renormalization and heat kernel

renormalization as partial integral

$$\int_{-\infty}^{\infty} dy\, e^{-y^2/2} \quad = \quad \sqrt{2\pi}$$

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\ e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2y^2}$$

$$\epsilon = 1 \quad \Rightarrow \quad 4.96145$$

$$\epsilon = 0.1 \quad \Rightarrow \quad 6.02068$$

$$\epsilon = 0.01 \quad \Rightarrow \quad 6.25245$$

$$\epsilon = 0.001 \quad \Rightarrow \quad 6.28005$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\ e^{-\frac{1}{2}x^2-\frac{1}{2}y^2} = \left[\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}x^2}\right]^2 = 2\pi$$

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\ e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2y^2} = \int dx\ e^{-\frac{1}{2}x^2}\int dy\ e^{-\frac{1}{2}y^2(1+\epsilon x^2)}$$

using

$$
\begin{aligned}
\int_{-\infty}^{\infty} dy\, e^{-\frac{m}{2}y^2} &= \sqrt{\frac{2}{m}} \int_{-\infty}^{\infty} d\tilde{y}\, e^{-\tilde{y}^2} \\
&= \sqrt{\frac{2}{m}} \int_{0}^{\infty} d\tilde{y}^2\, \tilde{y}^{-1} e^{-\tilde{y}^2} \\
&= \sqrt{\frac{2}{m}} \int_{0}^{\infty} ds\, s^{-1/2} e^{-s} \\
&= \sqrt{\frac{2}{m}} \times \Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{2\pi}{m}}
\end{aligned}
$$

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\ e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2y^2} = \int dx\ e^{-\frac{1}{2}x^2}\int dy\ e^{-\frac{1}{2}y^2(1+\epsilon x^2)}$$

$$= \int dx\ e^{-\frac{1}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{1+\epsilon x^2}}$$

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\ e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2y^2} = \int dx\ e^{-\frac{1}{2}x^2}\int dy\ e^{-\frac{1}{2}y^2(1+\epsilon x^2)}$$

$$= \int dx\ e^{-\frac{1}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{1+\epsilon x^2}}$$

$$= \sqrt{2\pi}\int dx\ e^{-\frac{1}{2}x^2-\frac{1}{2}\log((1+\epsilon x^2))}$$

$$= \sqrt{2\pi}\int dx\ e^{-\frac{1}{2}(1+\epsilon)x^2-O(\epsilon^2x^4)}$$

textbook renormalization = selective integration + truncation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2 y^2} = \int dx\; e^{-\frac{1}{2}x^2}\int dy\; e^{-\frac{1}{2}y^2(1+\epsilon x^2)}$$

$$= \int dx\; e^{-\frac{1}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{1+\epsilon x^2}}$$

$1\,:\;$ bare

$$= \sqrt{2\pi}\int dx\; e^{-\frac{1}{2}x^2-\frac{1}{2}\log((1+\epsilon x^2))}$$

$1+\epsilon\,:\;$ renormalized

$$= \sqrt{2\pi}\int dx\; e^{-\frac{1}{2}(1+\epsilon)x^2-O(\epsilon^2 x^4)}$$

$$\simeq \sqrt{2\pi}\int dx\; e^{-\frac{1}{2}(1+\epsilon)x^2}$$

textbook renormalization = selective integration + truncation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\ e^{-\frac{1}{2}x^2-\frac{1}{2}y^2-\frac{1}{2}\epsilon x^2y^2} \qquad \simeq \sqrt{2\pi}\int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}(1+\epsilon)x^2}$$

$\epsilon = 1$	$\Rightarrow$	4.96145	4.44288
$\epsilon = 0.1$	$\Rightarrow$	6.02068	5.99078
$\epsilon = 0.01$	$\Rightarrow$	6.25245	6.25200
$\epsilon = 0.001$	$\Rightarrow$	6.28005	6.28005

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{M}{2}x^2-\frac{m}{2}y^2-\frac{\epsilon}{2}x^2y^2} = \int dx\; e^{-\frac{M}{2}x^2}\int dy\; e^{-\frac{m+\epsilon x^2}{2}y^2}$$

$$= \int dx\; e^{-\frac{M}{2}x^2}\times\frac{\sqrt{2\pi}}{\sqrt{m+\epsilon x^2}}$$

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{M}{2}x^2-\frac{1}{2}\log(1+\epsilon x^2/m)}$$

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{1}{2}(M+\epsilon/m)x^2-O((\epsilon^2/m^2)x^4)}$$

renormalization = selective integration

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{M}{2}x^2-\frac{m}{2}y^2-\frac{\epsilon}{2}x^2y^2} \quad = \int dx\; e^{-\frac{M}{2}x^2}\int dy\; e^{-\frac{m+\epsilon x^2}{2}y^2}$$

$$= \int dx\; e^{-\frac{M}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{m+\epsilon x^2}}$$

M : bare

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{M}{2}x^2-\frac{1}{2}\log(1+\epsilon x^2/m)}$$

$M+\epsilon/m$: renormalized

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{1}{2}(M+\epsilon/m)x^2-O((\epsilon^2/m^2)x^4)}$$

textbook renormalization = selective integration + truncation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy\; e^{-\frac{M}{2}x^2-\frac{m}{2}y^2-\frac{\epsilon}{2}x^2y^2} = \int dx\; e^{-\frac{M}{2}x^2}\int dy\; e^{-\frac{m+\epsilon x^2}{2}y^2}$$

$$= \int dx\; e^{-\frac{M}{2}x^2} \times \frac{\sqrt{2\pi}}{\sqrt{m+\epsilon x^2}}$$

M : bare

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{M}{2}x^2-\frac{1}{2}\log(1+\epsilon x^2/m)}$$

