iEt + i\mathbf{p}\cdot\mathbf{x}}, \qquad E = \pm\sqrt{\mathbf{p}^2 + m^2}.$$

- With $\partial_t \phi = -iE\phi$, $\nabla\phi = i\mathbf{p}\phi$ and $j^\mu = i(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*)$,

$$j^0 = i\big(\phi^* \partial_t \phi - \phi\, \partial_t \phi^*\big) = 2E\,|A|^2,$$

$$\mathbf{j} = -\,i\big(\phi^* \nabla\phi - \phi\, \nabla\phi^*\big) = 2\mathbf{p}\,|A|^2.$$

- Since E has both signs in relativistic theory, $j^0 = 2E|A|^2$ can be negative $\Rightarrow$ j^0 **is not a probability density.**

Why j^0 is not positive: second order in time

- **Schrödinger (first order in time):**

$$i\,\partial_t\psi = H\psi \;\Rightarrow\; \partial_t|\psi|^2 + \nabla\cdot\mathbf{j} = 0, \quad \rho_S = |\psi|^2 \geq 0.$$

- **Klein–Gordon (second order in time):**

$$(\partial_t^2 - \nabla^2 + m^2)\,\phi = 0 \;\Rightarrow\; \partial_\mu j^\mu = 0, \quad j^0 = i\big(\phi^*\partial_t\phi - \partial_t\phi^*\,\phi\big).$$

- Because KG is *second order*, the conserved density necessarily involves $\partial_t\phi$, so it is not $|\phi|^2$ and need not be positive.
- Hence the non-positivity of j^0 is a structural consequence of the second-order time evolution, *independent* of any particular solution.

Interpretation & Nonrelativistic Limit

- **Interpretation:**
  - For a complex scalar, j^μ is the conserved **U(1) charge current**.
  - For a real scalar ($\phi = \phi^*$), $j^\mu \equiv 0$ (no global phase symmetry).
- **NR limit (recover Schrödinger):**

$$\phi(x) = \frac{e^{-imt}}{\sqrt{2m}}\,\Psi(t,\mathbf{x}) \quad \Longrightarrow \quad j^0 \approx |\Psi|^2, \qquad \mathbf{j} \approx \frac{1}{m}\,\mathrm{Im}\big(\Psi^*\nabla\Psi\big).$$

- Thus, in the nonrelativistic regime the KG current reduces to the familiar **probability density and current**.

9. Dirac: Motivation & Linearization

Historical Prelude (1926–1933)

- **1926:** Klein & Gordon propose the first relativistic wave eq. for a scalar. Good Lorentz covariance, but j^0 not positive.
- **Dirac's aim (1928):** Find a *first-order in time* & *first-order in space* relativistic equation with a positive density
  $\Rightarrow$ linearize $E^2 = \mathbf{p}^2 + m^2$.
- **Negative energies:** Dirac's hole theory ($\sim$1930)
  $\Rightarrow$ prediction of antiparticles.
- **1932:** Anderson discovers the *positron*, confirming Dirac's picture.
- **Modern view:** Negative-frequency solutions $\Rightarrow$ antiparticles in QFT (no hole sea needed).

Refs: P. A. M. Dirac, Proc. R. Soc. A **117** (1928) 610; Dirac, Proc. R. Soc. A 126 (1930) 360; C. D. Anderson, Science **76** (1932) 238.

Motivation: First Order in Time

- The Klein–Gordon (KG) equation is second order in time:

$$(\partial_t^2 - \nabla^2 + m^2)\,\phi = 0,$$

  whose conserved density j^0 is *not* positive definite.
- **Goal:** Find a *relativistic* wave equation that is **first order in time derivative**, so we can have a positive-definite density and a probability interpretation, like Schrödinger's theory.
- **Strategy:** *Linearize* the relativistic energy:

$$E^2 = \mathbf{p}^2 + m^2 \;\Rightarrow\; E = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m,$$

  with suitable matrices $\boldsymbol{\alpha}, \beta$.

