The 9th KIAS School and Workshop on
Quantum Information and Thermodynamics
Nov. 2 - 5, 2025 | Co-op City Hotel Seongsan @ Jeju

Fluctuation-response inequality for open quantum systems

*__Hyun-Myung Chun__, Euijoon Kwon, Hyunggyu Park, and Jae Sung Lee

KIAS KOREA INSTITUTE FOR ADVANCED STUDY
SEOUL NATIONAL UNIVERSITY
KIAS KOREA INSTITUTE FOR ADVANCED STUDY
KIAS KOREA INSTITUTE FOR ADVANCED STUDY

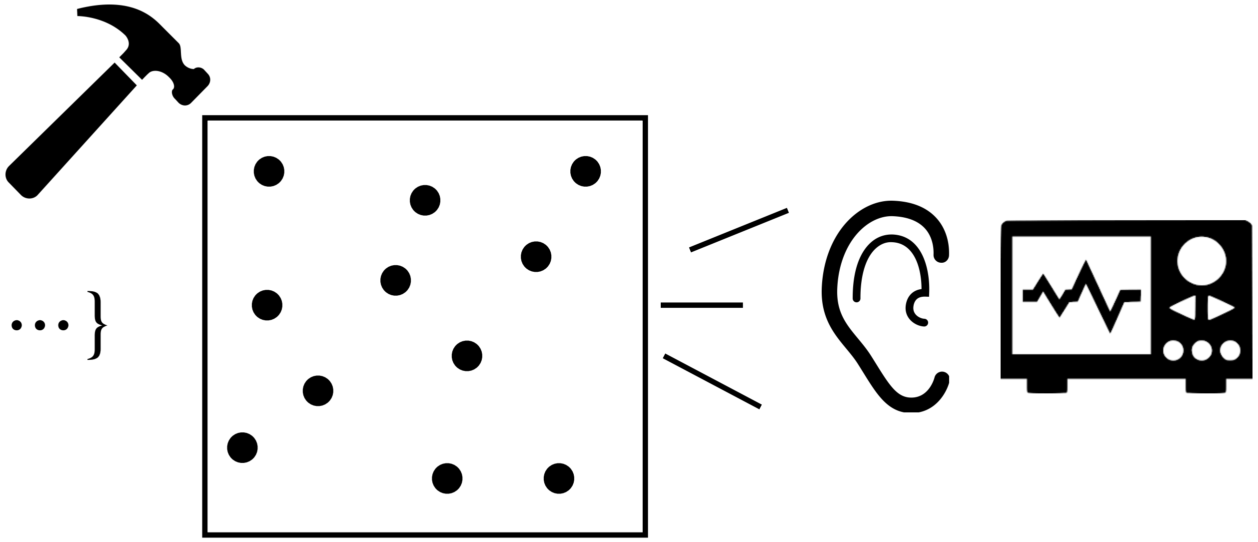
Reference: Physical Review Letters **135**, 097101 (2025)

Table of Contents

Fluctuation-dissipation theorem

Near equilibrium,

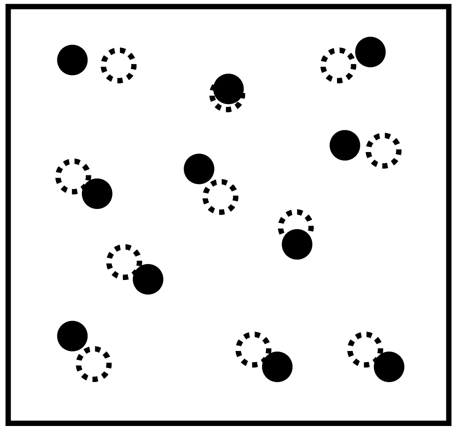
$\boldsymbol{F} \subset \{\Delta T,\ \Delta\mu,\ \cdots\}$



$\boldsymbol{J} \subset \{\dot{E},\ \dot{N},\ \cdots\}$

$\boldsymbol{J} = \mathsf{L} \cdot \boldsymbol{F}$ $\quad$ (thermodynamic currents) $\propto$ (thermodynamic forces)

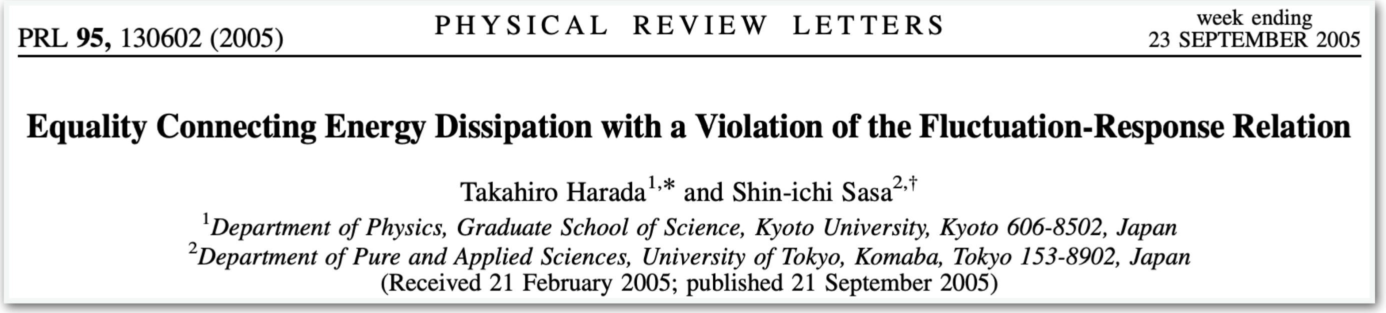
(dissipative) response



$\mathsf{C} \subset \left\{ \langle\langle \dot{N} \rangle\rangle_{\mathrm{eq}},\ \langle\langle \dot{E} \rangle\rangle_{\mathrm{eq}},\ \langle\langle \dot{N}, \dot{E} \rangle\rangle_{\mathrm{eq}},\ \cdots \right\}$ **fluctuation**

$\mathsf{C} = 2\mathsf{L}$ **: Fluctuation-Dissipation Theorem (FDT)**

Fluctuation-response relations for NESS



PRL **95,** 130602 (2005) PHYSICAL REVIEW LETTERS week ending 23 SEPTEMBER 2005

Equality Connecting Energy Dissipation with a Violation of the Fluctuation-Response Relation

Takahiro Harada[1,*] and Shin-ichi Sasa[2,†]
[1]*Department of Physics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan*
[2]*Department of Pure and Applied Sciences, University of Tokyo, Komaba, Tokyo 153-8902, Japan*
(Received 21 February 2005; published 21 September 2005)

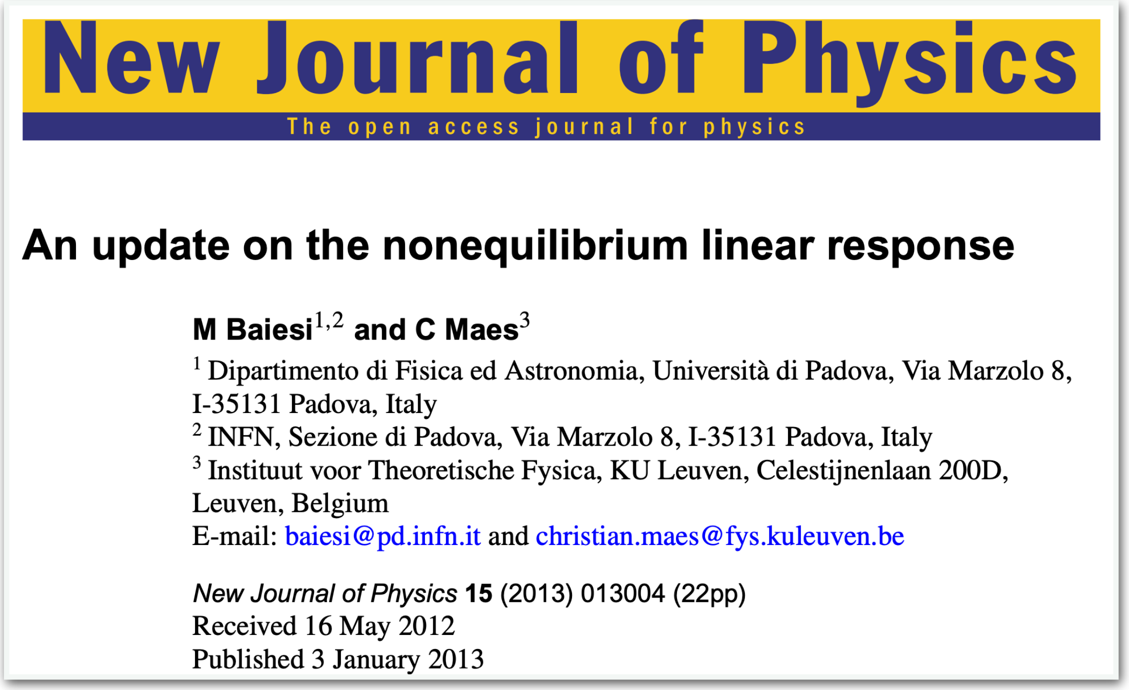
(violation of FDT) ∝ (heat dissipation)

