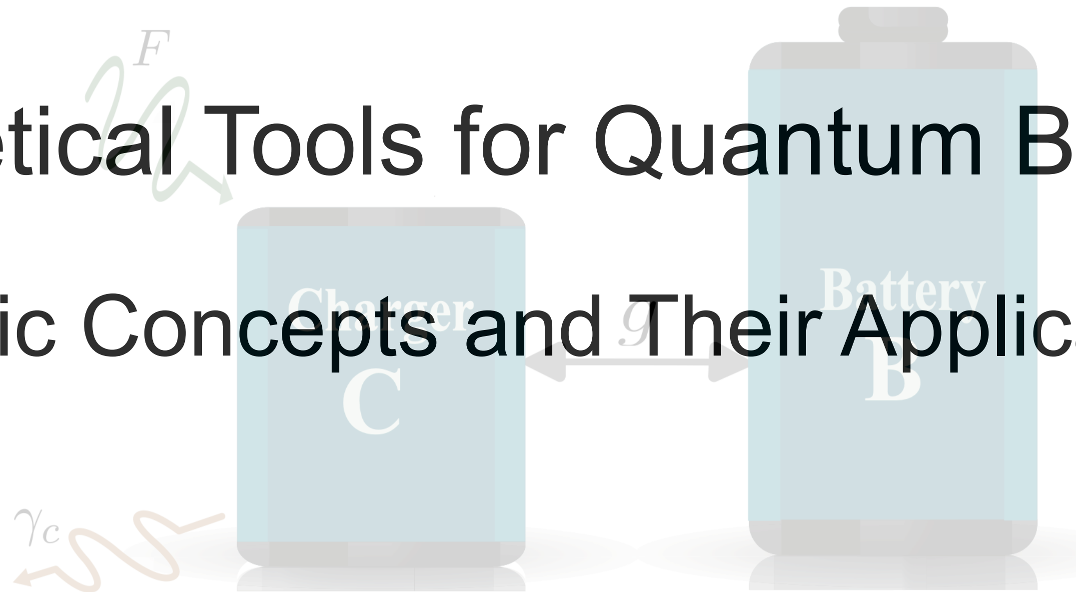
Theoretical Tools for Quantum Batteries:

Basic Concepts and Their Applications



Lecture 3

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Lecture 3: Dicke model—A workhorse of QB research

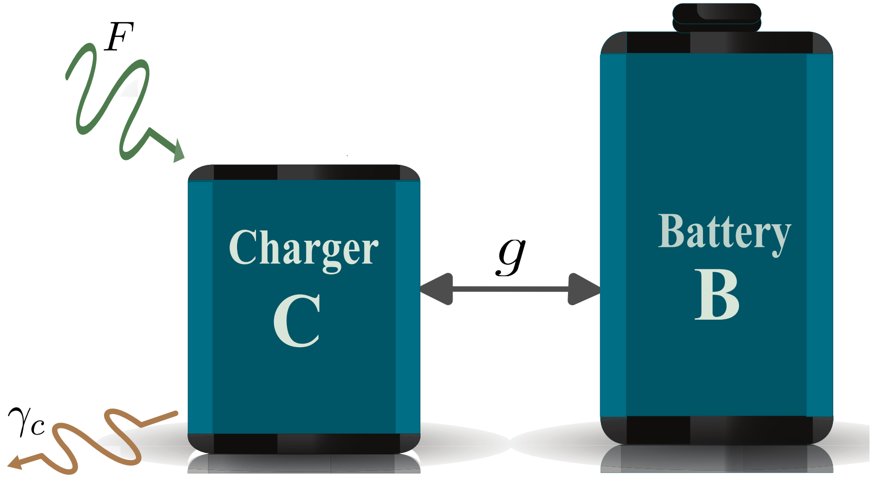
A paradigm model for QB research

Two-level sys. (TLS) & harmonic oscillator (HO):

Simplest quantum systems with finite & infinite dimensions.

A composite sys. consisting of a TLS & a HO:

A prototype system for a charger-battery setup.