Linearization Ansatz: $E = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m$

- Postulate a Hamiltonian linear in $\mathbf{p}$:

$$H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m, \qquad \boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3).$$

- Require that squaring reproduces the dispersion:

$$H^2 = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)^2 = \mathbf{p}^2 + m^2\,\mathbf{1}.$$

Squaring the Linear Hamiltonian: Expansion

- Start with

$$H = \alpha^i p_i + \beta m \qquad (i = 1, 2, 3),$$

and expand:

$$H^2 = (\alpha^i p_i)(\alpha^j p_j) + \alpha^i p_i \, \beta m + \beta m \, \alpha^j p_j + \beta^2 m^2.$$

- Since

$$(\alpha^i p_i)(\alpha^j p_j) = \alpha^i \alpha^j \, p_i p_j, \quad \alpha^i p_i \, \beta m + \beta m \, \alpha^j p_j = m p_i (\alpha^i \beta + \beta \alpha^i),$$

we have

$$H^2 = \underbrace{\alpha^i \alpha^j \, p_i p_j}_{\text{quadratic in } \mathbf{p}} + \underbrace{m p_i (\alpha^i \beta + \beta \alpha^i)}_{\text{linear in } \mathbf{p}} + \beta^2 m^2.$$

Why the linear term must vanish (and the mass term)

- To match $H^2 = \mathbf{p}^2 + m^2\mathbf{1}$, there must be *no* term linear in p_i:

$$\begin{aligned} m\,p_i(\alpha^i\beta + \beta\alpha^i) &= 0 \;\text{ for all } \mathbf{p} \\ \Longrightarrow &\quad \boxed{\{\alpha^i, \beta\} = 0, \quad \text{for each } i} \end{aligned}$$

  where the anti-commutator is $\{A, B\} = AB + BA$.
- The mass term must equal $m^2\mathbf{1}$:

$$\beta^2 m^2 = m^2\mathbf{1} \;\Longrightarrow\; \boxed{\beta^2 = \mathbf{1}}.$$

For the p^2 piece: using commutator & anticommutator

- Quadratic part:
$$(\boldsymbol{\alpha}\cdot\mathbf{p})^2 = \alpha^i\alpha^j\, p_i p_j.$$
- Decompose a product of matrices into symmetric/antisymmetric parts:
$$\alpha^i\alpha^j = \tfrac{1}{2}\{\alpha^i,\alpha^j\} + \tfrac{1}{2}[\alpha^i,\alpha^j].$$
- Since $p_i p_j = p_j p_i$ is **symmetric** in (i,j), the antisymmetric part drops:
$$\alpha^i\alpha^j\, p_i p_j = \tfrac{1}{2}\{\alpha^i,\alpha^j\}\, p_i p_j.$$
- To reproduce $\mathbf{p}^2\mathbf{1} = \delta^{ij} p_i p_j\,\mathbf{1}$ for *all* $\mathbf{p}$:

$$\tfrac{1}{2}\{\alpha^i,\alpha^j\} = \delta^{ij}\mathbf{1} \iff \boxed{\{\alpha^i,\alpha^j\} = 2\,\delta^{ij}\,\mathbf{1}}.$$

Sufficiency: Algebra ⇒ Dispersion

- Assume $\{\alpha^i, \alpha^j\} = 2\delta^{ij}\mathbf{1},\ \{\alpha^i, \beta\} = 0,\ \beta^2 = \mathbf{1}$.
  - The quadratic term becomes

$$(\boldsymbol{\alpha}\cdot\mathbf{p})^2 = \alpha^i\alpha^j\, p_i p_j = \tfrac{1}{2}\{\alpha^i, \alpha^j\}\, p_i p_j = \delta^{ij} p_i p_j\, \mathbf{1} = \mathbf{p}^2\, \mathbf{1}.$$

  - The cross term vanishes:

$$\alpha^i p_i\, \beta m + \beta m\, \alpha^j p_j = m p_i \{\alpha^i, \beta\} = 0.$$

  - And the mass term: $\beta^2 m^2 = m^2\mathbf{1}$.
- Therefore

$$\boxed{H^2 = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)^2 = \mathbf{p}^2 + m^2\ \mathbf{1}}.$$

10. Dirac Matrices & Representations

Why 4×4 Matrices?