fluctuation

$$\langle J\rangle_0 = \gamma\left\{v_s^2 + \int_{-\infty}^{\infty}\left[\tilde{C}(\omega) - 2T\tilde{R}'(\omega)\right]\frac{d\omega}{2\pi}\right\}. \qquad (5)$$

dissipation

response



New Journal of Physics
The open access journal for physics

An update on the nonequilibrium linear response

M Baiesi[1,2] and C Maes[3]
[1] Dipartimento di Fisica ed Astronomia, Università di Padova, Via Marzolo 8, I-35131 Padova, Italy
[2] INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padova, Italy
[3] Instituut voor Theoretische Fysica, KU Leuven, Celestijnenlaan 200D, Leuven, Belgium
E-mail: baiesi@pd.infn.it and christian.maes@fys.kuleuven.be

New Journal of Physics **15** (2013) 013004 (22pp)
Received 16 May 2012
Published 3 January 2013

(FDT can be modified in a more general form)

$$\boxed{R(\tau) = \left\langle \frac{\mathcal{L}_1\rho}{\rho}(0)\; A(\tau)\right\rangle} = \langle B(0)\, A(\tau)\rangle \qquad (13)$$

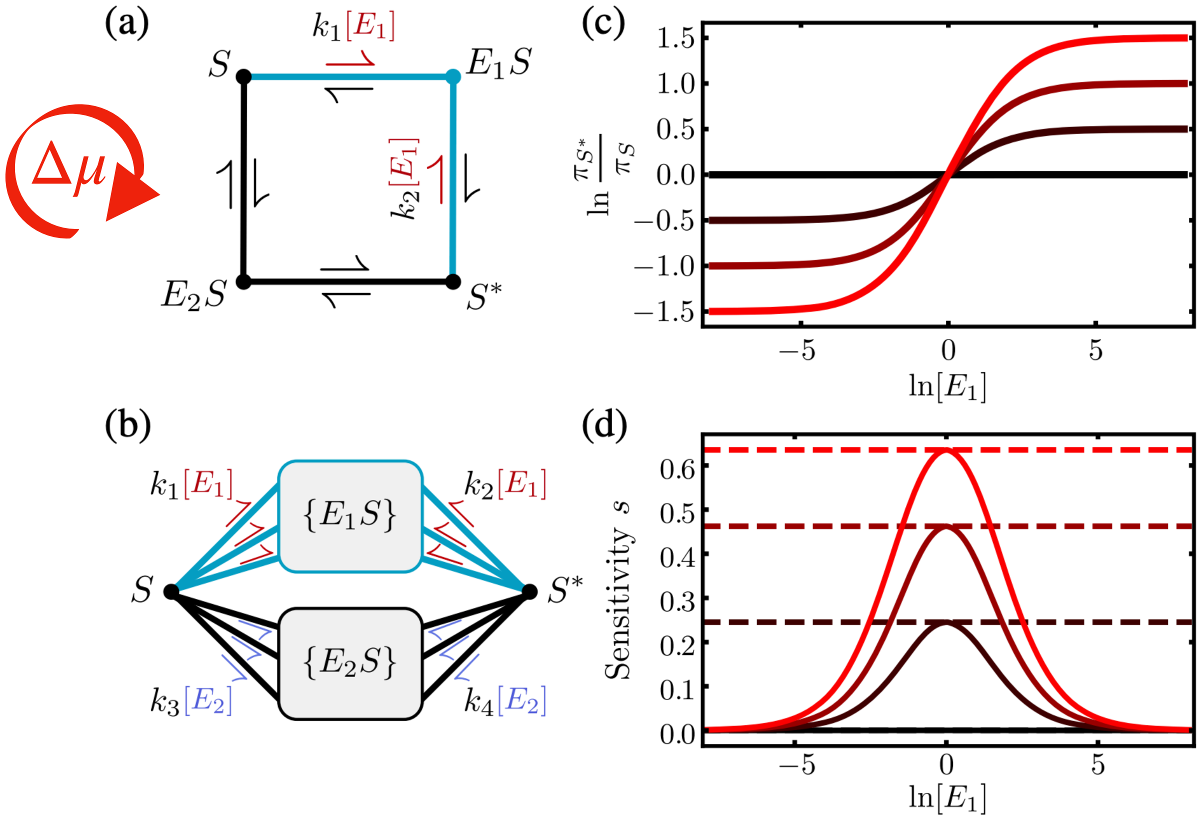
response

fluctuation

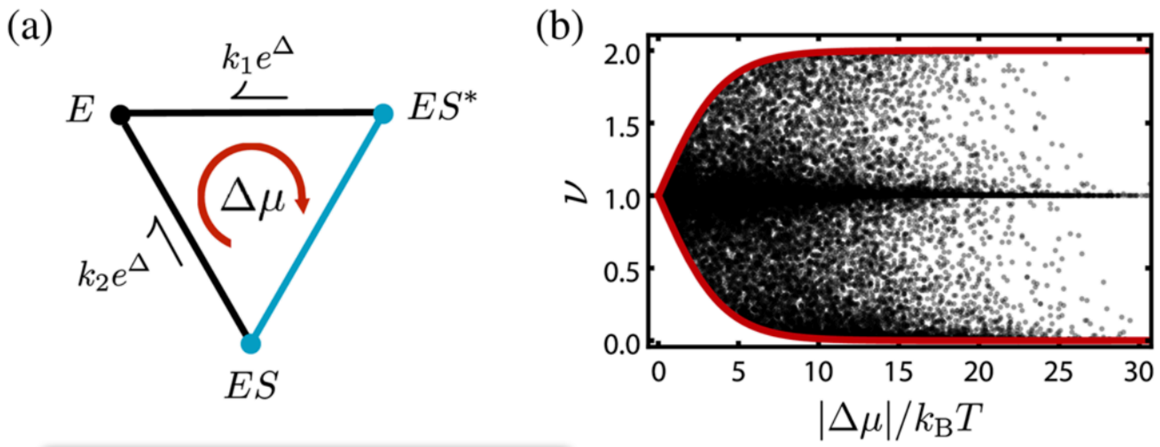
steady-state distribution

Thermodynamic bounds on response

Owen/Gingrich/Horowitz, PRX (2020)



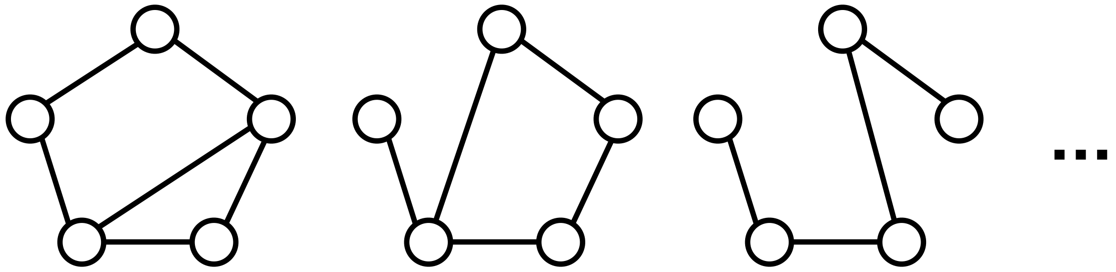
$$s = \left|\frac{\partial \ln\left(\pi_{S^*}/\pi_S\right)}{\partial \ln[E_1]}\right| \le \tanh(\Delta\mu/4k_BT),$$