$M+\epsilon/m$: renormalized

$$= \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{1}{2}(M+\epsilon/m)x^2-O((\epsilon^2/m^2)x^4)}$$

$$\simeq \sqrt{\frac{2\pi}{m}}\int dx\; e^{-\frac{1}{2}(M+\epsilon/m)x^2}$$

renormalization = selective path-integral

$$\int [d\phi] e^{-\int \mathcal{L}_{\mathrm{renormalized}}(\phi)} \equiv \int [d\phi] \int [d\psi]\ e^{-\int \mathcal{L}_{\mathrm{bare}}(\phi;\psi)}$$

the momentum shell integration, more standard in quantum field theory textbook, is merely one special case

$$\phi_\Lambda = \phi_\mu + \phi_{\mu;\Lambda}$$

$$\int_{p^2<\mu^2} [d\phi_\mu]\; e^{-\int \mathcal{L}_\mu(\phi_\mu)} \;=\; \int_{p^2<\Lambda^2} [d\phi_\Lambda]\; e^{-\int \mathcal{L}_\Lambda(\phi_\Lambda)}$$

the momentum shell integration, more standard in quantum field theory textbook, is merely one special case

$$\phi_\Lambda = \phi_\mu + \phi_{\mu;\Lambda}$$

$$\int_{p^2<\mu^2} [d\phi_\mu] \boxed{e^{-\int \mathcal{L}_\mu(\phi_\mu)}} = \int_{p^2<\Lambda^2} [d\phi_\Lambda]\, e^{-\int \mathcal{L}_\Lambda(\phi_\Lambda)}$$

$$= \int_{p^2<\mu^2} [d\phi_\mu] \boxed{\int_{\mu^2<p^2<\Lambda^2} [d\phi_{\mu;\Lambda}]\, e^{-\int \mathcal{L}_\Lambda(\phi_\mu+\phi_{\mu;\Lambda})}}$$

functional determinant and functional trace

$$Q \equiv -\partial^2 + \cdots + m^2 \qquad\qquad Q|\psi_n\rangle = \lambda_n^2|\psi_n\rangle$$

$$\mathrm{Det}Q \equiv \prod_n \lambda_n^2$$

$$\log \mathrm{Det}Q = \log\left(\prod_n \lambda_n^2\right) = \sum_n \log(\lambda_n^2) = \mathrm{Tr}\log(Q)$$

log of determinant = trace of log

$$\log \mathrm{Det} Q = \mathrm{Tr} \log Q$$

$$
\begin{aligned}
-\int_{1/\Lambda^2}^{\infty} \frac{ds}{s}\ e^{-\lambda^2\cdot s} \quad &= -\int_{\lambda^2/\Lambda^2}^{\infty} \frac{d(\lambda^2\cdot s)}{\lambda^2\cdot s}\ e^{-s\cdot\lambda^2} \\
&= -\int_{\lambda^2/\Lambda^2}^{\infty} \frac{dy}{y}\ e^{-y} \\
&= -\log(y)e^{-y}\Big|_{\lambda^2/\Lambda^2}^{\infty} + \int_{\lambda^2/\Lambda^2}^{\infty} dy\ \log(y)\ e^{-y} \\
&= \ \log(\lambda^2/\Lambda^2) + O(\lambda^2/\Lambda^2)
\end{aligned}
$$

log of functional determinant of operator
= functional trace of log of operator

$$\log \mathrm{Det} Q = \mathrm{Tr} \log Q$$

$$= -\mathrm{Tr} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left[-sQ\right]$$

$$= -\sum_n \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \left\langle \psi_n \right| \exp\left[-sQ\right] \left| \psi_n \right\rangle$$

$$= -\int dx^d \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \left\langle x \right| \exp\left[-sQ\right] \left| x \right\rangle$$

UV cut-off

is there a way to incorporate the momentum shell integration, or at least some partial integration in the same spirit, as well ?

$$\int_{p^2<\mu^2} [d\phi_\mu] e^{-\int \mathcal{L}_\mu(\phi_\mu)} = \int_{p^2<\Lambda^2} [d\phi_\Lambda]\, e^{-\int \mathcal{L}_\Lambda(\phi_\Lambda)}$$

$$= \int_{p^2<\mu^2} \int_{\mu^2<p^2<\Lambda^2} [d\phi_\Lambda]\, e^{-\int \mathcal{L}_\Lambda(\phi_\Lambda)}$$

cut off the momentum integral smoothly
by cutting off the integral range of the "proper-time"

$$\log \mathrm{Det} Q = \mathrm{Tr} \log Q$$

$$= -\mathrm{Tr} \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s} \exp\left[-sQ\right]$$

$$= -\sum_n \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s} \langle \psi_n | \exp\left[-sQ\right] | \psi_n \rangle$$

$$= -\int dx^d \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s} \langle x | \exp\left[-sQ\right] | x \rangle$$

the integral is independent of small eigenvalues below the IR cut-off, so their potential contribution to the trace is effectively removed

IR cut-off

heat kernel expansion

Schwinger proper time method,
or Schwinger-de Witt method

heat kernel

$$G_s(x;y) = \langle x| \exp\left[-sQ\right] |y\rangle$$

heat kernel

$$-\frac{\partial}{\partial s} G_s(x;y) = Q G_s(x;y)$$

$$G_s(x;y) = \langle x| \exp\left[-sQ\right] |y\rangle$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s(x;y) = Q G_s(x;y) \; = \left(Q^{(0)} + Q^{(1)}\right) G_s(x;y)$$

$$Q^{(0)} + Q^{(1)} = -\partial^2 + m^2 + Q^{(1)}$$

$$G_s = G_s^{(0)} + G_s^{(1)} + G_s^{(2)} + \cdots$$

$$G_s(x;y) = \langle x | \exp\left[-sQ\right] | y \rangle$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s(x;y) = Q G_s(x;y) \; = \left(Q^{(0)} + Q^{(1)} \right) G_s(x;y)$$

$$Q^{(0)} + Q^{(1)} = -\partial^2 + m^2 + Q^{(1)}$$

$$G_s = G_s^{(0)} + G_s^{(1)} + G_s^{(2)} + \cdots$$

$$-\frac{\partial}{\partial s} G_s^{(0)}(x;y) = Q^{(0)} G_s^{(0)}(x;y)$$

$$G_s^{(0)}(x;y) = \langle x | e^{-s(-\partial^2 + m^2)} | y \rangle = \frac{1}{(4\pi s)^{d/2}} e^{-(x-y)^2/4s} e^{-sm^2} \quad \text{for } R^d$$