- We need four independent, mutually anticommuting Hermitian matrices: $\alpha^1, \alpha^2, \alpha^3, \beta$, with $(\alpha^i)^2 = \beta^2 = 1$.
- 2×2 Pauli matrices provide at most three anticommuting matrices.
- Hence the **minimal representation** is 4×4
- Dirac spinors have **four components**.

Explicit Matrices (Dirac / Standard Representation)

- Choose

$$\beta = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}, \qquad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad (i = 1, 2, 3).$$

- Quick checks (using $\sigma^i \sigma^j = \delta^{ij} \mathbf{1}_2 + i\epsilon^{ijk} \sigma^k$):

$$(\alpha^i)^2 = \begin{pmatrix} \sigma^i \sigma^i & 0 \\ 0 & \sigma^i \sigma^i \end{pmatrix} = \mathbf{1}_4, \qquad \beta^2 = \mathbf{1}_4,$$

$$\alpha^i \alpha^j + \alpha^j \alpha^i = \begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix} = 2\delta^{ij} \mathbf{1}_4,$$

$$\alpha^i \beta + \beta \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} = 0.$$

From Hamiltonian to Covariant Dirac Equation

- Covariant notation (from the Hamiltonian form):

$$\gamma^0 \equiv \beta, \qquad \gamma^i \equiv \beta\,\alpha^i \quad \Rightarrow \quad \alpha^i = \gamma^0\gamma^i,\ \beta = \gamma^0.$$

- **Start with the Hamiltonian form**

$$i\,\partial_t\psi \;=\; \Big(-\,i\,\boldsymbol{\alpha}\!\cdot\!\nabla + \beta\,m\Big)\psi.$$

- Multiply on the left by $\gamma^0 = \beta$ and use $\alpha^i = \gamma^0\gamma^i$:

$$\gamma^0(i\partial_t)\psi = \Big(-\,i\,\gamma^0\alpha^i\partial_i + \gamma^0\beta\,m\Big)\psi = \Big(-\,i\,\gamma^i\partial_i + m\Big)\psi.$$

- Bring all terms to the left:

$$(i\,\gamma^0\partial_t + i\,\gamma^i\partial_i - m)\psi = 0 \quad \Rightarrow \quad \boxed{(i\,\gamma^\mu\partial_\mu - m)\,\psi(x) = 0}.$$

Dirac Matrices: Representations & Equivalence

- Covariant notation (from the Hamiltonian form):

$$\gamma^0 \equiv \beta, \qquad \gamma^i \equiv \beta\,\alpha^i.$$

- Clifford algebra (metric $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$):

$$\{\gamma^\mu, \gamma^\nu\} = 2\,g^{\mu\nu}\,\mathbf{1}.$$

- **Not unique:** any set $\{\gamma^\mu\}$ obeying the Clifford algebra is a valid *representation*.

Two Common Representations (Examples)

- **Dirac (standard) representation**

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

(Here γ^5 is *off-diagonal*.)

- **Chiral (Weyl) representation**

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

(Here γ^5 is *diagonal*.)

