Ultracold atomic quantum simulator

1. Bose-Einstein condensate and saturation in the excited state

(a) Consider N number of identical particles that has only two levels. In thermal equilibrium, show that the excited state population N_e is given by

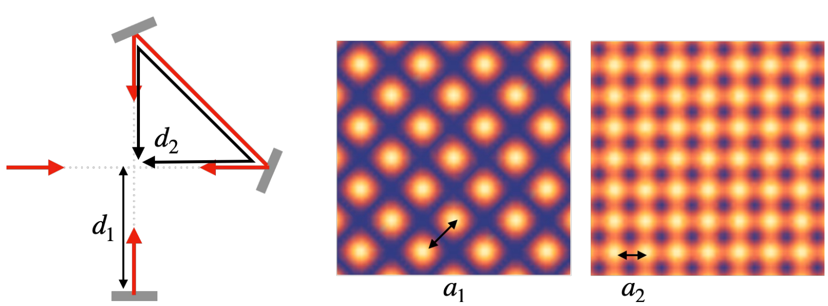
$$N_e = \frac{p}{1-p} - \frac{(N+1)p^{N+1}}{1-p^{N+1}}$$

(b) Real particles are unbounded level spectrum, and in three dimensions, they can display Bose-Einstein condenation (BEC) at a critical temperature. Show that the BEC cannot possible in two-dimensional non-interacting Bose gases.

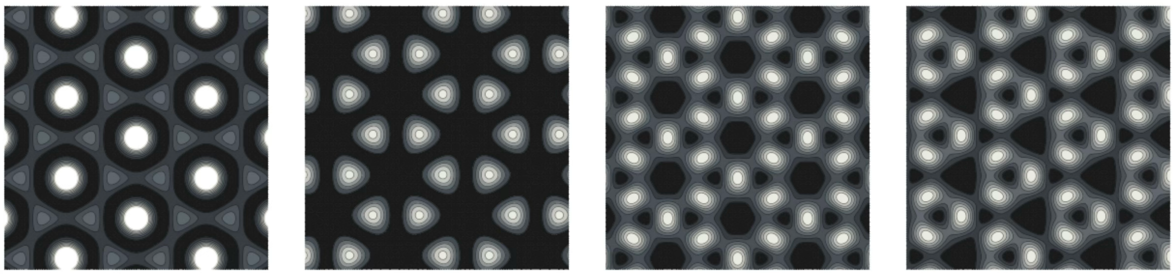
(c) When we trap the particles in a two-dimensional isotropic harmonic potential, $V(x, y) = \frac{1}{2}m\omega^2(x^2 + y^2)$ can the BEC occur? If so, what is the critical temepratur e?

2. Making optical lattices

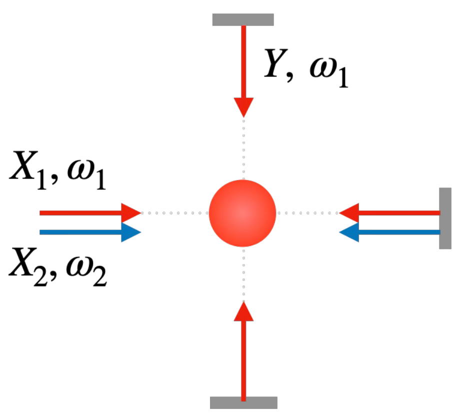
(a) Consider a two-dimensional lattice formed by interfering four laser beams. By adjusting the polarization of the laser light, you can make two-dimensional square lattice geometry at different lattice spacing (below figure). Find the laser condition for each configuration. If the distance d_1 and d_2 changed, how the lattice structure change?



(b) Devise an optical system that can generate a lattice with 3-fold or 6-fold symmetry. For example, we want to have change lattice geometries from triangular, Honeycomb, Kagome, canted-triangular by tuning a polarization or relative phase of the laser beams.

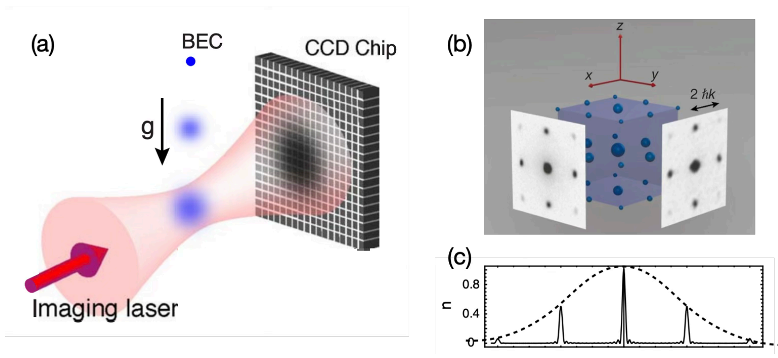


(c) Consider the following laser beam configuration: two laser beams (X_1 and X_2) along the *x*-axis and one laser beam (Y) along the *y*-axis. Both laser beams have linear polarization at the *z*-axis and are retro-reflected from mirrors at normal incidence (assuming the mirrors have a reflectance, R=1). Here, we use the same light frequency for X_1 and Y beam ω_1, and the laser beam X2 has a different frequency $\omega_2 = \omega_1 + \Delta\omega$, with $\Delta\omega = 2\pi \times 300$ MHz. The difference of the light frequency is much faster than the atomic motion, such that you may assume they feel the time-averaged potential of the laser light X_1 and X_2 in interference. List up all possible lattice geometry that you can find in this configuration.



3. Probing superfluid phase in optical lattices

As discussed in the lecture, the time-of-flight images reflect the first-order coherence of many-body wave function. Assume that a superfluid phase of atomic could is prepared in a shallow cubic optical lattice with its spacing a_{lat}=0.532 nm. Suddenly turning off the lattice beam, they expand in a free space. After ~ 20 *ms* of free expansion we obtained the Bragg peaks [below figure (b)]. The cross-section of the density profile [figure (c)] displays three length scales: peak distance, peak width, and the overall envelope of the profile (dashed line). How can you explain each length scale?



Reference: Rev. Mod. Phys. **80**, 885 (2008). Page 897-898

Before turning off the trap, you can construct the field operator From the Bloch function,

$$\hat{\psi}(\mathrm{r}) = \sum_{k,n} W_n\,(\mathbf{r} - \mathbf{r_k})\hat{b}_k,$$

Where $\hat{b}_j^\dagger(\hat{b}_j)$ is a bosonic creation (annihilation) operator at *j*-th lattice site and $W_n(\mathbf{r})$ is the Wannier wave function of the band *n*. Assuming that the atoms are in the lowest band, where the Wannier function approximately follows a 3D Gaussian envelop. During the free expansion, this gaussian envelop becomes,

$$W(\mathbf{r},t) = \frac{1}{[2\pi r_0(t)]^{3/4}} e^{-\frac{r^2}{2r_0^2(t)}}, r_0(t) = \sqrt{r_0(0) + \frac{\hbar^2 t^2}{r_0^2(0) m^2}}$$

Then, one can dertive the density operator can be written as,

$$\hat{n}(\mathbf{r},t) \simeq |W(\mathbf{r})| \sum_{j,k} \exp\left(i\frac{m}{2\hbar t}\left(-\left(\mathbf{r} - \mathbf{r}_j\right)^2 + (\mathbf{r} - \mathbf{r}_k)^2\right)\right)\hat{b}_j^\dagger \hat{b}_k.$$

In the superfluid phase the expectation value of the one-particle density matric leads to a constant value, $\langle \hat{b}_j^\dagger \hat{b}_k \rangle = n_0$, condensate density $n_0 = N_0/V$.