$$\lim_{s \to 0} G_s^{(0)}(x;y) = \delta(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial s}G_s(x;y) = QG_s(x;y) = \left(Q^{(0)} + Q^{(1)}\right)G_s(x;y)$$

$$Q^{(0)} + Q^{(1)} = -\partial^2 + m^2 + Q^{(1)}$$

$$G_s = G_s^{(0)} + G_s^{(1)} + G_s^{(2)} + \cdots$$

$$\Updownarrow$$

$$-\frac{\partial}{\partial s}G_s^{(n+1)} = Q^{(0)}G_s^{(n+1)} + Q^{(1)}G_s^{(n)}$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s^{(n+1)}(x;y) = Q^{(0)} G_s^{(n+1)}(x;y) + Q^{(1)} G_s^{(n)}(x;y)$$

$$Q^{(0)} + Q^{(1)} = -\partial^2 + m^2 + Q^{(1)}$$

$$G_s = G_s^{(0)} + G_s^{(1)} + G_s^{(2)} + \cdots$$

heat kernel expansion

$$-\frac{\partial}{\partial s}G_s^{(n+1)}(x;y) = Q^{(0)}G_s^{(n+1)}(x;y) + Q^{(1)}G_s^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial s}G_s^{(n+1)}(x;y) = Q^{(0)}G_s^{(n+1)}(x;y) + Q^{(1)}G_s^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \, G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

$$\frac{\partial}{\partial\beta}G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \, \frac{\partial}{\partial\beta}G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

$$- \lim_{s\to\beta}\int_z \, G_{\beta-s}^{(0)}(x;z)Q^{(1)}(z)G_s^{(n)}(z;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s^{(n+1)}(x;y) = Q^{(0)} G_s^{(n+1)}(x;y) + Q^{(1)} G_s^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z) Q^{(1)}(z) G_s^{(n)}(z;y)$$

$$\frac{\partial}{\partial \beta} G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z \frac{\partial}{\partial \beta} G_{\beta-s}^{(0)}(x;z) Q^{(1)}(z) G_s^{(n)}(z;y)$$

$$-Q^{(1)}(x) G_\beta^{(n)}(x;y)$$

$$\lim_{\beta \to 0} G_\beta^{(0)}(x;y) = \delta(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s^{(n+1)}(x;y) = Q^{(0)} G_s^{(n+1)}(x;y) + Q^{(1)} G_s^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z) Q^{(1)}(z) G_s^{(n)}(z;y)$$

$$\frac{\partial}{\partial \beta} G_\beta^{(n+1)}(x;y) = -Q^{(0)} \left[-\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z) Q^{(1)}(z) G_s^{(n)}(z;y) \right]$$

$$-Q^{(1)}(x) G_\beta^{(n)}(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s^{(n+1)}(x;y) = Q^{(0)} G_s^{(n+1)}(x;y) + Q^{(1)} G_s^{(n)}(x;y)$$

$$G_\beta^{(n+1)}(x;y) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z) Q^{(1)}(z) G_s^{(n)}(z;y)$$

$$\frac{\partial}{\partial \beta} G_\beta^{(n+1)}(x;y) = -Q^{(0)} \left[-\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(x;z) Q^{(1)}(z) G_s^{(n)}(z;y) \right]$$

$$-Q^{(1)}(x) G_\beta^{(n)}(x;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s(x;y) = Q G_s(x;y) \; = \left(Q^{(0)} + Q^{(1)}\right) G_s(x;y)$$

$$G_\beta^{(n)}(x;y) = -\int_0^\beta ds \int_z \; G_{\beta-s}^{(0)}(x;z) Q^{(1)}(z) G_s^{(n-1)}(z;y)$$

$$= (-1)^n \int_0^\beta ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \int_{z_1} \cdots \int_{z_n}$$

$$G_{\beta-s_1}^{(0)}(x;z_1) Q^{(1)} G_{s_1-s_2}(z_1;z_2) \cdots Q^{(1)} G_{s_n}^{(0)}(z_n;y)$$

heat kernel expansion: power counting β

1. each $G^{(0)}$ → $\beta^{-d/2}$
2. each x-integral → $\beta^{d/2}$
3. each s-integral → β

each iteration induces at least one factor of β

4. each derivative → $\beta^{-1/2}$
5. each x → $\beta^{1/2}$

$[x]^a[\partial]^b$ in $Q^{(1)}$ → $\beta^{1+(a-b)/2}$ at each iteration

heat kernel expansion

$$-\frac{\partial}{\partial s}G_s(x;y) = QG_s(x;y) \;= \left(Q^{(0)} + Q^{(1)}\right) G_s(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_s^{(0)}(x;y) = \frac{1}{(4\pi s)^{d/2}} e^{\,-(x-y)^2/4s} e^{-sm^2}$$

$$G_\beta^{(1)}(x;y) = -\int_0^\beta ds \int_z \; G_{\beta-s}^{(0)}(x;z) Q^{(1)} G_s^{(0)}(z;y)$$

$$G_\beta^{(2)}(x;y) = -\int_0^\beta ds \int_z \; G_{\beta-s}^{(0)}(x;z) Q^{(1)} G_s^{(1)}(z;y)$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s(x;y) = Q G_s(x;y) \; = \left(Q^{(0)} + Q^{(1)}\right) G_s(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu} \delta x^\mu \partial^\nu + c_{\mu\nu} \delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;y) \; = -\int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d ((\beta - s)s)^{d/2}} \int d^d z \; e^{-(x-z)^2/4(\beta-s)} Q^{(1)} e^{-(z-y)^2/4s} \right\}$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s(x;y) = Q G_s(x;y) = \left(Q^{(0)} + Q^{(1)}\right) G_s(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu} \delta x^\mu \partial^\nu + c_{\mu\nu} \delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;y) = -\int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d ((\beta - s)s)^{d/2}} \int d^d z \, e^{-(x-z)^2/4(\beta-s)} Q^{(1)} e^{-(z-y)^2/4s} \right\}$$