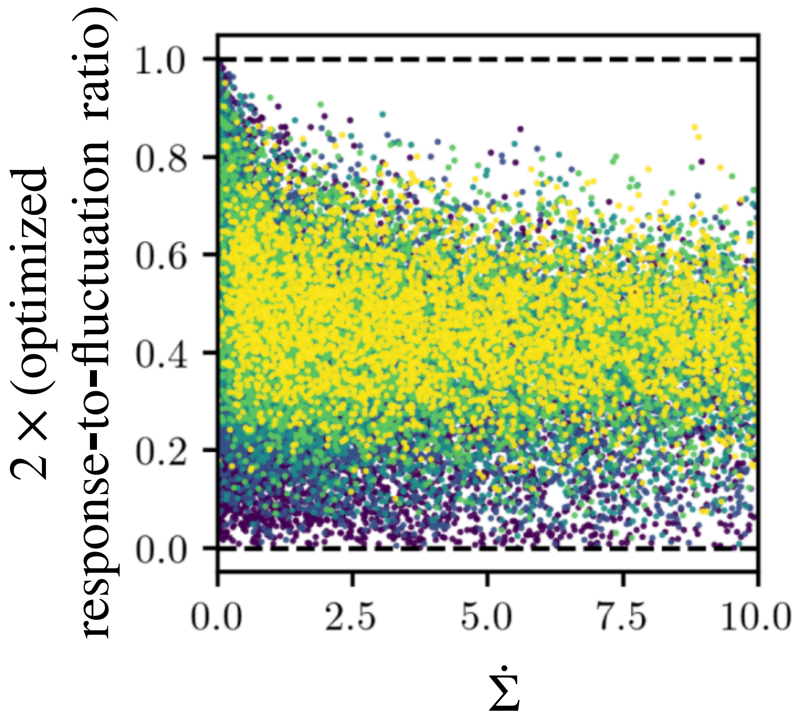


$$\nu = -\frac{\partial \ln(\pi_E/\pi_{ES})}{\partial \Delta}.$$

$$|\nu - 1| \le \tanh(\Delta\mu/4k_BT)$$

Ptaszyński/Aslyamov/Esposito, PRL (2024)





response

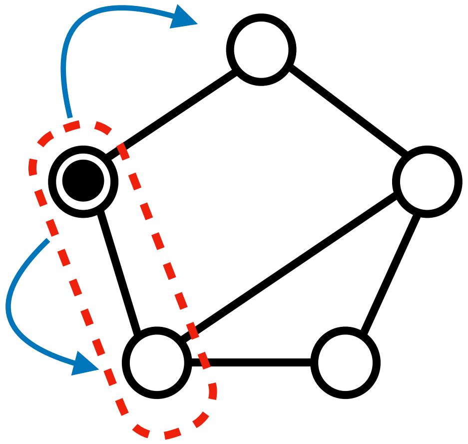
$$\frac{(\partial_{B_{ij}}\langle J\rangle)^2}{\langle\langle J\rangle\rangle} \le \frac{\dot\Sigma}{2}$$

dissipation

fluctuation

Response TUR (R-TUR)

Kinetic and entropic perturbations



$$\frac{dp_i}{dt} = \sum_{j(\neq i)} \left(W_{ij}p_j - W_{ji}p_i \right)$$

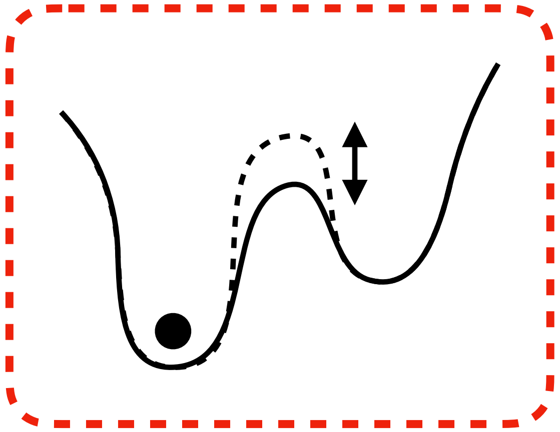
unique steady state

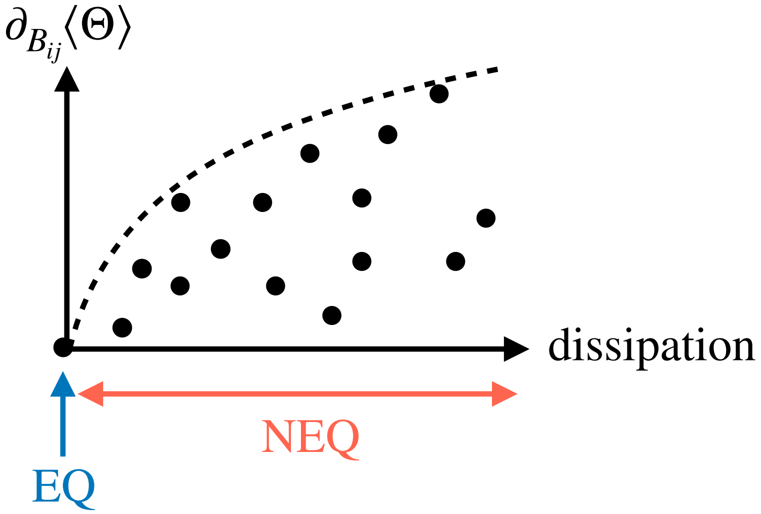
$$\lim_{t\to\infty} p_i = \pi_i$$

[1] kinetic perturbation

$$\frac{\partial}{\partial B_{ij}} \equiv W_{ij}\frac{\partial}{\partial W_{ij}} + W_{ji}\frac{\partial}{\partial W_{ji}}$$

$$\begin{cases} W_{ij} \mapsto W_{ij}(1+\delta) \\ W_{ji} \mapsto W_{ji}(1+\delta) \end{cases}$$

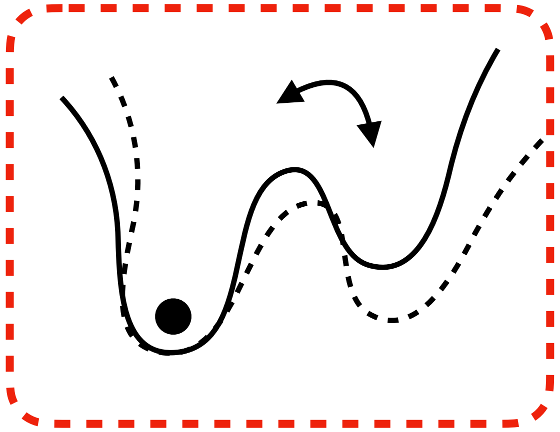




[2] entropic perturbation

$$\frac{\partial}{\partial F_{ij}} \equiv \frac{1}{2}\left(W_{ij}\frac{\partial}{\partial W_{ij}} - W_{ji}\frac{\partial}{\partial W_{ji}} \right)$$