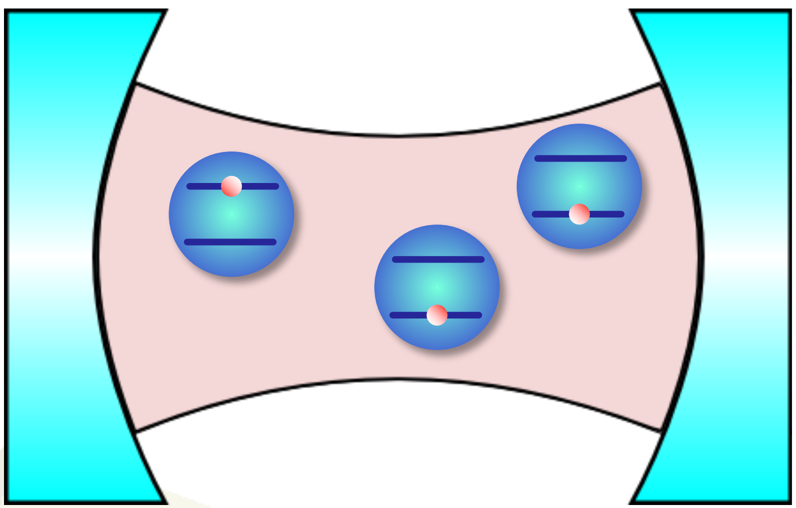
Widely used for proof-of-principle studies of QBs.

A “workhorse” of quantum battery research.

Dicke model

Dicke, Phys. Rev. **93**, 99 (1954)

Dicke model: A quantum model describing light-matter interaction in an optical cavity with atoms.



System:

A collection of N two-level atoms

+

A single-mode photon field in a cavity.

$$\hat{H}_{\mathrm{Dicke}} = \omega_c \hat{a}^\dagger \hat{a} + \frac{\omega_z}{2}\sum_{i=1}^{N} \hat{\sigma}_i^z + \frac{g}{\sqrt{N}}\sum_{i=1}^{N} \hat{\sigma}_i^x (\hat{a} + \hat{a}^\dagger)$$

cavity photons — atoms — atom-photon int. (dipole int.)

Remarks

$$\hat{H}_{\text{Dicke}} = \omega_c \hat{a}^\dagger \hat{a} + \frac{\omega_z}{2}\sum_{i=1}^{N} \hat{\sigma}_i^z + \frac{g}{\sqrt{N}}\sum_{i=1}^{N} \hat{\sigma}_i^x (\hat{a} + \hat{a}^\dagger)$$

- Both the light & atoms are treated quantum mechanically.

- Motional DOFs of atoms are ignored.

  Atoms are treated as immobile electric dipoles described by a TLS with $\{|g\rangle, |e\rangle\}$.

- All atoms are assumed to be localized in a small region of $\ll \lambda$.

  (λ: wavelength of cavity photons)

- The factor $1/\sqrt{N}$ in the int. term guarantees the well-defined thermodynamic limit. ("Kac scaling" or "Dicke scaling")

  Keeping $\langle \hat{H} \rangle / N$ finite in the limit of $N \to \infty$. ($\because \langle \hat{a} \rangle \sim \sqrt{N}$)

  Corresponding to keeping the atomic density N/L in the cavity fixed in the limit of $N \to \infty$.

Derivation of $\widehat{H}_{\rm int}$: Dipole interaction

Standing $\boldsymbol{E}$-field (along z-direction):

$$\widehat{\boldsymbol{E}}(z) = \mathbb{e}\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}\left(\hat{a}+\hat{a}^\dagger\right)\sin kz = \boldsymbol{\mathcal{E}}(z)\left(\hat{a}+\hat{a}^\dagger\right)$$

Consider an atom interacting with the $\boldsymbol{E}$-field through its electric dipole moment $\widehat{\boldsymbol{d}} \equiv q\,\hat{\mathbf{r}}$:

$$\widehat{H}_{\rm int} = -\widehat{\boldsymbol{d}}\cdot\widehat{\boldsymbol{E}} = -\underbrace{\widehat{\boldsymbol{d}}\cdot\mathbb{e}}\;\mathcal{E}(z)\left(\hat{a}+\hat{a}^\dagger\right)$$

$\hat{d}_\parallel$: parallel component along $\mathbb{e}$.