we must now decide what is the most useful decomposition of Q

→ we will eventually take $G_s(x;x)$ for determinant,
so Q should be expanded around this position

heat kernel expansion

$$-\frac{\partial}{\partial s}G_s(x;y) = QG_s(x;y) \; = \left(Q^{(0)} + Q^{(1)}\right) G_s(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G^{(1)}_\beta(x;x) \; = -\int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^dz \; e^{-(x-z)^2/4(\beta-s)} Q^{(1)}\Big|_z e^{-(z-x)^2/4s} \right\}$$

$$= \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^dz \; \left[\frac{1}{2s}b_{\mu\nu}(x) - c_{\mu\nu}(x)\right] (z-x)^\mu (z-x)^\nu e^{-\beta(z-x)^2/4s(\beta-s)} \right\}$$

$$= \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d\tilde{z} \; \left[\frac{1}{2s}b_{\mu\nu}(x) - c_{\mu\nu}(x)\right] \tilde{z}^\mu \tilde{z}^\nu e^{-\beta\tilde{z}^2/4s(\beta-s)} \right\}$$

heat kernel expansion

$$-\frac{\partial}{\partial s}G_s(x;y) = QG_s(x;y) \; = \left(Q^{(0)} + Q^{(1)}\right) G_s(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G^{(1)}_\beta(x;x) \; = -\int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d z \; e^{-(x-z)^2/4(\beta-s)} Q^{(1)}\Big|_z e^{-(z-x)^2/4s} \right\}$$

$$= \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d \tilde{z} \; \left[\frac{1}{2s} b_{\mu\nu}(x) - c_{\mu\nu}(x)\right] \tilde{z}^\mu \tilde{z}^\nu e^{-\beta \tilde{z}^2/4s(\beta-s)} \right\}$$

$$= \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d \tilde{z} \left[\frac{1}{2s} b_{\mu\nu}(x) - c_{\mu\nu}(x)\right] \delta_{\mu\nu} \frac{\tilde{z}^2}{d} e^{-\tilde{z}^2\beta/4s(\beta-s)} \right\}$$

heat kernel expansion

$$-\frac{\partial}{\partial s}G_s(x;y) = QG_s(x;y) = \left(Q^{(0)} + Q^{(1)}\right)G_s(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;x) \simeq \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \int d^d\tilde{z} \left[\frac{1}{2s}b_{\mu\nu} - c_{\mu\nu}\right]\Big|_x \delta_{\mu\nu}\frac{\tilde{z}^2}{d}e^{-\tilde{z}^2\beta/4s(\beta-s)} \right\}$$

$$\simeq \int_0^\beta ds \left\{ \frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \left[\frac{1}{2s}b_\mu{}^\mu - c_\mu{}^\mu\right]\Big|_x \int d^d\tilde{z}\, \frac{\tilde{z}^2}{d}e^{-\tilde{z}^2\beta/4s(\beta-s)} \right\}$$

$$\simeq \int_0^\beta ds \left\{ \frac{4^{d/2+1}((\beta-s)s)e^{-\beta m^2}}{(4\pi)^d\beta^{d/2+1}} \left[\frac{1}{2s}b_\mu{}^\mu - c_\mu{}^\mu\right]\Big|_x \int d^d\tilde{z}\, \frac{\tilde{z}^2}{d}e^{-\tilde{z}^2} \right\}$$

heat kernel expansion

$$-\frac{\partial}{\partial s}G_s(x;y) = QG_s(x;y) = \left(Q^{(0)} + Q^{(1)}\right)G_s(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_\beta^{(1)}(x;x) \simeq \int_0^\beta ds \left\{ \frac{4^{d/2+1}((\beta-s)s)e^{-\beta m^2}}{(4\pi)^d \beta^{d/2+1}} \left[\frac{1}{2s}b_\mu{}^\mu - c_\mu{}^\mu\right]\Big|_x \int d^d\tilde{z}\ \frac{\tilde{z}^2}{d}e^{-\tilde{z}^2} \right\}$$

$$\simeq \int_0^\beta ds \left\{ \frac{4^{d/2+1}(\beta-s)se^{-\beta m^2}}{(4\pi)^d \beta^{d/2+1}} \left[\frac{1}{2s}b_\mu{}^\mu - c_\mu{}^\mu\right]\Big|_x \frac{\pi^{d/2}}{2} \right\}$$

$$\simeq \frac{e^{-\beta m^2}}{(4\pi\beta)^{d/2}} \left[\int_0^\beta ds \left[\frac{\beta-s}{\beta}b_\mu{}^\mu - \frac{2s(\beta-s)}{\beta}c_\mu{}^\mu\right]\Big|_x\right]$$

heat kernel expansion

$$-\frac{\partial}{\partial s} G_s(x;y) = Q G_s(x;y) \;=\; \left(Q^{(0)} + Q^{(1)}\right) G_s(x;y)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b_{\mu\nu}\delta x^\mu \partial^\nu + c_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G_s^{(1)}(x;x) \;=\; \frac{e^{-sm^2}}{(4\pi s)^{d/2}} \left[\frac{s}{2} b_\mu{}^\mu(x) - \frac{s^2}{3} c_\mu{}^\mu(x)\right]$$

heat kernel expansion, again, toward the 2nd order

$$-\frac{\partial}{\partial\beta}G_\beta(y;x) = QG_\beta(y;x) \;=\left(Q^{(0)}+Q^{(1)}\right)G_\beta(y;x)$$

$$Q^{(0)} = -\partial^2 + m^2$$

$$Q^{(1)} = b(x)_{\mu\nu}\delta x^\mu \partial^\nu + c(x)_{\mu\nu}\delta x^\mu \delta x^\nu$$

$$G^{(1)}_\beta(y;x) \;= -\int_0^\beta ds\left\{\frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}}\int d^dz\; e^{-(y-z)^2/4(\beta-s)}Q^{(1)}\Big|_z e^{-(z-x)^2/4s}\right\}$$