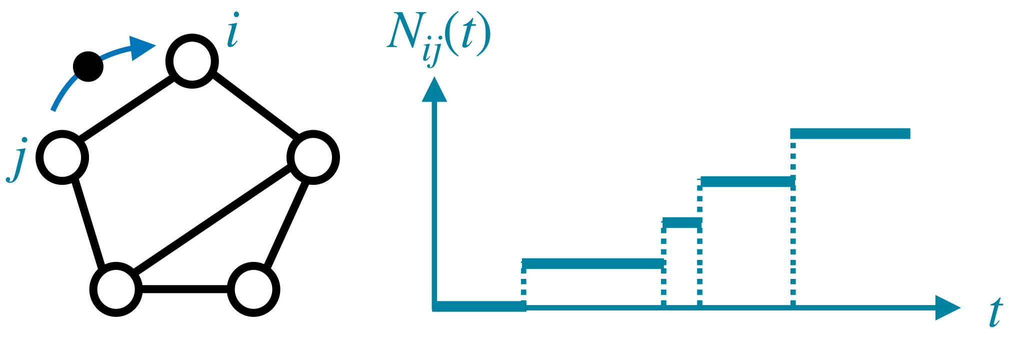
$$\begin{cases} W_{ij} \mapsto W_{ij}(1+\delta/2) \\ W_{ji} \mapsto W_{ji}(1-\delta/2) \end{cases}$$



$$\frac{\partial\langle\Theta\rangle}{\partial F_{ij}} \neq 0 \text{ in EQ}$$

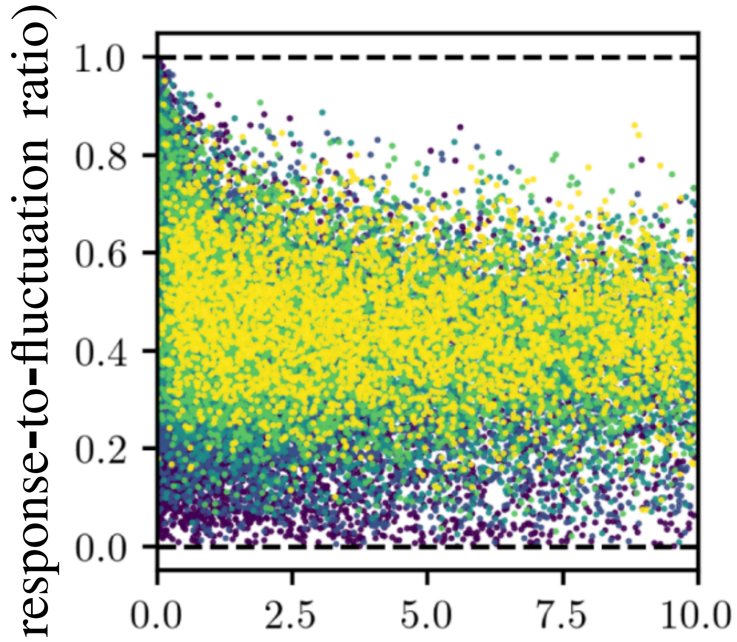
Observables of interest

$$J(\tau) \equiv \frac{1}{\tau}\int_0^{\tau} dt \sum_{i<j} \Lambda_{ij}(\dot{N}_{ij}(t) - \dot{N}_{ji}(t))$$

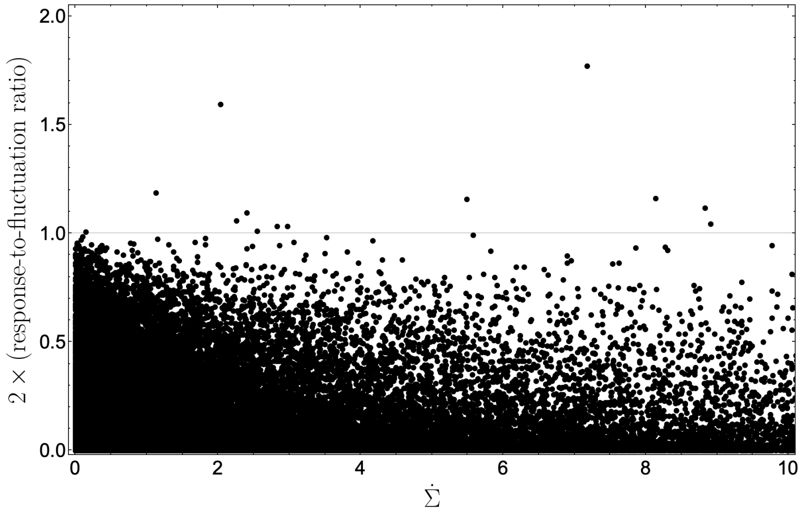


$$\langle J \rangle \equiv \lim_{\tau\to\infty} \langle J(\tau) \rangle = \sum_{i<j} \Lambda_{ij} \underbrace{(W_{ij}\pi_j - W_{ji}\pi_i)}_{\mathcal{J}_{ij}} = \sum_{i<j} \Lambda_{ij}\mathcal{J}_{ij},$$

$$\langle\langle J \rangle\rangle \equiv \lim_{\tau\to\infty} \tau \mathrm{Var}\{J(\tau)\}$$



$$\frac{(\partial_{B_{ij}}\langle J \rangle)^2}{\langle\langle J \rangle\rangle} \leq \frac{\dot{\Sigma}}{2} \quad (\tau \to \infty)$$

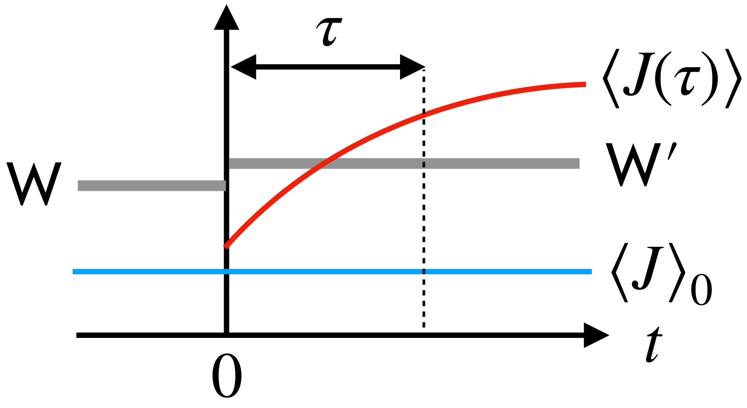


$$\frac{(\partial_{B_{ij}}\langle J \rangle)^2}{\tau \mathrm{Var}\{J\}} \cancel{\leq} \frac{\dot{\Sigma}}{2}$$

Cramér-Rao bound

$$\frac{(\partial_{\theta}\langle X \rangle)^2}{\mathrm{Var}\{X\}} \leq I(\theta)$$

Dynamic perturbation



kinetic

$$\begin{cases} W_{ij} \mapsto W'_{ij} = W_{ij}(1+\varepsilon) \\ W_{ji} \mapsto W'_{ji} = W_{ji}(1+\varepsilon) \end{cases}$$

entropic

$$\begin{cases} W_{ij} \mapsto W'_{ij} = W_{ij}(1+\delta/2) \\ W_{ji} \mapsto W'_{ji} = W_{ji}(1-\delta/2) \end{cases}$$

$$\langle J \rangle_0 \mapsto \langle J(\tau) \rangle = \langle J \rangle_0 + \Delta J(\tau)$$

$$R_{B_{ij}}(\tau) = \lim_{\varepsilon \to 0} \frac{\Delta J(\tau)}{\varepsilon}$$

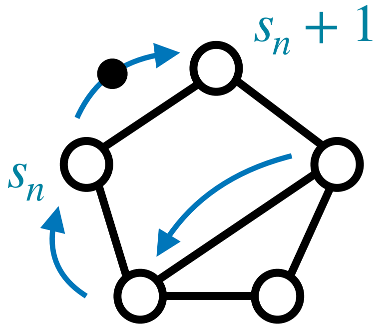
$$R_{F_{ij}}(\tau) = \lim_{\delta \to 0} \frac{\Delta J(\tau)}{\delta}$$

cf.) $\lim_{\tau \to \infty} R_{B_{ij}}(\tau) = \lim_{\varepsilon \to 0} \frac{\langle J(\infty) \rangle - \langle J \rangle_0}{\varepsilon} = \frac{\partial \langle J \rangle}{\partial B_{ij}}$