Since the dipole mom. ($\widehat{\boldsymbol{d}} \equiv q\,\hat{\mathbf{r}}$) has the odd parity,

$$\langle g|\widehat{\boldsymbol{d}}|g\rangle = \langle e|\widehat{\boldsymbol{d}}|e\rangle = 0.$$

$$\Rightarrow\quad \hat{d}_\parallel = \left(d_\parallel\,|g\rangle\langle e| + d_\parallel^*\,|e\rangle\langle g|\right) = \left(d_\parallel\,\hat{\sigma}_- + d_\parallel^*\,\hat{\sigma}_+\right)$$

For simplicity, assuming $d_\parallel$ is real: $\hat{d}_\parallel = d_\parallel(\hat{\sigma}_- + \hat{\sigma}_+) = d_\parallel\,\hat{\sigma}_x$

$$\Rightarrow\quad \boxed{\widehat{H}_{\rm int} = -\,\hat{d}_\parallel\,\mathcal{E}\left(\hat{a}+\hat{a}^\dagger\right) \equiv g\,\hat{\sigma}_x\left(\hat{a}+\hat{a}^\dagger\right)}$$

Rotating-wave approximation (RWA)

In the interaction picture:

$$\hat{a}(t) = e^{i\omega_c \hat{a}^\dagger \hat{a} t}\, \hat{a}\, e^{-i\omega_c \hat{a}^\dagger \hat{a} t} = \hat{a}\, e^{-i\omega_c t}$$

$$\hat{a}^\dagger(t) = e^{i\omega_c \hat{a}^\dagger \hat{a} t} \hat{a}^\dagger\, e^{-i\omega_c \hat{a}^\dagger \hat{a} t} = \hat{a}^\dagger\, e^{i\omega_c t}$$

$$\hat{\sigma}_\pm(t) = e^{i\frac{\omega_z}{2}\hat{\sigma}_z t}\, \hat{\sigma}_\pm\, e^{-i\frac{\omega_z}{2}\hat{\sigma}_z t} = \hat{\sigma}_\pm\, e^{\pm i\omega_z t}$$

$$\hat{H}_{\mathrm{int}} = g\,\hat{\sigma}_x \left(\hat{a} + \hat{a}^\dagger\right) = g\,(\hat{\sigma}_- + \hat{\sigma}_+)\left(\hat{a} + \hat{a}^\dagger\right)$$

$$\left.\begin{array}{l} \hat{\sigma}_+\hat{a} \sim e^{i(\omega_z - \omega_c)t} \\ \hat{\sigma}_-\hat{a}^\dagger \sim e^{-i(\omega_z - \omega_c)t} \end{array}\right\}$$ Slowly rotating when $\omega_z \simeq \omega_c$.

$$\left.\begin{array}{l} \hat{\sigma}_+\hat{a}^\dagger \sim e^{i(\omega_z + \omega_c)t} \\ \hat{\sigma}_-\hat{a} \sim e^{-i(\omega_z + \omega_c)t} \end{array}\right\}$$ Rapidly rotating when $\omega_z \simeq \omega_c$.

→ Negligible in coarse-grained time scale.

$(\hat{\sigma}_- + \hat{\sigma}_+)\left(\hat{a} + \hat{a}^\dagger\right) \simeq \hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger$: rotating-wave approx. (RWA)

Retain the terms that conserve the “excitation number”.

Single atom in a cavity

$$\hat{H}_{\mathrm{Rabi}} = \omega_c \hat{a}^\dagger \hat{a} + \frac{\omega_z}{2}\hat{\sigma}_z + g\,(\hat{\sigma}_- + \hat{\sigma}_+)\left(\hat{a} + \hat{a}^\dagger\right) \quad \text{(Rabi model)}$$

$$\hat{H}_{\mathrm{JC}} = \omega_c \hat{a}^\dagger \hat{a} + \frac{\omega_z}{2}\hat{\sigma}_z + g\,\left(\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger\right)$$

(Jaynes-Cummings model)

(Integrable: $\hat{N} \equiv \hat{a}^\dagger \hat{a} + \frac{1}{2}(\hat{\sigma}_z + 1)$ is conserved)

N atoms in a cavity

$$\hat{H}_{\mathrm{Dicke}} = \omega_c \hat{a}^\dagger \hat{a} + \frac{\omega_z}{2}\sum_{i=1}^{N} \hat{\sigma}_i^z + \frac{g}{\sqrt{N}}\sum_{i=1}^{N} \hat{\sigma}_i^x \left(\hat{a} + \hat{a}^\dagger\right)$$

(Dicke model)

$$\hat{H}_{\mathrm{TC}} = \omega_c \hat{a}^\dagger \hat{a} + \frac{\omega_z}{2}\sum_{i=1}^{N} \hat{\sigma}_i^z + \frac{g}{\sqrt{N}}\sum_{i=1}^{N} \left(\hat{\sigma}_i^+ \hat{a} + \hat{\sigma}_i^- \hat{a}^\dagger\right)$$