11. Spin & Lorentz Transformations of a Dirac spinor

Dirac Equation: Degrees of Freedom & Spin

- Relativistic wave equation with 4×4 matrices:

$$(i\,\gamma^\mu \partial_\mu - m)\,\psi(x) = 0.$$

- The energy and three-momentum must satisfy

$$E^2 = \mathbf{p}^2 + m^2$$

- **Four-component spinor** ψ yields, at fixed $\mathbf{p}$, *four* independent solutions:
  - two **positive-energy** solutions $E = +\sqrt{\mathbf{p}^2 + m^2}$ with spin $s = \pm\frac{1}{2}$;
  - two **negative-energy** solutions $E = -\sqrt{\mathbf{p}^2 + m^2}$ with spin $s = \pm\frac{1}{2}$.
- **Spin is automatic:** the γ^μ contain Pauli matrices; rotations act via

$$\Sigma^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \quad \Rightarrow \quad s = \pm\frac{1}{2}.$$

How Spinors Change under Lorentz Transformations

- Under a Lorentz transformation (a rotation or a boost), the spinor changes by a 4×4 matrix:

$$\psi \;\longrightarrow\; \psi' = S(\Lambda)\,\psi$$

(think: a matrix that mixes the 4 components of ψ.)

- Finite transformation generated by $\Sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$:

$$\boxed{S(\Lambda) \;=\; \exp\!\left(-\frac{i}{4}\,\omega_{\mu\nu}\,\Sigma^{\mu\nu}\right), \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}}$$

which leads to

$$\text{rotations: } J^i \equiv \frac{1}{2}\,\epsilon^{ijk}\,\Sigma^{jk}, \qquad \text{boosts: } K^i \equiv \Sigma^{0i}.$$

Explicit Σ^i in the Dirac Representation

- The **spin operator** is the rotation generator J^i.
- Using the γ matrices in the Dirac representation:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

one finds

$$\Sigma^{ij} = \frac{i}{2}[\gamma^i, \gamma^j] = \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.$$

- Therefore the rotation generators are

$$\boxed{J^i = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}} \quad \Rightarrow \quad \text{eigenvalues } s = \pm\tfrac{1}{2}.$$

The Triumph of Dirac: Spin (vs. Schrödinger)

- **Schrödinger (spinless):**

$$i\,\partial_t \psi = \left(-\frac{\nabla^2}{2m} + V\right)\psi,$$

where ψ is a *scalar*. Spin is *not* built in.

  - To include spin in NR QM, we *add it by hand*: $\psi \to \Psi = \begin{pmatrix}\psi_\uparrow \\ \psi_\downarrow\end{pmatrix}$

- **Dirac (spin built in):**

$$(i\,\gamma^\mu \partial_\mu - m)\,\psi = 0, \qquad \psi: \text{ four-component spinor.}$$

As shown earlier, the rotation generators J^i for the Dirac spinor have eigenvalues $\pm\frac{1}{2}$. **Intrinsic spin-$\frac{1}{2}$ appears automatically**.

Boost vs. Rotation Generators: Hermiticity & Unitarity

- **Rotation generators** $J^i \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^{jk}$:

$$J^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \quad \Rightarrow \quad (J^i)^\dagger = J^i \ \text{ (Hermitian).}$$

- **Boost generators** $K^i \equiv \Sigma^{0i}$:

$$K^i = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad \Rightarrow \quad (K^i)^\dagger = -K^i \ \text{ (anti-Hermitian).}$$

- Finite Lorentz transformations on spinors, $S(\Lambda) = e^{-\frac{i}{4}\,\omega_{\mu\nu}\,\Sigma^{\mu\nu}}$
  - **Pure rotation** ($\omega_{ij} = \theta_{ij}$): Hermitian generator $\Rightarrow S$ **unitary**.
  - **Pure boost** ($\omega_{0i} = \varphi_i$ rapidity): anti-Hermitian generator $\Rightarrow S$ is **not unitary** (Lorentz group is non-compact).

