Mathematical applications to QIT

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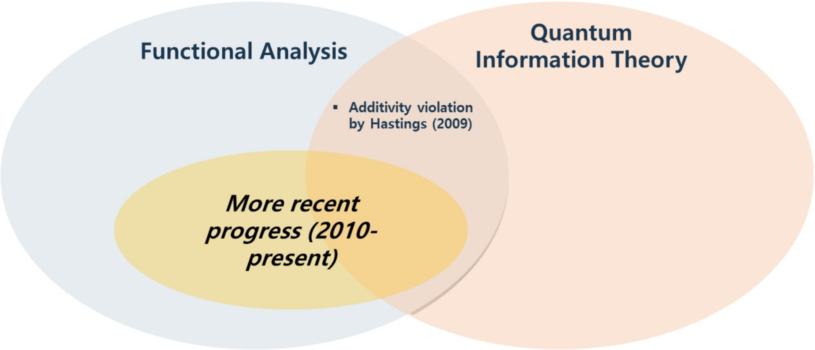
Additivity question for $C(\Phi)$


Functional Analysis
Quantum Information Theory
Additivity violation by Hastings (2009)
More recent progress (2010-present)

Theorem (Hastings, 2009)

There exist two quantum channels Φ *and* Ψ *such that*

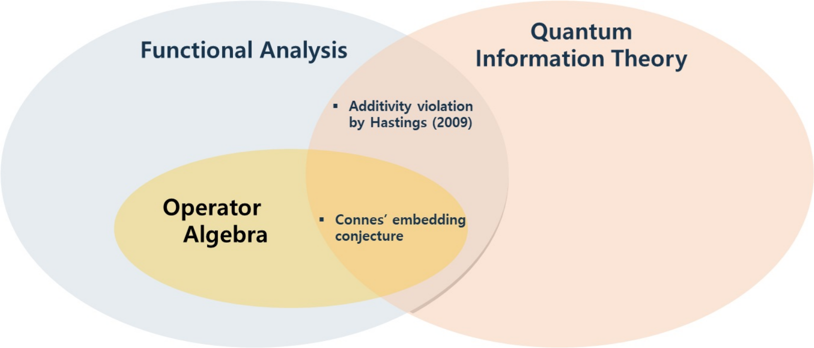
$$\chi(\Phi \otimes \Psi) > \chi(\Phi) + \chi(\Psi)$$

where $\chi(\cdot)$ *is the Holevo information.*

- Holevo's result implies that the classical capacity

$$C(\Phi) = \lim_{n\to\infty} \frac{\chi(\Phi^{\otimes n})}{n}$$

  is very hard to compute.
- Holevo's counterexample uses random unitary matrices.


Functional Analysis
Quantum Information Theory
Additivity violation by Hastings (2009)
Operator Algebra
Connes' embedding conjecture

Recent developments between OAT and QIT

- **Equivalence between two long-standing open questions**

Connes' embedding conjecture in OAT

Any ***quantum probability space*** can be understood as a ***certain limit of matrix algebras***.