(Tavis-Cummings model)

(Integrable: $\hat{N} \equiv \hat{a}^\dagger \hat{a} + \frac{1}{2}\sum_{i=1}^{N} \hat{\sigma}_i^z + \frac{N}{2}$ & $\hat{S}^2$ are conserved)

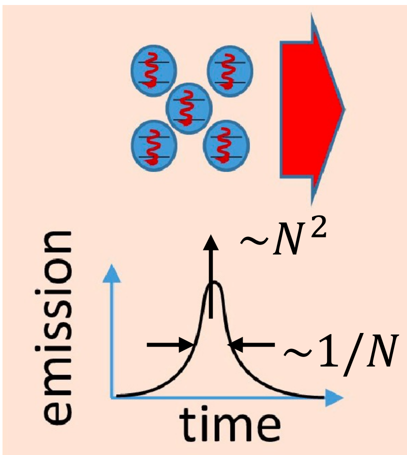
Family of the Dicke model : Summary

RWA	Yes	No
1 atom	Jaynes-Cummings	Rabi
N atoms	Tavis-Cummings	Dicke

Integrable

Superradiance

Dicke superrad.: Cooperative & phase-coherent spontaneous emission from a collection of indistinguishable atoms coupled to a common radiation mode in free space.



Roses & Dalla Torre,
PLoS ONE **15**, e0235197 (2020)

Atoms indistinguishable from the viewpoint of the radiation field.

→ Each acts as an identical radiation center.

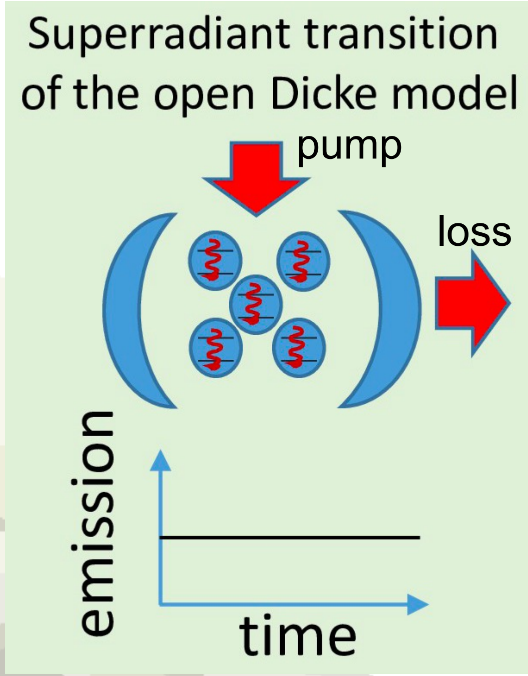
→ Phase synchronization & constructive interf.

Radiation field amplitude $\propto N$ → Intensity $I \propto N^2$

(cf. $I \propto N$ for indep. atoms)

Superradiant phase transition:

Steady st. of a collection of indistinguishable atoms in an optical cavity under pump & loss exhibits a nonzero photon population $\langle \hat{a}^\dagger \hat{a} \rangle \neq 0$ when the coupling exceeds a critical value ($g \geq g_c$).



Roses & Dalla Torre,
PLoS ONE **15**, e0235197 (2020)

Dicke superradiance & superradiant phase transition are distinct phenomena, yet closely related.

$\langle \hat{a} \rangle \sim \sqrt{N}$ in superradiant phase (see later)

(cf. $E \sim N$ in the Dicke superradiance)

MF analysis of superradiant phase transition

Assumption: Photon field is in a coherent st. $|\alpha\rangle$ with

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

Mean-fied Hamiltonian:

$$\hat{H}_{\mathrm{Dicke}} \longrightarrow \hat{H}^{\mathrm{MF}}_{\mathrm{Dicke}} \equiv \omega_c|\alpha|^2 + \frac{\omega_z}{2}\sum_{i=1}^{N}\hat{\sigma}_i^z + \frac{g}{\sqrt{N}}\sum_{i=1}^{N}\hat{\sigma}_i^x(\alpha+\alpha^*)$$

We can take $\alpha =$ real without loss of generality.