$$=\int_0^\beta ds\left\{\frac{e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}}\int d^d\tilde{z}\;\left[\frac{1}{2s}b_{\mu\nu}(x)-c_{\mu\nu}(x)\right]\tilde{z}^\mu\tilde{z}^\nu e^{-(y-x-\tilde{z})^2/4(\beta-s)}e^{-\tilde{z}^2/4s}\right\}$$

$$=\int_0^\beta ds\left\{\frac{e^{-(y-x)^2/4\beta}e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}}\int d^d\tilde{z}\;\left[\frac{1}{2s}b_{\mu\nu}(x)-c_{\mu\nu}(x)\right]\tilde{z}^\mu\tilde{z}^\nu e^{-\beta(\tilde{z}-(s/\beta)(y-x))^2/(4s(\beta-s))}\right\}$$

heat kernel expansion, again, toward the 2nd order

$$G_\beta^{(1)}(y;x) = \int_0^\beta ds \left\{ \frac{e^{-(y-x)^2/4\beta} e^{-\beta m^2}}{(4\pi)^d((\beta-s)s)^{d/2}} \left[\frac{1}{2s} b_{\mu\nu}(x) - c_{\mu\nu}(x) \right] \int d\tilde{z}^d \; \tilde{z}^\mu \tilde{z}^\nu e^{-\beta(\tilde{z}-(s/\beta)(y-x))^2/(4s(\beta-s))} \right\}$$

$$\int dw^d \left(\delta^{\mu\nu} \frac{w^2}{d} + \frac{s^2}{\beta^2}(y-x)^\mu (y-x)^\nu \right) e^{-w^2/4t}$$

$$= \delta^{\mu\nu}(2t)(4\pi t)^{d/2} + \frac{s^2}{\beta^2}(y-x)^\mu (y-x)^\nu (4\pi t)^{d/2}$$

$$G_\beta^{(1)}(y;x) = \frac{e^{-(y-x)^2/4\beta} e^{-\beta m^2}}{(4\pi\beta)^{d/2}} \int_0^\beta ds \left[\frac{1}{2s} b_{\mu\nu}(x) - c_{\mu\nu}(x) \right] \left(\frac{2s(\beta-s)}{\beta} \delta^{\mu\nu} + \frac{s^2}{\beta^2}(y-x)^\mu (y-x)^\nu \right)$$

$$= \int_0^\beta ds \left[\frac{(\beta-s)}{\beta} b^\mu_\mu(x) + \frac{s}{2\beta^2} b_{\mu\nu}(x)(y-x)^\mu(y-x)^\nu - \frac{2s(\beta-s)}{\beta} c^\mu_\mu(x) - \frac{s^2}{\beta^2} c_{\mu\nu}(x)(y-x)^\mu(y-x)^\nu \right]$$

$$G_\beta^{(1)}(y;x) = \frac{e^{-(y-x)^2/4\beta} e^{-\beta m^2}}{(4\pi\beta)^{d/2}} \left[\left(\frac{\beta}{2} b^\mu_\mu(x) - \frac{\beta^2}{3} c^\mu_\mu(x) \right) + \left(\frac{1}{4} b_{\mu\nu}(x) - \frac{\beta}{3} c_{\mu\nu}(x) \right)(y-x)^\mu (y-x)^\nu \right]$$

gauge coupling renormalization & continuum field theory

U(1) gauge theory with massive charged field

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \tilde{D}_\mu\Phi^*\tilde{D}^\mu\Phi + m^2|\Phi|^2\right]$$

$$\tilde{F}_{\mu\nu} = \partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu$$

$$\tilde{D}_\mu = \partial_\mu - ig\tilde{A}_\mu$$

$$A_\mu = g\tilde{A}_\mu$$

$$D_\mu = \partial_\mu - iA_\mu$$

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi^*D^\mu\Phi + m^2|\Phi|^2\right]$$

U(1) gauge theory with massive charged field

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2 \right]$$

$$A_\mu \quad \to \quad A_\mu + \partial_\mu \Theta$$

$$\Phi \quad \to \quad e^{i\Theta} \Phi$$

$$F_{\mu\nu} \quad \to \quad F_{\mu\nu}$$

$$(\partial_\mu - i A_\mu)\Phi \quad \to \quad e^{i\Theta}(\partial_\mu - i A_\mu)\Phi$$

renormalization is selective path-integral

$$\int [dA][d\Phi][d\Phi^*]\ e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi^* D^\mu\Phi + m^2|\Phi|^2\right]}$$

$$= \int [dA] e^{-\int \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}} \times \int [d\Phi][d\Phi^*]\ e^{-\int \left[D_\mu\Phi^* D^\mu\Phi + m^2|\Phi|^2\right]}$$

$$\downarrow$$

$$= \int [dA] e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}\right] - \Delta\mathcal{W}(F; m, \Lambda)}$$

integrating out heavy modes will invariably imply integrating out the massive scalar first, and this renormalizes the coupling

$$\int [dA][d\Phi][d\Phi^*]\ e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]}$$

$$= \int [dA] e^{-\int \frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}} \times \frac{1}{\mathrm{Det}_\Lambda(-(\partial - iA)^2 + m^2)}$$

$$\Delta \mathcal{W}(F; m, \Lambda) = \log \mathrm{Det}_\Lambda(-(\partial - iA)^2 + m^2)$$

$$= \int [dA] e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu}\right] - \Delta \mathcal{W}(F; m, \Lambda)}$$

usual textbook renormalization is
selective path-integral + truncation

$$\int [dA][d\Phi][d\Phi^*]\ e^{-\int\ \left[\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}+D_\mu\Phi^* D^\mu\Phi+m^2|\Phi|^2\right]}$$

$$= \int [dA] e^{-\int\ \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu}} \times \frac{1}{\mathrm{Det}_\Lambda(-(\partial - iA)^2 + m^2)}$$

$$\downarrow$$

$$\Delta\mathcal{W}(F; m, \Lambda) = \log \mathrm{Det}_\Lambda(-(\partial - iA)^2 + m^2)$$

$$\simeq \#\log(\Lambda/m) \int F_{\mu\nu}F^{\mu\nu} + \cdots$$

$$\simeq \int [dA] e^{-\int \left[\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \#\log(\Lambda/m) F_{\mu\nu}F^{\mu\nu}\right]}$$

heat kernel expansion for U(1) R.G.