Fluctuation-response inequalities

multivariate Cramér-Rao bound

$$\sum_{\alpha,\beta} R_{\theta_\alpha}(\tau)[\mathtt{I}^{-1}(\tau)]_{\theta_\alpha\theta_\beta} R_{\theta_\beta}(\tau) \le \mathrm{Var}\{J(\tau)\} \quad \text{where } \mathtt{I}_{\theta_\alpha\theta_\beta}(\tau) = -\left\langle \partial_{\theta_\alpha}\partial_{\theta_\beta} \ln \mathscr{P}[\{s(t)\}_{t=0}^{\tau}] \right\rangle_0$$



$$\{s(t)\}_{t=0}^{\tau} = \{(s_0, t_0) \to (s_1, t_1) \to (s_2, t_2) \to \cdots \to (s_N, t_N) \to (s_N, \tau)\}$$

$$\mathscr{P}[\{s(t)\}_{t=0}^{\tau}] = \pi_{s_0} \underbrace{e^{(t_1 - t_0)W_{s_0 s_0}}}_{\text{waiting at } s_0} \underbrace{W_{s_1 s_0}}_{s_0 \to s_1} \underbrace{e^{(t_2 - t_1)W_{s_1 s_1}}}_{\text{waiting at } s_1} \underbrace{W_{s_2 s_1}}_{s_1 \to s_2} \cdots$$

$$\theta_\alpha \mapsto B_{ij} : \ \mathtt{I}_{B_{ij}B_{i'j'}}(\tau) = \tau \delta_{ii'} \delta_{jj'} a_{ij}$$

$$(= W_{ij}\pi_j + W_{ji}\pi_i)$$

$$\Downarrow$$

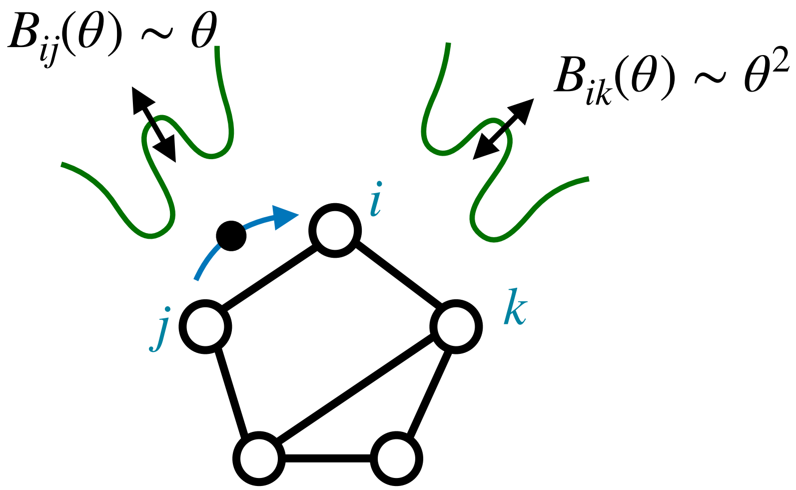
$$\boxed{\sum_{i<j} \frac{R_{B_{ij}}^2(\tau)}{\tau a_{ij}} \le \mathrm{Var}\{J(\tau)\}}$$

$$\theta_\alpha \mapsto F_{ij} : \ \mathtt{I}_{F_{ij}F_{i'j'}}(\tau) = \frac{1}{4} \tau \delta_{ii'} \delta_{jj'} a_{ij}$$

$$\Downarrow$$

$$\boxed{\sum_{i<j} \frac{4R_{F_{ij}}^2(\tau)}{\tau a_{ij}} \le \mathrm{Var}\{J(\tau)\}}$$

Response uncertainty relations



$$\left.\begin{aligned}
&\theta \mapsto \theta + \Delta\theta \\
&B_{ij} \mapsto B_{ij} + \underbrace{1}_{b_{ij}}\Delta\theta \\
&B_{ik} \mapsto B_{ik} + \underbrace{2\theta}_{b_{ik}}\Delta\theta \\
&\quad\vdots
\end{aligned}\right) \longrightarrow R_\theta(\tau) = \sum_{i<j} b_{ij} R_{B_{ij}}(\tau)$$

$$\tau \mathrm{Var}\{\Theta(\tau)\} \underset{\substack{\uparrow\\ \text{FRI}}}{\geq} \sum_{i<j}\frac{R^2_{B_{ij}}(\tau)}{a_{ij}} = \sum_{i<j}\frac{b_{ij}^2 R^2_{B_{ij}}(\tau)}{b_{ij}^2 a_{ij}} \underset{\substack{\uparrow\\ \text{CS ineq.}}}{\geq} \frac{\{\overbrace{\textstyle\sum_{i<j} b_{ij}R_{B_{ij}}(\tau)}^{R_\theta}\}^2}{\sum_{i<j} b_{ij}^2 a_{ij}} \underset{\substack{\uparrow\\ b_{ij}\leq b_{\max}\ \forall(ij)}}{\geq} \frac{R_\theta^2(\tau)}{b_{\max}^2 \underbrace{\textstyle\sum_{i<j} a_{ij}}_{\dot A}} = \frac{R_\theta^2(\tau)}{b_{\max}^2 \dot A}$$

$$\boxed{\frac{R_\theta^2(\tau)}{\tau \mathrm{Var}\{\Theta(\tau)\}} \leq b_{\max}^2 \dot A}$$

Response **K**inetic **U**ncertainty **R**elation (R-KUR)

$$\frac{(\partial_\theta \langle J \rangle)^2}{\langle\langle J \rangle\rangle} \geq \frac{b_{\max}^2 \dot\Sigma}{2}$$ 🤔

R-TUR
for infinite observation times

Ptaszyński/Aslyamov/Esposito, PRL (2024)

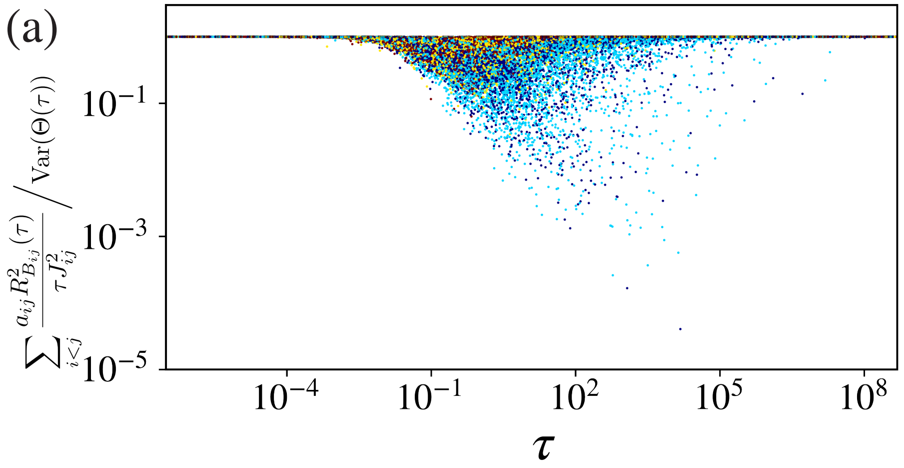
Response uncertainty relations

$$\frac{R_{B_{ij}}(\tau)}{R_{F_{ij}}(\tau)} = \frac{2\mathcal{J}_{ij}}{a_{ij}} \quad \Rightarrow$$

$$\sum_{i<j}\frac{4R^2_{F_{ij}}(\tau)}{\tau a_{ij}} \le \mathrm{Var}\{J(\tau)\}$$

$$\updownarrow$$

$$\sum_{i<j}\frac{a_{ij}R^2_{B_{ij}}(\tau)}{\tau\mathcal{J}^2_{ij}} \le \mathrm{Var}\{J(\tau)\}$$