12. Dirac Adjoint & Conserved Current

Lorentz–covariant density via the Dirac adjoint $\bar{\psi}$

- $\psi^\dagger\psi$ is not Lorentz invariant. Why?
- Under a Lorentz transformation Λ, a Dirac spinor changes

$$\psi \longrightarrow \psi' = S(\Lambda)\,\psi.$$

- Then

$$\psi'^\dagger\psi' = \psi^\dagger S^\dagger S\,\psi \quad \text{equals } \psi^\dagger\psi \text{ only if } S^\dagger S = \mathbf{1}.$$

- For **pure rotations**, S *is* unitary $(S^\dagger S = \mathbf{1})$.
- For **boosts**, S is *not* unitary, so $\psi'^\dagger\psi' \neq \psi^\dagger\psi$.
- What can we do?

Spinor Boost Along z from the Explicit Generator

- In the Dirac representation,

$$K^3 \equiv \Sigma^{03} = i \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \equiv i\, M\,, \qquad (M)^\dagger = M \quad \text{(Hermitian).}$$

- A boost of rapidity η along $+z$ acts as $\psi' \;=\; S_z(\eta)\,\psi$,

$$S_z(\eta) \;=\; \exp\!\left(\frac{\eta}{2}\,M\right) \;=\; \cosh\frac{\eta}{2}\,\mathbf{1} \;+\; M\,\sinh\frac{\eta}{2}.$$

- Since M is Hermitian, $S_z^\dagger(\eta) = S_z(\eta)$ and

$$S_z^\dagger S_z = \exp(\eta M) \neq \mathbf{1} \text{ for } \eta \neq 0$$

If $S^{-1} \neq S^\dagger$, what is S^{-1}?

- For a z–boost with $M = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}$ and $S_z(\eta) = \exp\left(\frac{\eta}{2}M\right)$:

$$\gamma^0 S_z^\dagger \gamma^0 = \gamma^0 S_z \gamma^0 = \exp\left(\frac{\eta}{2}\,\gamma^0 M \gamma^0\right) = \exp\left(-\frac{\eta}{2}M\right) = S_z^{-1},$$

  since $\gamma^0 M \gamma^0 = -M$.
- This relation holds true in general:

$$\boxed{S^{-1} = \gamma^0 S^\dagger \gamma^0}.$$

Dirac adjoint and covariant bilinears

- Define the **Dirac adjoint**: $\bar{\psi} \equiv \psi^\dagger \gamma^0$.
- Using $S^{-1} = \gamma^0 S^\dagger \gamma^0$, we can show that

$$\begin{aligned} \bar{\psi}' \psi' &= \psi^\dagger S^\dagger \gamma^0 S \psi \\ &= \psi^\dagger \gamma^0 \gamma^0 S^\dagger \gamma^0 S \psi \\ &= \bar{\psi} S^{-1} S \psi = \bar{\psi} \psi. \end{aligned}$$

- Define a proper Lorentz 4-vector current:

$$j^\mu = \bar{\psi} \gamma^\mu \psi$$

Current Conservation from the Dirac Equation

- Dirac equation and its adjoint:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \qquad \partial_\mu \bar{\psi}\, i\gamma^\mu + m\,\bar{\psi} = 0.$$

- Compute the divergence of $j^\mu \equiv \bar{\psi}\gamma^\mu\psi$:

$$\partial_\mu j^\mu = (\partial_\mu \bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi).$$

- Use the two equations:

$$(\partial_\mu \bar{\psi})\gamma^\mu = -\,i\,m\,\bar{\psi}, \qquad \gamma^\mu\partial_\mu\psi = i\,m\,\psi.$$

- Hence the **continuity equation** holds true:

$$\partial_\mu j^\mu = (-im\,\bar{\psi})\psi + \bar{\psi}(im\,\psi) = 0.$$

- Identify the **probability density** and **current**:

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi \geq 0, \qquad \text{unlike Klein–Gordon}$$