Fritz (2010)
Junge et el. (2010)
Ozawa (2013)
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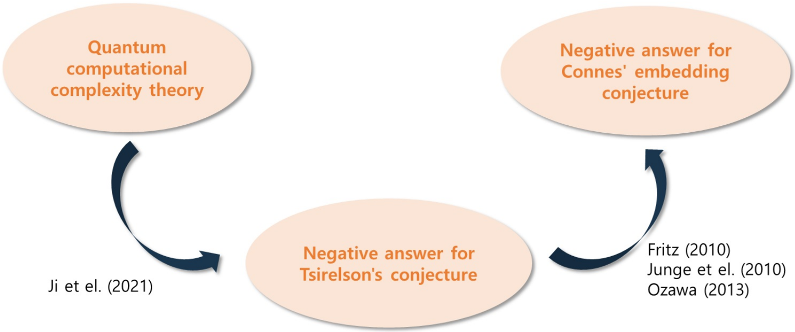
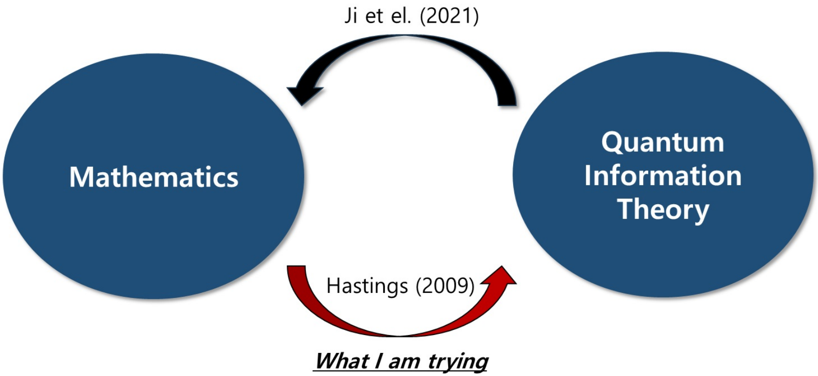
⇑
Kirchberg (1993)

Tsirelson's conjecture in QIT

Correlations from ***commuting operator setup*** can be approximated by correlations from ***tensor product setup***.


Quantum computational complexity theory
Negative answer for Connes' embedding conjecture
Negative answer for Tsirelson's conjecture
Ji et el. (2021)
Fritz (2010)
Junge et el. (2010)
Ozawa (2013)


Ji et el. (2021)
Mathematics
Quantum Information Theory
Hastings (2009)
What I am trying

For any quantum channels Φ and Ψ, we have

$$C(\Phi \otimes \Psi) \geq C(\Phi) + C(\Psi)$$

where $C(\cdot)$ is the classical capacity.

Question

Do there exist Φ and Ψ such that

$$C(\Phi \otimes \Psi) > C(\Phi) + C(\Psi)?$$

Entanglement under (quantum) group symmetry

- Quantifying quantum entanglement is a crucial problem in QIT.
- The so-called Schmidt number $\mathrm{SN}(\rho)$ is a natural measure of entanglement of a quantum state ρ.

  If $\rho \in M_n(\mathbb{C})$, then $\mathrm{SN}(\rho)$ is one of $1, 2, \cdots, n$.
- In particular, $\mathrm{SN}(\rho) = 1 \iff \rho$ is separable.

- A linear map $\Phi : M_d(\mathbb{C}) \to M_{d'}(\mathbb{C})$ is called k-positive $(1 \le k \le d)$ if the following map

$$\mathrm{id}_k \otimes \Phi : M_k(\mathbb{C}) \otimes M_d(\mathbb{C}) \to M_k(\mathbb{C}) \otimes M_{d'}(\mathbb{C}) \text{ is positive.}$$

- There is a general machinery to transfer k-positivity to quantification of quantum entanglement, namely

k-positivity of general linear maps

$\rightsquigarrow$ Schmidt numbers of general quantum states.

- HOWEVER, $(1$-$)$positivity is already NP-hard to check.

Question

Can we compute SN($\rho_{a,b}^{(d)}$) for all quantum states of the form

$$\rho_{a,b}^{(d)} = \frac{1-a-b}{d^2} I_d \otimes I_d + a|\Omega_d\rangle\langle\Omega_d| + \frac{b}{d} F_d ?$$

Here,

$$\begin{cases} |\Omega_d\rangle\langle\Omega_d| = \frac{1}{d}\sum_{i,j=1}^{d} |ii\rangle\langle jj| = \frac{1}{d}\sum_{i,j=1}^{d} e_{ij} \otimes e_{ij} \\ \frac{1}{d}F_d = \frac{1}{d}\sum_{i,j=1}^{d} |ij\rangle\langle ji| = \frac{1}{d}\sum_{i,j=1}^{d} e_{ij} \otimes e_{ji} \end{cases}.$$

Theorem (Park and Y., 2024)

Under compact group symmetries,

k-positivity of (π_A, π_B)-covariant linear maps

$\longleftrightarrow$ *Schmidt numbers of $\overline{\pi_A} \otimes \pi_B$-invariant quantum states.*

