Heat Kernel Expansion

Given a quantum mechanical Hamiltonian, H, one can define a heat kernel G as a sum over the complete basis

$$G_s(x;y) \equiv \sum_n \psi_n(x)^* e^{-sH} \psi_n(y) \tag{1}$$

which obeys

$$-\frac{\partial}{\partial s} G_s(x;y) = H G_s(x;y) \ , \qquad \lim_{s\to 0} G_s(x;y) = \delta(x-y) \tag{2}$$

for an appropriate delta function. Here, H can be taken as acting on y or x, thanks to its Hermitian nature. Such a bilocal function is called the heat kernel. If ψ_n's happen to be the energy eigenstates, the heat kernel can also be written as

$$G_s(x;y) \equiv \sum_n e^{-sE_n} \psi_n(x)^* \psi_n(y) \tag{3}$$

For physicists, the technique based on it is widely known as the Schwinger-DeWitt method which competed with Feynman diagrams in 1950's. Its importance survived this initial purpose in many corners of quantum field theories and mathematical physics.

This project asks you to study the heat kernel and one or two of its diverse applications. If this is too vague, one possible recourse is to follow the exercises below and investigate the physics content therein. You are welcome to pursue other directions, however, for the completion of the project, as long as you find a substantial story to tell your fellow students in the end. Another problem where the heat kernel was used very usefully is the Atiyah-Singer index theorem, for example.

1) Show that if $H = H_0 \equiv -\partial^2 + m^2$ in a flat R^d, the heat kernel becomes

$$G_s^{(0)}(x;y) \equiv \frac{1}{(4\pi s)^{d/2}} e^{-sm^2-(x-y)^2/4s} \tag{4}$$

2) Derive $G_s^{(0)}(x;y)$ if $H = H_0 \equiv -(\partial_x - iBy/2)^2 - (\partial_y + iBx/2)^2 + m^2(x^2+y^2)$ in a

flat R^2. In particular $G_s^{(0)}(x;x)$ is a very simple expression in terms of s and B.

3) Show that, given a decomposition $H = H_0 + H_1$, one can find a perturbative expansion

$$G_s(x;y) = \sum_n G_s^{(n)}(x;y) \tag{5}$$

Show that

$$G_s^{(n+1)}(x;y) = -\int_0^s dt \int_z G_{s-t}^{(0)}(x,z) H_1 G_t^{(n)}(z;y) \tag{6}$$

and use this to express $G_s^{(n+1)}(x;y)$ entirely in terms of H_1's and $G^{(0)}$'s. You will see that this is an expansion in integer powers of s, although generally $G_s^{(n)}/G_s^{(0)}$ for each fixed $n > 0$ could contain terms of several distinct positive powers of s.

4) Perform the above heat kernel expansion for $G_s(x;x)$ with

$$H = -(\partial_\mu + iF_{\mu\alpha}(x)\delta x^\alpha/2)(\partial_\nu + iF_{\nu\beta}(x)\delta x^\beta/2)\delta^{\mu\nu} + M^2 \tag{7}$$

for $x^\mu = x^{1,2,3,4}$ and small δx^μ, and

$$H_0 = -\partial_\mu\partial_\nu\delta^{\mu\nu} + M^2 \ , \tag{8}$$

under the assumption that the small field strength $F_{\mu\nu} = -F_{\nu\mu}$ is considered uniform, i.e., we ignore its gradient. You should find a Taylor expansion in F augmented by functions of M, μ, Λ, with first two terms being "divergent" if we insist on $\Lambda \to \infty$.

5) One often uses the heat kernel for computing a functional determinant of an "elliptic" H as

$$\log(\det H) \simeq \log(\det H)_{\mu;\Lambda} \equiv -\int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s} \int_x G_s(x;x) \tag{9}$$

An elliptic H means that its eigenvalues are mostly positive, allowing a finite number of zero-eigenmodes at most. This relation is not really an equality; the formula on the right-hand side merely captures an "essential" part of the determinant. What do you think this last claim means? What would be your interpretation of Λ and μ?

(Try first with a Hermitian matrix H of large but finite rank.)

6) The above exercise 5) is one possible starting point for the renormalization process for Quantum Electrodynamics. Consider the functional integral

$$\int [d\phi]_\Lambda \, e^{-S_{\text{scalar QED}}} \,, \qquad S_{\text{scalar QED}} = \frac{1}{4e^2} \int d^4x \; F_{\mu\nu} F^{\mu\nu} + \int d^4x \; \phi^* H \phi \tag{10}$$

where the subscript Λ is a cut-off in the same sense as in equation (7), with

$$H = -(\partial_\mu - i n_e A_\mu)(\partial_\nu - i n_e A_\nu)\delta^{\mu\nu} + M^2 \tag{11}$$

For this exercise, we are effectively pretending that the electron with $M \simeq 0.5MeV$ is a scalar particle, which does not affect the qualitative conclusion we draw next.

Approximating $A_\mu(x + \delta x) \simeq -F_{\mu\nu}(x)\delta x^\nu/2$ with a slowly-varying and small F, again meaning that we ignore its gradient in the subsequent heat kernel expansion, we find that the path integral reduces to

$$\sim \int [d\phi]_\mu \, e^{-S_{\text{scalar QED}}} \times \frac{1}{(\det H)_{\mu;\Lambda}} \tag{12}$$

The exponent of the resulting path integral

$$S_{\text{scalar QED}} + \log(\det H)_{\mu;\Lambda} \tag{13}$$

is called the effective action. Extract from it a term of type

$$\frac{1}{4e(\mu/\Lambda;\mu/M)^2} \int d^4x \; F_{\mu\nu} F^{\mu\nu} \tag{14}$$

and isolate $1/e(\mu/\Lambda;\mu/M)^2$.

7) The quantity e may be considered an electric coupling constant; why is that so? Then, $e(\mu/\Lambda;\mu/M)$ is what Feynman would call the one-loop-corrected "running" coupling. What would you do to determine this scale-dependent coupling, say, by comparing to Coulomb's experiment? Once you do this, you can plot $1/e(\mu/\Lambda;\mu/M)^2$ and find an unavoidable disease. What conclusion would you draw from such a disease?