13. Negative Energies & the Positron

Dirac Plane Waves: Positive & Negative Energy

- Seek plane-wave solutions of $(i\gamma^\mu \partial_\mu - m)\psi = 0$:

$$\psi(x) = u_s(\mathbf{p})\, e^{-ip\cdot x} \quad \text{or} \quad \psi(x) = v_s(\mathbf{p})\, e^{+ip\cdot x}, \qquad p\cdot x \equiv p_\mu x^\mu .$$

- With $p^0 \equiv E = \sqrt{\mathbf{p}^2 + m^2} > 0$,

$$(\gamma^\mu p_\mu - m)\, u_s(\mathbf{p}) = 0 \quad (\text{"positive-energy" branch}),$$

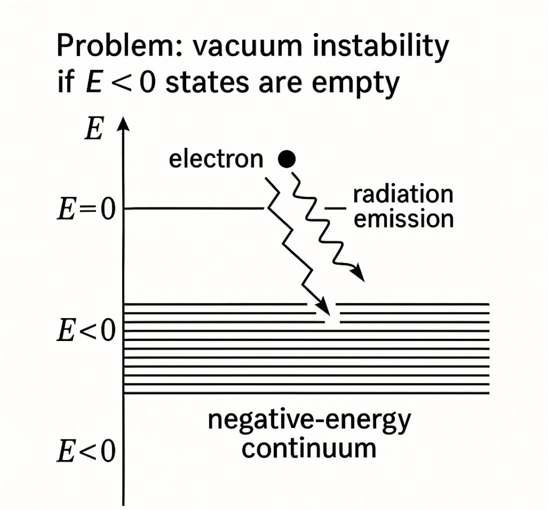
$$(\gamma^\mu p_\mu + m)\, v_s(\mathbf{p}) = 0 \quad (\text{"negative-energy" branch}).$$

- **Key point:** despite the negative-energy eigenvalue, the density is always positive:

$$j^0 = \psi^\dagger \psi \geq 0.$$

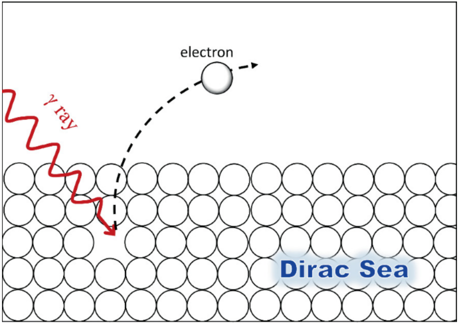
The issue is *stability*: in a one-particle picture, why not fall into ever-lower (negative) energies?

If negative-energy states exist, why doesn't an electron radiate down to $E < 0$?



Dirac's Idea: the "Sea" of Negative-Energy Electrons

- **Dirac's proposal (historical):**
  1. All negative-energy electron states are *filled* in the vacuum (Pauli exclusion blocks further decays).
  2. Removing one electron from the sea leaves a **hole**: it behaves like a particle with charge $+e$ and energy $+E$ (later identified as the *positron*).



Modern View (QFT): Antiparticles without a Literal Sea

- Promote ψ to a **field operator**; expand in modes:

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[b_s(\mathbf{p})\, u_s(\mathbf{p})\, e^{-ip\cdot x} + d_s^\dagger(\mathbf{p})\, v_s(\mathbf{p})\, e^{+ip\cdot x} \right].$$

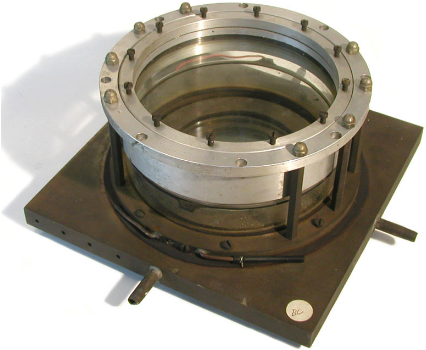
- **Interpretation:** $b_s^\dagger$ creates an electron; $d_s^\dagger$ creates an *antiparticle* (positron). Negative-frequency modes are reinterpreted as *creation of antiparticles*, not negative-energy electrons.
- Charge and density:

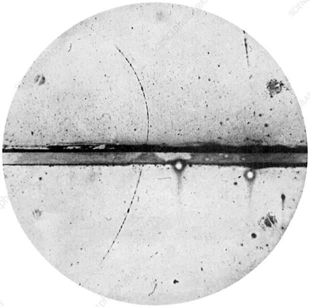
$$Q = \int d^3x\, j^0 \;\propto\; \sum_{\mathbf{p},s} \left[b_s^\dagger b_s \;-\; d_s^\dagger d_s \right].$$

- **Bottom line:** QFT keeps $j^0 = \psi^\dagger\psi \geq 0$, preserves Lorentz covariance, and explains antiparticles *without* an infinitely filled sea—just creation/annihilation operators and a stable vacuum.

Anderson (1932): Cloud Chamber Setup & Strategy

- **Apparatus:** vertical expansion (Wilson) cloud chamber in a strong magnetic field B
- Cosmic rays provide charged particles.
- A **lead plate** is mounted midway through the chamber ($\sim$6 mm thick). (Energy loss in Pb makes tracks curve more after crossing the lead plate.)





Why the Lead Plate? Direction from Curvature Change

- In a uniform B field, track radius r is set by momentum:

$$r = \frac{p}{|q|B}\,.$$

- Crossing the lead plate $\Rightarrow$ energy loss ($\Delta E < 0$) and thus momentum loss ($\Delta p < 0$).

$$p_{\text{after}} < p_{\text{before}} \quad \Rightarrow \quad r_{\text{after}} < r_{\text{before}}.$$

- **Read the direction:** the track goes from the *larger-radius* arc (before plate) to the *smaller-radius* arc (after plate).
- **Then read the sign:** with the known B direction, the sense of bending gives $\mathrm{sign}(q)$.

Evidence from the Track: Light, Positively Charged

- **Famous photograph:** track curves *more* above the plate (lower momentum), and bends in the direction for *positive* charge.
  (Upward-going, left-curving in the published image.)
- **Mass inference: Proton?**
  - *Range/energy loss* in Pb and gas: protons with the same curvature would stop within mm, but the observed tracks are cm-scale.
- **Conclusion:** a light ($\approx m_e$) particle with *positive* charge.
- It is a **positron**.

14. Helicity and Chirality

Helicity

- **Helicity**: projection of spin on momentum direction

$$h = \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|}, \quad \vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

From Dirac to Weyl: Setup (Chiral Rep.)

- Start from the momentum-space Dirac equation:

$$(\not{p} - m)\, u(p) = 0, \qquad \not{p} = \gamma^\mu p_\mu$$

- Chiral (Weyl) representation:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

- Spinor decomposition and Pauli 4-vectors:

$$u(p) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \qquad \sigma^\mu = (\mathbf{1}, \vec{\sigma}), \quad \bar{\sigma}^\mu = (\mathbf{1}, -\vec{\sigma}).$$

- With metric $\mathrm{diag}(+,-,-,-)$ and $p_\mu = (E, -\vec{p})$:

$$p \cdot \sigma = E\,\mathbf{1} - \vec{p} \cdot \vec{\sigma}, \qquad p \cdot \bar{\sigma} = E\,\mathbf{1} + \vec{p} \cdot \vec{\sigma}.$$

Block Form of $\not{p}$ and $(\not{p} - m)$

- Using the chiral γ^μ:

$$\not{p} = p_\mu \gamma^\mu = \begin{pmatrix} 0 & p \cdot \sigma \\ p \cdot \bar{\sigma} & 0 \end{pmatrix}$$

- Therefore,

$$(\not{p} - m) = \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix}.$$

- Acting on $u(p) = \binom{\psi_L}{\psi_R}$:

$$\begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} -m\psi_L + (p \cdot \sigma)\psi_R \\ (p \cdot \bar{\sigma})\psi_L - m\psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Extracting the Two Weyl Equations