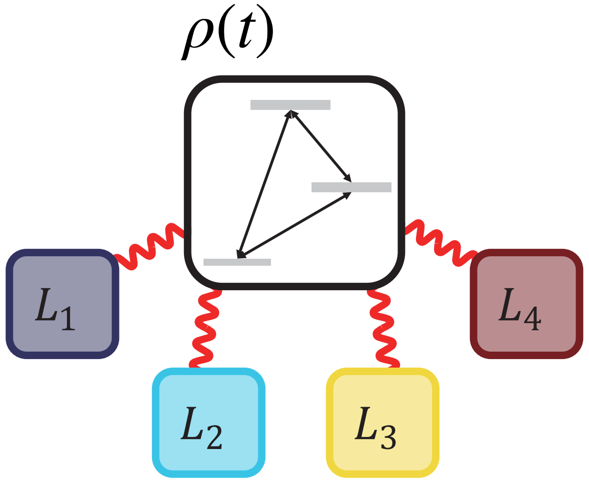


cf.) $\sum_{i<j}\frac{R^2_{B_{ij}}(\tau)}{\tau a_{ij}} \le \mathrm{Var}\{J(\tau)\}$

$$\tau\mathrm{Var}\{\Theta(\tau)\} \ge \frac{R^2_\theta(\tau)}{b^2_{\max}\underbrace{\sum_{i<j}\mathcal{J}^2_{ij}/a_{ij}}_{\le\min\{\dot{A},\dot{\Sigma}/2\}}} \longrightarrow \boxed{\frac{R^2_\theta(\tau)}{\tau\mathrm{Var}\{\Theta(\tau)\}} \le \frac{b^2_{\max}}{2}\min\{\dot{\Sigma},\, 2\dot{A}\}}$$

R-TKUR for finite observation times

Open quantum system



Lindblad master equation

$$\dot{\rho}(t) = -i[H, \rho(t)] + \sum_{k=1}^{K} \underbrace{\left(L_k \rho(t) L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho(t) - \frac{1}{2} \rho(t) L_k^\dagger L_k \right)}_{\mathscr{D}(L_k)\rho(t)} \equiv \mathscr{L}\rho(t)$$

$$L_k \mapsto L_k^{\theta_k} = e^{\theta_k/2} L_k$$

$$\uparrow$$

quantum jump unravelling

$$\begin{cases} M_k = \sqrt{dt} L_k & \text{jump of type } k \\ M_0 = 1 - iH_{\text{eff}} dt & \text{no jump} \end{cases}$$

$$^*H_{\text{eff}} = H - (i/2) \sum_k L_k^\dagger L_k$$

observables of interest: $\Theta(\tau) = \sum_{k=1}^{K} \Lambda_k N_k(\tau)$

Quantum Fisher information

$$\sum_{\alpha,\beta} R_{\theta_\alpha}(\tau)[\mathrm{I}^{-1}(\tau)]_{\theta_\alpha\theta_\beta} R_{\theta_\beta}(\tau) \le \mathrm{Var}\{\Theta(\tau)\} \quad \text{where } \mathrm{I}_{\theta_\alpha\theta_\beta}(\tau) = -\left\langle \partial_{\theta_\alpha}\partial_{\theta_\beta} \ln \mathscr{P}[\{s(t)\}_{t=0}^{\tau}] \right\rangle_0$$

$$\mathscr{P}[\{s(t)\}_{t=0}^{\tau}] = \mathrm{tr}\left(\mathscr{M}_{\alpha_N}\cdots\mathscr{M}_{\alpha_2}\mathscr{M}_{\alpha_1}\rho_{\mathrm{ss}}\right) \quad \text{with } \mathscr{M}_\alpha \bullet = M_\alpha \bullet M_\alpha^\dagger$$

dependence on POVM

Lindblad equation

- jump unraveling
- homodyne unraveling
- heterodyne unraveling
- arbitrary POVM

$$\mathrm{I}(\tau) \le \mathrm{I}_Q(\tau): \ \text{quantum Fisher info}.$$

$$\le \mathrm{I}_Q^{\mathrm{global}}(\tau)$$

$$D_B(\rho_{\boldsymbol{\theta}}, \rho_{\boldsymbol{\theta}+d\boldsymbol{\theta}}) \approx \frac{1}{4}\sum_{i,j}[I_Q(\theta)]_{ij} d\theta_i d\theta_j$$

$$D_B(\rho,\sigma) = 2[1-\sqrt{F(\rho,\sigma)}] \quad \text{where } F(\rho,\sigma) = (\mathrm{tr}\{\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\})^2$$

Bures distance

fidelity

$$|\psi;\boldsymbol{\theta}\rangle = U_{t_{N-1}}\cdots U_{t_0}|\psi_S^0\rangle \otimes |\mathrm{init.\ env.}\rangle = \sum_{\alpha_0,\cdots\alpha_N} M_{\alpha_N}\cdots M_{\alpha_0}|\psi_S^0\rangle \otimes |\mathrm{measured\ env.}\rangle$$

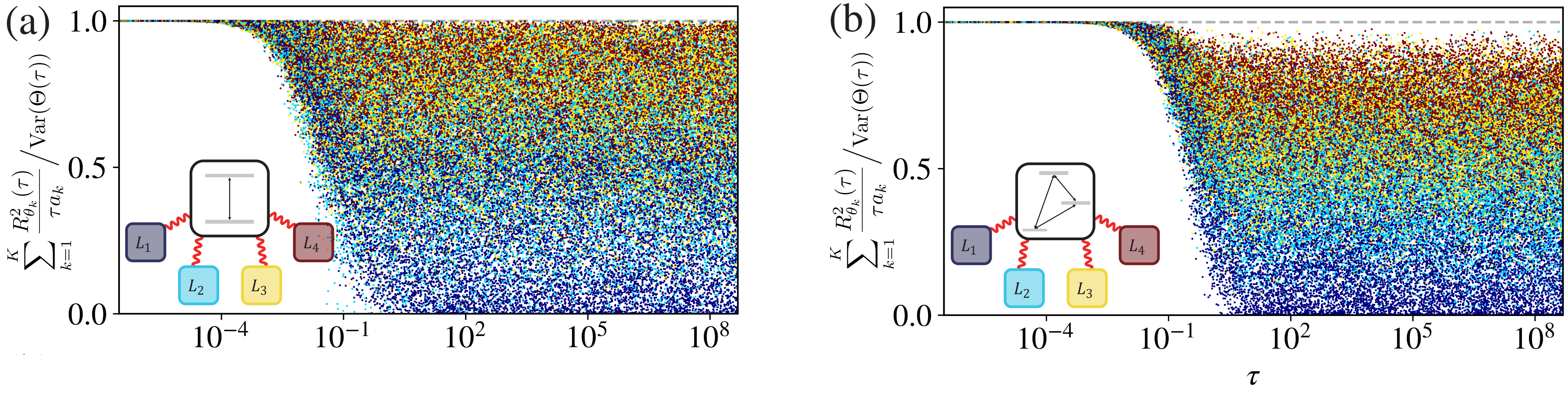
Quantum fluctuation-response inequality

$$[\mathbf{I}_Q^{\mathrm{global}}(\tau)]_{\theta_\alpha\theta_\beta} = 4\partial_\alpha^1\partial_\beta^2 \ln\{\mathrm{tr}[\rho_{\boldsymbol{\theta}^1\boldsymbol{\theta}^2}(\tau)]\}_{\boldsymbol{\theta}^1=\boldsymbol{\theta}^2=\boldsymbol{\theta}} = \cdots = \cdots = \cdots = \cdots = \tau\delta_{\alpha\beta}\underbrace{\mathrm{tr}[L_\alpha^{\theta_\alpha}\rho(L_\alpha^{\theta_\alpha})^\dagger]}_{=a_\alpha}$$

$$\Big\uparrow$$

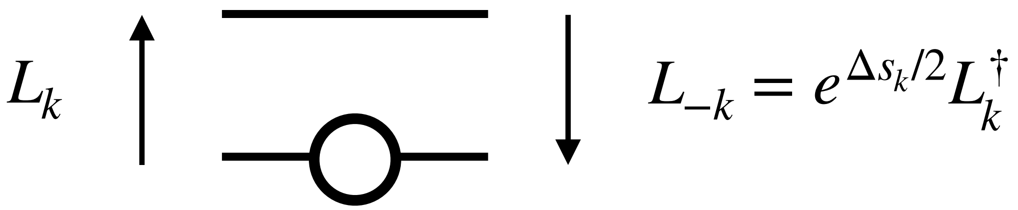
$$\dot{\rho}_{\boldsymbol{\theta}_1\boldsymbol{\theta}_2} = -i[H,\rho(t)] + \sum_{k=1}^{K}\underbrace{\left(L_k^{\theta_k^1}\rho(t)(L_k^{\theta_k^2})^\dagger - \frac{1}{2}(L_k^{\theta_k^1})^\dagger L_k^{\theta_k^1}\rho(t) - \frac{1}{2}\rho(t)(L_k^{\theta_k^2})^\dagger L_k^{\theta_k^2}\right)}_{\mathscr{D}(L_k)\rho(t)}$$

$$\boxed{\sum_k \frac{R_{\theta_k}^2(\tau)}{\tau a_k} \le \mathrm{Var}\{\Theta(\tau)\}} \xrightarrow{\theta_k = \theta_k(\epsilon)} \frac{R_{\theta_\epsilon}^2(\tau)}{\mathrm{Var}\{\Theta(\tau)\}} \le \tau(\Delta\theta_{\max})^2\dot{A}$$: quantum R-KUR



A, L_k : random matrices, $H = (A + A^\dagger)/2$

Any relation to dissipation?