$$= \omega_c\alpha^2 + \frac{\omega_z}{2}\sum_{i=1}^{N}\hat{\sigma}_i^z + \frac{2g\alpha}{\sqrt{N}}\sum_{i=1}^{N}\hat{\sigma}_i^x$$

partition func.: $Z(\alpha) = \mathrm{Tr}\, e^{-\beta\hat{H}^{\mathrm{MF}}_{\mathrm{Dicke}}}$

$$= e^{-\beta\omega_c\alpha^2}\prod_i \mathrm{Tr}_i\, e^{-\beta\left(\frac{\omega_z}{2}\hat{\sigma}_i^z + \frac{2g\alpha}{\sqrt{N}}\hat{\sigma}_i^x\right)}$$

$$= e^{-\beta\omega_c\alpha^2}\left\{\mathrm{Tr}\exp\left[-\beta\underbrace{\left(\frac{\omega_z}{2}\hat{\sigma}_z + \frac{2g\alpha}{\sqrt{N}}\hat{\sigma}_x\right)}_{\text{single-atom Hamiltonian: } \hat{h}(\alpha)}\right]\right\}^N$$

single-atom Hamiltonian: $\hat{h}(\alpha)$

MF analysis of superradiant phase transition

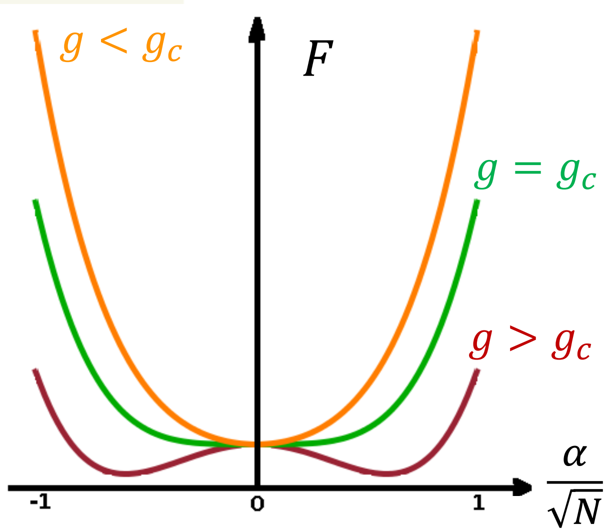
Single-atom Hamiltonian: $\hat{h}(\alpha) \equiv \frac{\omega_z}{2}\,\hat{\sigma}_z + \frac{2g\alpha}{\sqrt{N}}\,\hat{\sigma}_x$

Eigenvalues $\pm E$ of $\hat{h}(\alpha)$: $E = \sqrt{\frac{{\omega_z}^2}{4} + \frac{4g^2}{N}\alpha^2}$

$$\mathrm{Tr}e^{-\beta\hat{h}(\alpha)} = 2\cosh\beta E$$

$$Z(\alpha) = e^{-\beta\omega_c\alpha^2}\left(\mathrm{Tr}e^{-\beta\hat{h}(\alpha)}\right)^N = e^{-\beta\omega_c\alpha^2}(2\cosh\beta E(\alpha))^N$$

$$F(\alpha) = -\beta^{-1}\ln Z(\alpha) = \omega_c\alpha^2 - \beta^{-1}N\ln(2\cosh\beta E(\alpha))$$