Example

k-positivity of $\mathcal{L}_{p,q}^{(d)} = (1-p-q)\frac{\mathrm{Tr}(X)}{d}\mathrm{Id}_d + pX + qX^t$

$\longleftrightarrow$ Schmidt numbers of $\rho_{a,b}^{(d)} = \frac{1-a-b}{d^2} I_d \otimes I_d + a|\Omega_d\rangle\langle\Omega_d| + \frac{b}{d}F_d.$

From now on, let us focus on linear maps of the form

$$\mathcal{L}^{(d)}_{p,q}(X) = (1-p-q)\frac{\mathrm{Tr}(X)}{d}\mathrm{Id}_d + pX + qX^t.$$

Example (Tomiyama, 1985)

- $\mathcal{L}^{(d)}_{p,0}(X)$ is k-positive $\Leftrightarrow -\frac{1}{kd-1} \leq p \leq 1$.
- $\begin{cases} \mathcal{L}^{(d)}_{0,q}(X) \text{ is 1-positive} & \Leftrightarrow -\frac{1}{d-1} \leq q \leq 1. \\ \mathcal{L}^{(d)}_{0,q}(X) \text{ is } k\text{-positive } (2 \leq k \leq d) & \Leftrightarrow -\frac{1}{d-1} \leq q \leq \frac{1}{d+1} \end{cases}$

Theorem (Park and Y., 2024)

A complete characterization of k-positivity of linear maps

$$\mathcal{L}_{p,q}^{(d)}(X) = (1-p-q)\frac{Tr(X)}{d} Id_d + pX + qX^t$$

is given by

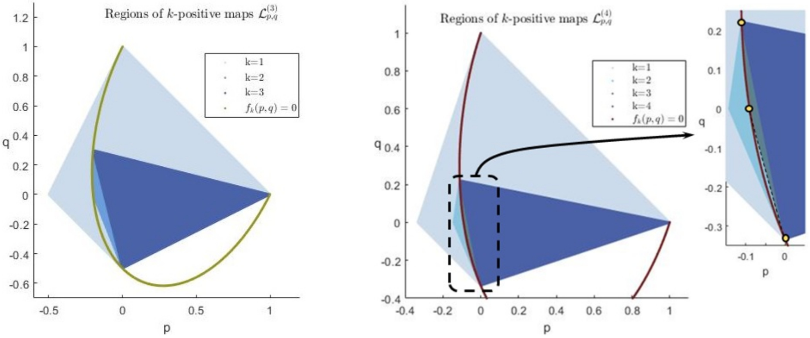


Figure: The regions of k-positive maps $\mathcal{L}_{p,q}^{(d)}$ for $d = 3, 4$

Theorem (Park and Y., 2024)

We have a complete characterization for the Schmidt numbers $SN(\rho_{a,b}^{(d)})$ of all quantum states of the form

$$\rho_{a,b}^{(d)} = \frac{1-a-b}{d^2} I_d \otimes I_d + a|\Omega_d\rangle\langle\Omega_d| + \frac{b}{d} F_d.$$

with explicit algebraic descriptions.

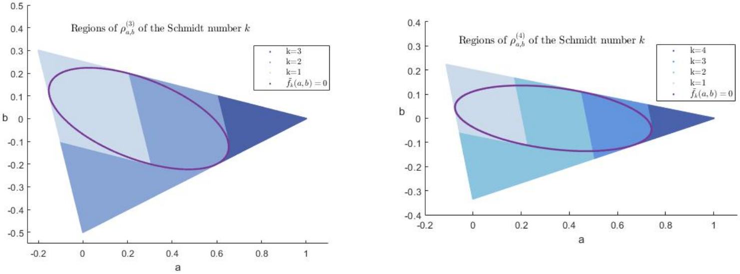


Figure: Regions of $\rho_{a,b}^{(d)}$ with $\mathrm{SN}(\rho_{a,b}^{(d)}) = k$ for $d = 3, 4$

Thank you very much for your attention.