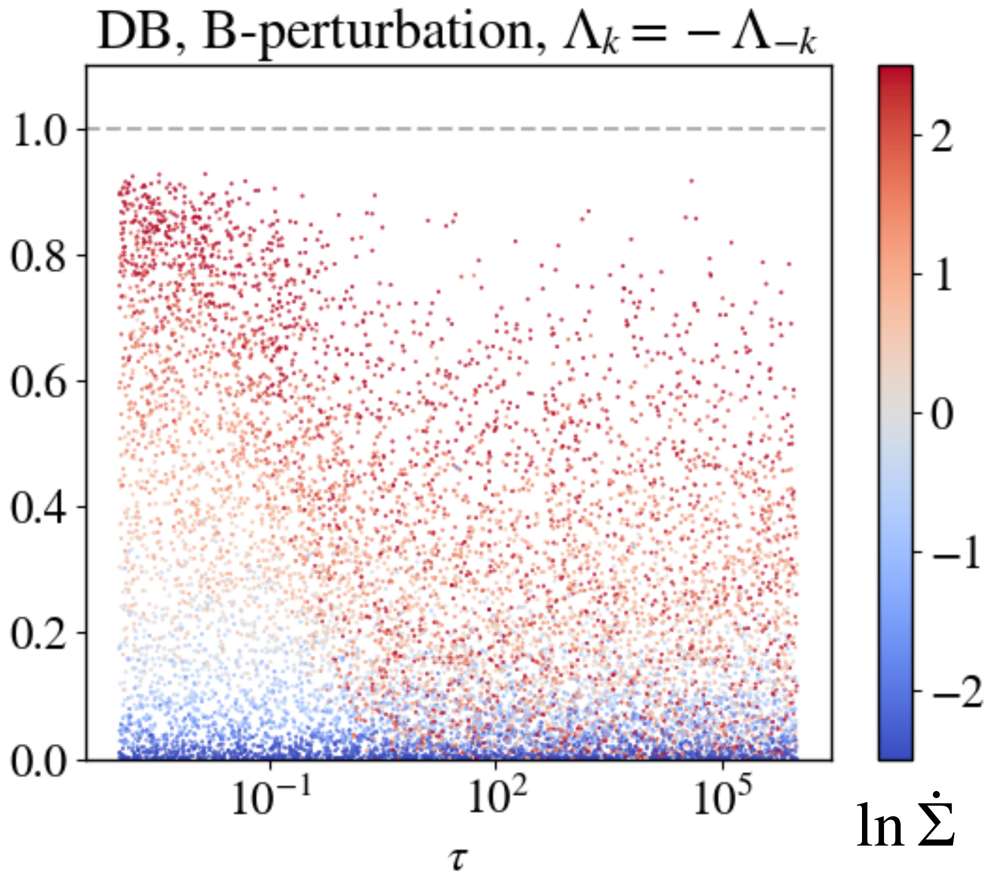


L_k

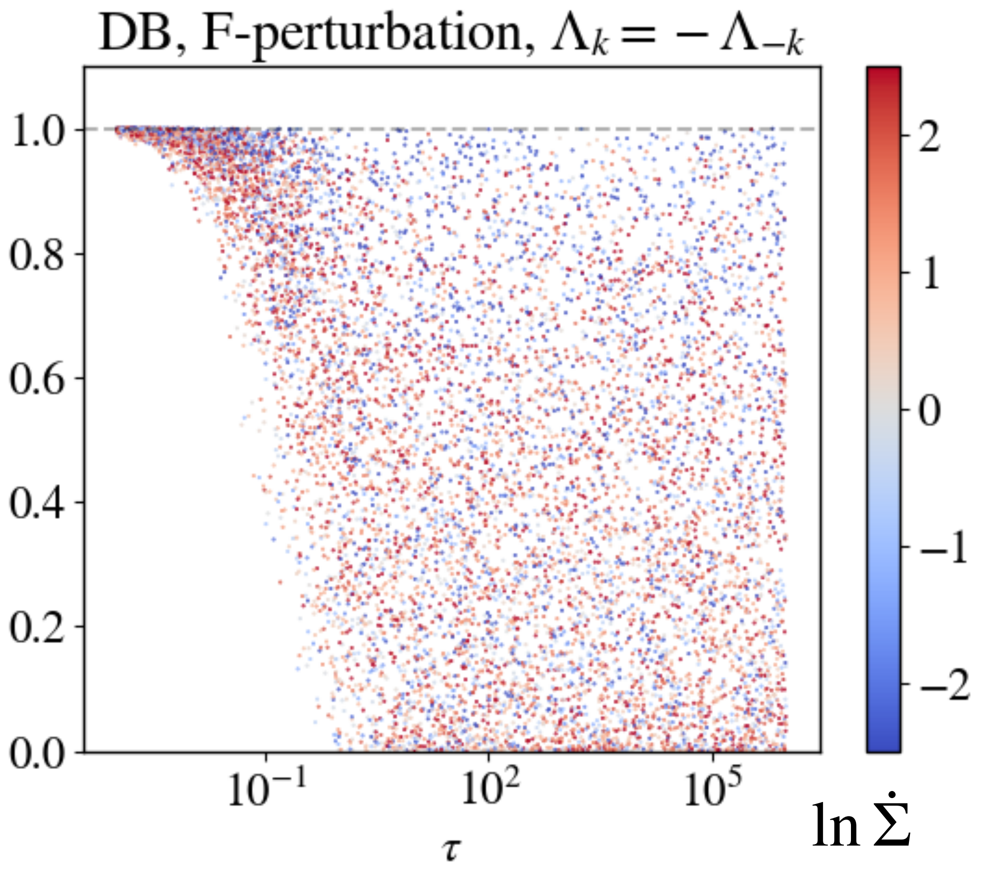
$L_{-k} = e^{\Delta s_k/2} L_k^\dagger$

$$\sum_k \frac{R^2_{\theta_k}(\tau)}{\tau a_k} \le \mathrm{Var}\{J(\tau)\} \xrightarrow{\text{CS ineq.}} \textbf{R-KUR}$$

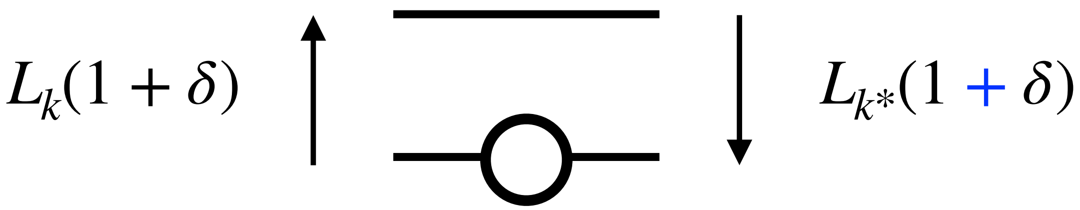
$$J(\tau) = \sum_{k>0} \Lambda_k \{N_k(\tau) - N_{-k}(\tau)\}$$