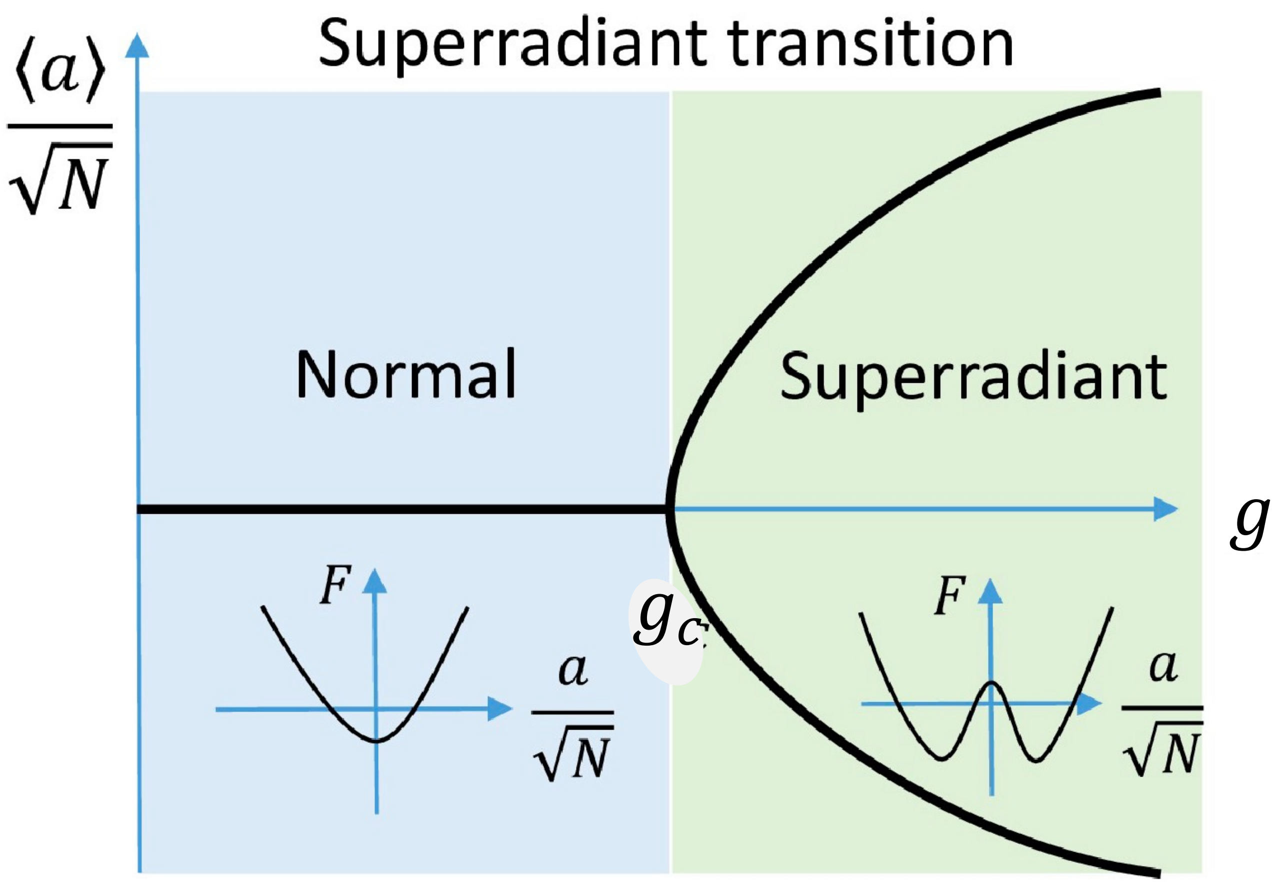


Minima of $F(\alpha)$: Equilibrium st.

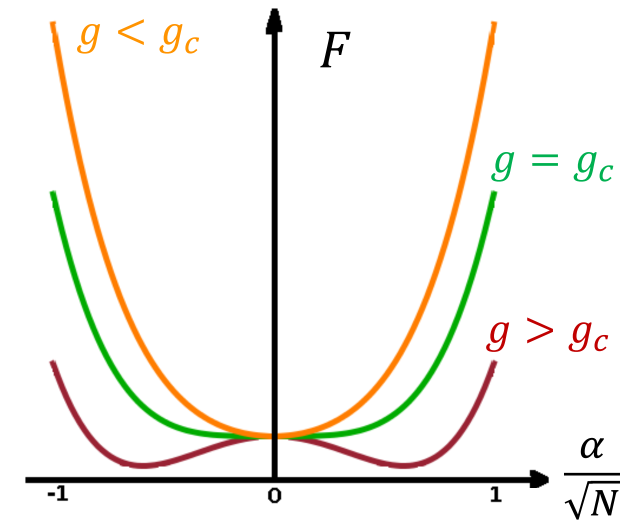
critical coupling g_c : $\frac{\partial^2 F(0)}{\partial\alpha^2} = 0$

$$g_c = \frac{1}{2}\sqrt{\omega_c\omega_z\coth\frac{\beta\omega_z}{2}}$$

MF phase diagram



Roses & Dalla Torre, PLoS ONE **15**, e0235197 (2020)



$$g_c = \frac{1}{2}\sqrt{\omega_c \omega_z \coth\frac{\beta\omega_z}{2}}$$

For $g > g_c$: $\alpha \equiv \langle \hat{a} \rangle \neq 0$ and $\alpha \sim \sqrt{N}$. (superradiant phase)

If no $1/\sqrt{N}$-scaling in $\hat{H}_{\mathrm{int}}$. ➡ $g_c \to \sim 1/\sqrt{N}$

SR phase at $\forall g$ in $N \to \infty$.

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