- From upper two components:

$$(p \cdot \sigma)\, \psi_R = m\, \psi_L.$$

- From lower two components:

$$(p \cdot \bar{\sigma})\, \psi_L = m\, \psi_R.$$

- **Massless limit** $(m = 0)$ decouples the chiralities:

$$(p \cdot \sigma)\, \psi_R = 0, \qquad (p \cdot \bar{\sigma})\, \psi_L = 0,$$

i.e. two independent **Weyl equations**.

From Weyl Equations to Helicity Eigenstates

- Massless Weyl equations:

$$(p \cdot \sigma)\,\psi_R = 0, \qquad (p \cdot \bar{\sigma})\,\psi_L = 0.$$

- For $\vec{p} = |\vec{p}|\hat{\vec{p}}$, $E = |\vec{p}|$:

$$(E\mathbf{1} - \vec{p} \cdot \vec{\sigma})\psi_R = 0 \implies (\vec{\sigma} \cdot \hat{\vec{p}})\psi_R = +\psi_R,$$

$$(E\mathbf{1} + \vec{p} \cdot \vec{\sigma})\psi_L = 0 \implies (\vec{\sigma} \cdot \hat{\vec{p}})\psi_L = -\psi_L.$$

- Thus, ψ_R and ψ_L are helicity eigenstates with eigenvalues $+1$ and -1.
- Interpreting $\vec{\sigma}/2$ as spin for spin-$\frac{1}{2}$:

$$\psi_R : \; h = +\tfrac{1}{2} \text{ (spin parallel to } \vec{p}), \quad \psi_L : \; h = -\tfrac{1}{2} \text{ (spin anti-parallel)}.$$

Chirality: Massless vs Massive Fermions

- **Massless case** $(m = 0)$: chiralities decouple

$$p_\mu \sigma^\mu \psi_R = 0, \qquad p_\mu \bar{\sigma}^\mu \psi_L = 0,$$

so chirality is conserved and

$$\text{chirality} = \text{helicity}.$$

- **Massive case** $(m \neq 0)$: the mass term couples chiralities

$$p_\mu \sigma^\mu \psi_R = m\,\psi_L, \qquad p_\mu \bar{\sigma}^\mu \psi_L = m\,\psi_R,$$

so chirality is *not* conserved.

Left-handed Neutrinos in the Standard Model

- Experiment: weak interactions violate parity maximally.
  - Wu experiment (1957): beta decay asymmetry.
  - Goldhaber experiment (1958): neutrinos are left-helical.
- Theory: weak force is an $SU(2)_L$ gauge interaction.

$$L_\ell = \begin{pmatrix} \nu_\ell \\ \ell^- \end{pmatrix}_L, \qquad \ell = e, \mu, \tau$$

  Only left-chiral doublets couple to $W^\pm$, Z bosons.
- No right-handed neutrino field exists in the minimal SM.
- $\Rightarrow$ In the SM: only **left-handed neutrinos** exist.

Summary: Why Chirality Matters

- **Helicity**: intuitive picture — spin along momentum.
- **Chirality**: fundamental in the Standard Model.
- Electroweak gauge structure:
  - Left-chiral fermions form $SU(2)_L$ **doublets**:

$$L_\ell = \begin{pmatrix} \nu_\ell \\ \ell^- \end{pmatrix}_L, \qquad Q_q = \begin{pmatrix} u \\ d \end{pmatrix}_L$$

  - Right-chiral fermions are $SU(2)_L$ **singlets**:

$$e_R,\ u_R,\ d_R \quad \text{(no } \nu_R \text{ in SM)}$$

- This chiral asymmetry explains:
  - **Maximal parity violation** in weak interactions.
  - Different coupling strengths of left- vs. right-handed fields.