DB, B-perturbation, $\Lambda_k = -\Lambda_{-k}$

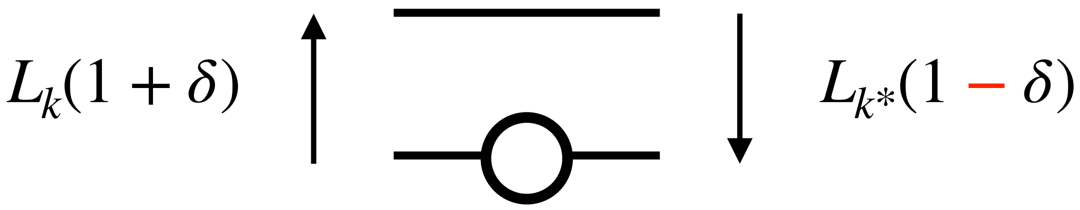


DB, F-perturbation, $\Lambda_k = -\Lambda_{-k}$



$L_k(1+\delta)$

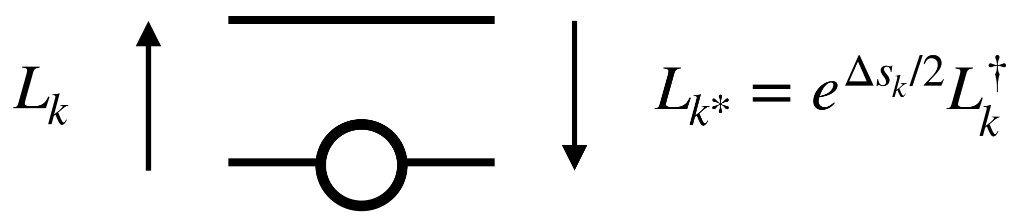
$L_{k^*}(1+\delta)$



$L_k(1+\delta)$

$L_{k^*}(1-\delta)$

Any relation to dissipation?



$$J(\tau) = \sum_{k>0} \Lambda_k \{N_k(\tau) - N_{-k}(\tau)\}$$

$$\sum_k \frac{R_{F_k}^2(\tau)}{\tau a_k} \leq \mathrm{Var}\{J(\tau)\} \xrightarrow{\text{CS ineq.}} \textbf{R-KUR}$$

$$\Big\downarrow \quad \text{cf. } R_{B_k}(\tau)/R_{F_k}(\tau) = 2\mathcal{J}_k/a_k \text{ for CL}$$

$$\sum_k \frac{a_k R_{B_k}^2(\tau)}{\tau \mathcal{J}_k^2} \ \textbf{?} \leq \mathrm{Var}\{J(\tau)\} \xrightarrow{\text{CS ineq.}} \textbf{R-TUR ?}$$

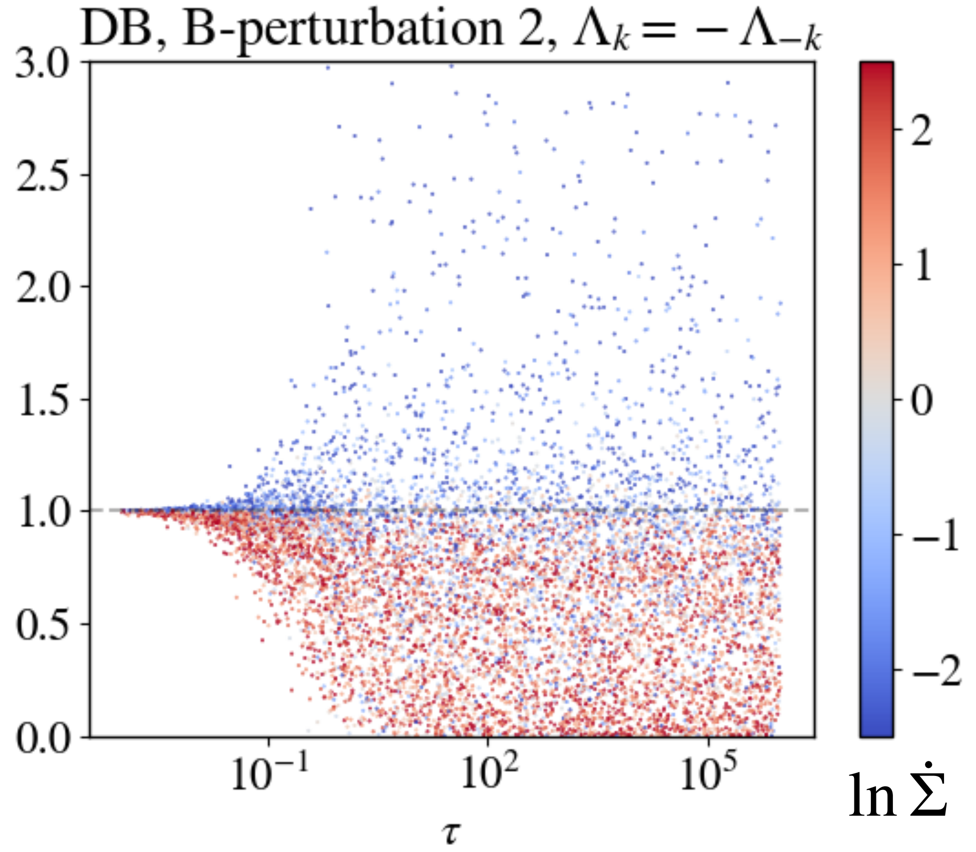
quantum TURs

$$\frac{\mathrm{var}[J]}{\langle J\rangle^2} \geq \frac{2(1+\tilde{\delta J})^2}{\Sigma_\tau + 2\mathcal{Q}_1}, \qquad (12)$$

Van Vu/Saito, PRL (2022)

$$\frac{F_\phi}{(1+\delta_\phi)^2} \geq \frac{4a}{\sigma^2}\Phi\left(\frac{\sigma}{2a}\right)^2 \geq \max\left(\frac{2}{\sigma}, \frac{1}{a}\right), \qquad (23)$$

Van Vu, PRX Quantum (2025)


DB, B-perturbation 2, $\Lambda_k = -\Lambda_{-k}$

Summary

Classical systems

FRIs

$$\sum_{i<j} \frac{a_{ij} R^2_{B_{ij}}(\tau)}{\tau \mathscr{J}^2_{ij}} \leq \mathrm{Var}\{J(\tau)\}$$

$$\sum_{i<j} \frac{4R^2_{F_{ij}}(\tau)}{\tau a_{ij}} \leq \mathrm{Var}\{J(\tau)\}$$

↓

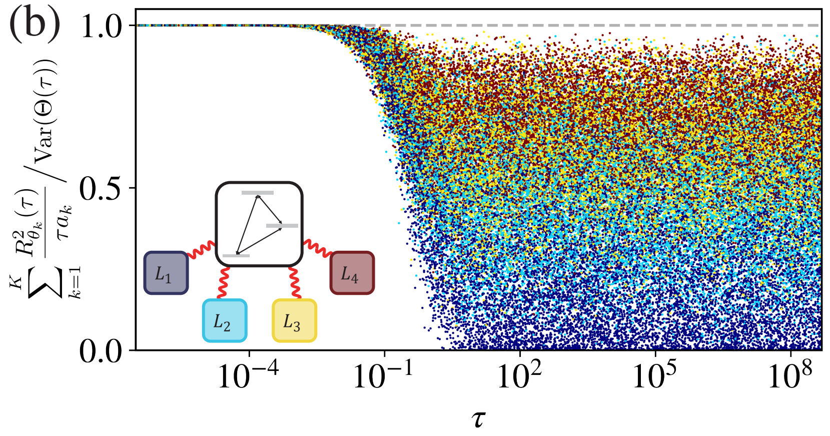
R-TKUR

$$\frac{R^2_{\theta}(\tau)}{\mathrm{Var}\{J(\tau)\}} \leq \frac{\tau b^2_{\max}}{2} \min\{\dot{\Sigma}, 2\dot{A}\}$$

Open quantum systems

FRI

$$\sum_{k} \frac{R^2_{\theta_k}(\tau)}{\tau a_k} \leq \mathrm{Var}\{\Theta(\tau)\}$$



↓

R-KUR

$$\frac{R^2_{\epsilon}(\tau)}{\mathrm{Var}\{\Theta(\tau)\}} \leq \tau(\Delta\theta_{\max})^2 \dot{A}$$

Thank you for your attention!

Reference: Physical Review Letters **135**, 097101 (2025)