$$Q \equiv -(\partial - iA)^2 + m^2$$

$$= -\partial^2 + m^2 + 2iA \cdot \partial + i(\partial \cdot A) + A^2$$

$$A_\mu \to \frac{1}{2} F_{\alpha\mu} x^\alpha$$

if derivative of field strength can be ignored or is irrelevant

$$\simeq \boxed{-\partial^2 + m^2} + \boxed{iF_{\mu\nu} x^\mu \partial^\nu + \frac{1}{4} F_{\sigma\mu} F^{\sigma\nu} x^\mu x_\nu}$$

$$Q^{(0)} \qquad Q^{(1)} = b_{\mu\nu} \delta x^\mu \partial^\nu + c_{\mu\nu} \delta x^\mu \delta x^\nu$$

so for scalar fields charged under U(1), 1st order suffices

$$G_s^{(1)}(y;x) = \frac{e^{-(y-x)^2/4s}e^{-sm^2}}{(4\pi s)^{d/2}}\left[\left(\frac{s}{2}b_\mu^\mu(x) - \frac{s^2}{3}c_\mu^\mu(x)\right) + \left(b_{\mu\nu}(x) + \frac{s}{3}c_{\mu\nu}(x)\right)(y-x)^\mu(y-x)^\nu\right]$$

the sole contribution to $F_{\mu\nu}F^{\mu\nu}$ at logarithmic level

$$G_\beta^{(2)}(y;x) = -\int_0^\beta ds \int_z G_{\beta-s}^{(0)}(y;z)Q^{(1)}G_s^{(1)}(z;x) = \frac{e^{-(y-x)^2/4\beta}e^{-\beta m^2}}{(4\pi\beta)^{d/2}} \times O(\beta^3)$$

so for scalar fields charged under U(1), 1st order suffices

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

$$= -\mathrm{Tr} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left[-s(-(\partial - iA)^2 + m^2)\right]$$

$$= -\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \int dx^4 \, \langle x| \exp\left[-s(-(\partial - iA)^2 + m^2)\right] |x\rangle$$

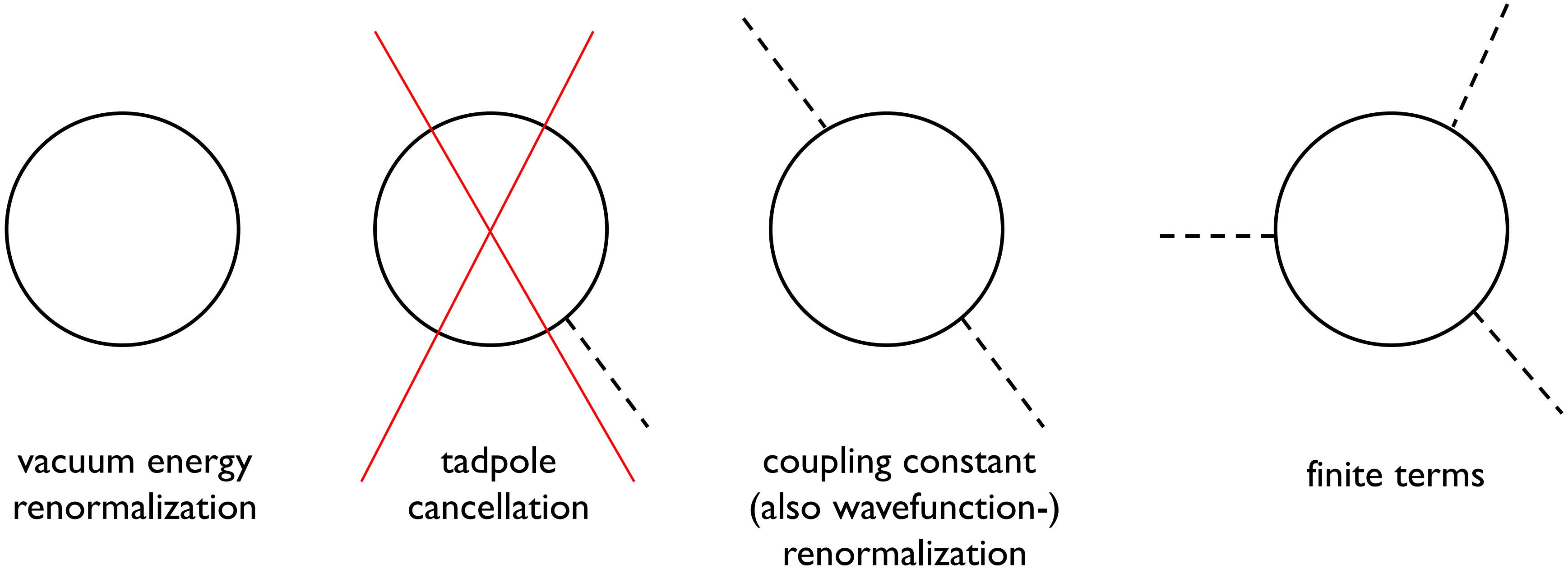
$$= -\frac{1}{16\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^3} \, e^{-m^2 s} \int dx^4$$

$$A_\mu \to \frac{1}{2} F_{\alpha\mu} x^\alpha \qquad + \frac{1}{192\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \, e^{-m^2 s} \times \int dx^4 F_{\mu\nu} F^{\mu\nu} \quad + \cdots$$

U(1) gauge theory with massive charged field

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

$$= \Lambda^4 a(m/\Lambda) + b(m/\Lambda)\frac{1}{4}F^2 + \frac{c(m/\Lambda)}{\Lambda^2}(\partial F)^2 + \cdots$$



U(1) gauge theory with massive charged field

$$\Lambda^{4-n} I_n(m/\Lambda) \equiv \int_{1/\Lambda^2}^{\infty} ds\ s^{n/2-3}\ e^{-m^2 s} = \Lambda^{4-n} \int_1^{\infty} d\tilde{s}\ \tilde{s}^{n/2-3}\ e^{-(m^2/\Lambda^2)\tilde{s}}$$

$$I_0(m/\Lambda) = 2\left(1 - m^2/\Lambda^2 + m^4/\Lambda^4 I_4(m/\Lambda)\right)$$

$$I_4(m/\Lambda) \simeq \begin{cases} -\gamma - \log(m^2/\Lambda^2) + O(m^2/\Lambda^2) & \text{when } m^2 \ll \Lambda^2 \\ e^{-m^2/\Lambda^2} & \text{when } m^2 \gg \Lambda^2 \end{cases}$$

U(1) gauge theory with massive charged field

$$\int [dA] e^{-W_{eff}(A)} \equiv \int [dA][d\Phi][d\Phi^*] \, e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2 |\Phi|^2\right]}$$

$$W_{eff} = \int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \Delta\mathcal{L}(F; m, \Lambda)\right]$$

$$= \int \left[\cdots + \left(\frac{1}{4g^2} + \frac{1}{192\pi^2} I_4(m/\Lambda)\right) F_{\mu\nu}F^{\mu\nu} + \cdots\right]$$

$$= \frac{1}{4g^2(0; m, \Lambda)_{ren}}$$

heat kernel expansion for a unit-charged massive dirac spinor

$$\int \Delta\mathcal{L}(F; m, \Lambda) \;=\; \boxed{-\frac{1}{2}}\, \mathrm{Tr}_\Lambda \log Q_{spinor}$$

$$\begin{aligned} Q_{spinor} \;&\equiv\; (i\gamma^\mu(\partial_\mu - iA_\mu) + im)^\dagger (i\gamma^\mu(\partial_\mu - iA_\mu) + im) \\ &=\; (i\gamma^\mu(\partial_\mu - iA_\mu) - im)(i\gamma^\mu(\partial_\mu - iA_\mu) + im) \\ &=\; -\left[\gamma^\mu(\partial_\mu - iA_\mu)\right]^2 + m^2 \end{aligned}$$

'i'$\times m$ is necessary for the reality of the action in the Lorentzian signature

more generally, complex mass can be introduced as $im + M\gamma^5$ which merely shifts $m^2 \;\to\; m^2 + M^2$

heat kernel expansion for a unit-charged massive dirac spinor

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \boxed{-\frac{1}{2}} \mathrm{Tr}_\Lambda \log Q_{spinor}$$

$$Q_{spinor} = -\left[\gamma^\mu(\partial_\mu - iA_\mu)\right]^2 + m^2$$

$$= 1_{2^{d/2}\times 2^{d/2}} \times \left[-(\partial - iA)^2 + m^2\right] + \frac{i}{2} F_{\mu\nu}\gamma^{\mu\nu}$$

$$= \underbrace{\boxed{1_{2^{d/2}\times 2^{d/2}} Q^{(0)}_{scalar}}}_{Q^{(0)}_{spinor}} + \underbrace{\boxed{1_{2^{d/2}\times 2^{d/2}} Q^{(1)}_{scalar} + \frac{i}{2} F_{\mu\nu}\gamma^{\mu\nu}}}_{Q^{(1)}_{spinor}}$$

heat kernel expansion for a unit-charged massive dirac spinor

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \boxed{-\frac{1}{2}} \mathrm{Tr}_\Lambda \log Q_{spinor}$$

$$= \boxed{+\frac{1}{2}} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \int dx^4 \, \langle x| \exp\left[-s\left(1_{4\times 4} Q_{\text{scalar}} + \frac{i}{2} F\cdot\gamma\right)\right] |x\rangle$$

$$= \cdots + \frac{1}{2} \times \left(-4 \cdot \frac{1}{192\pi^2} + \frac{1}{16\pi^2}\right) I_4(m/\Lambda) F_{\mu\nu}F^{\mu\nu} + \cdots$$

$$Q_{spinor} = \underbrace{1_{2^{d/2}\times 2^{d/2}} Q^{(0)}_{scalar}}_{Q^{(0)}_{spinor}} + \underbrace{1_{2^{d/2}\times 2^{d/2}} Q^{(1)}_{scalar} + \frac{i}{2} F_{\mu\nu}\gamma^{\mu\nu}}_{Q^{(1)}_{spinor}}$$

heat kernel expansion for a unit-charged massive dirac spinor

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \boxed{-\frac{1}{2}} \mathrm{Tr}_\Lambda \log Q_{spinor}$$

$$\frac{1}{192\pi^2} I_4(m/\Lambda) F_{\mu\nu}F^{\mu\nu}$$

vs

$$-\frac{1}{2} \times \left(4 \cdot \frac{1}{192\pi^2} - \frac{1}{16\pi^2}\right) I_4(m/\Lambda) F_{\mu\nu}F^{\mu\nu}$$

$$= \frac{4}{192\pi^2} I_4(m/\Lambda) F_{\mu\nu}F^{\mu\nu}$$

$G_\beta^{(1)}$ $G_\beta^{(2)}$

$$Q_{spinor} = \boxed{1_{2^{d/2}\times 2^{d/2}} Q^{(0)}_{scalar}} + \boxed{1_{2^{d/2}\times 2^{d/2}} Q^{(1)}_{scalar} + \frac{i}{2} F_{\mu\nu}\gamma^{\mu\nu}}$$

$Q^{(0)}_{spinor}$ $Q^{(1)}_{spinor}$

U(1) gauge theory with massive charged fields

$$\frac{1}{4g^2(0;m,\Lambda)_{ren}} = \frac{1}{4g^2} + \frac{q_{scalar}^2}{192\pi^2} I_4(m_{scalar}/\Lambda)$$

$$+\frac{4q_{spinor}^2}{192\pi^2} I_4(m_{spinor}/\Lambda)$$

U(1) gauge theory with massive charged fields

more generally we may wish to integrate out partially, say, $\mu^2 < p^2 < \Lambda^2$

$$\int_{p^2<\mu^2} [dA][d\Phi][d\Phi^*][\cdots] e^{-\int \frac{1}{4g^2_{ren}(\mu;m,\Lambda)} F^2 + \cdots}$$

$$\equiv \int_{p^2<\Lambda^2} [dA][d\Phi][d\Phi^*][\cdots]\ e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + D_\mu \Phi^* D^\mu \Phi + m^2|\Phi|^2 + \cdots\right]}$$

U(1) gauge theory with massive charged fields

more generally we may wish to integrate out partially, say, $\mu^2 < p^2 < \Lambda^2$

$$\frac{1}{4g^2(\mu; m, \Lambda)_{ren}} = \frac{1}{4g^2} + \frac{q^2_{scalar}}{192\pi^2}\tilde{I}_4(m_{scalar}; \mu, \Lambda) + \frac{4q^2_{\text{spinor}}}{192\pi^2}\tilde{I}_4(m_{spinor}; \mu, \Lambda)$$

$$\tilde{I}_4(m; \mu, \Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s}\, e^{-m^2 s}$$

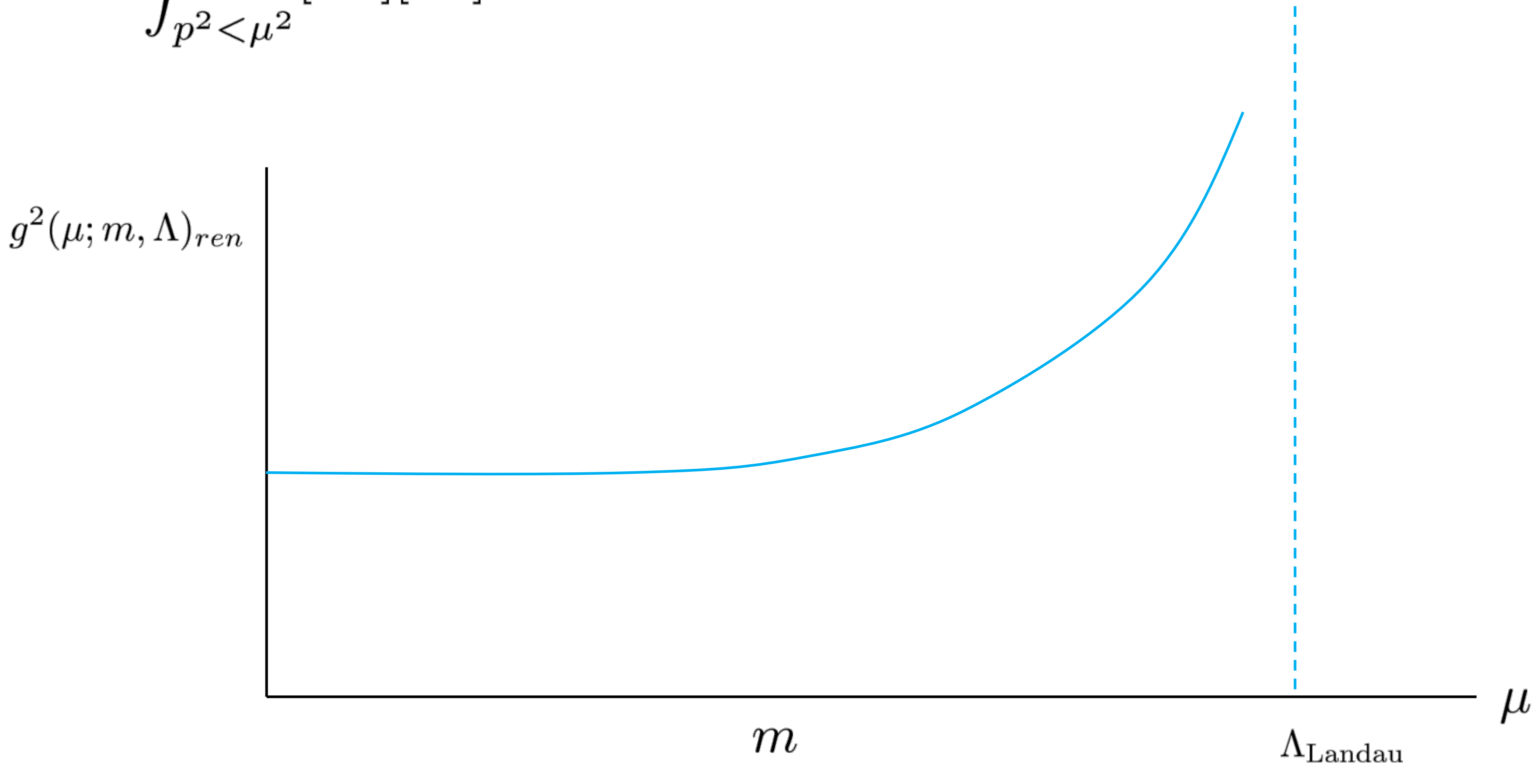
$$\sim \log(\Lambda^2/m^2) \quad \text{when } \mu^2 < m^2 < \Lambda^2$$

$$\sim \log(\Lambda^2/\mu^2) \quad \text{when } m^2 < \mu^2 < \Lambda^2$$

U(1) gauge theory with massive charged fields

$$\int_{p^2<\Lambda^2} [dA][\cdots]\, e^{-\int \left[\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \cdots\right]}$$

$$= \int_{p^2<\mu^2} [dA][\cdots] e^{-\int \left[\cdots + \frac{1}{4g(\mu;m,\Lambda)^2_{ren}} F_{\mu\nu}F^{\mu\nu} + \cdots\right]}$$



Yang-Mills theories

Yang-Mills theory renormalization & asymptotic freedom

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2}\mathrm{tr}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}\right]$$

$$\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu - ig[\tilde{A}_\mu, \tilde{A}_\nu]$$

$$A_\mu = g\tilde{A}_\mu \quad \downarrow$$

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A = A^\dagger \qquad B = B^\dagger$$

$$(i[A,B])^\dagger = i^*(AB - BA)^\dagger = -i(B^\dagger A^\dagger - A^\dagger B^\dagger) = -i[B,A] = i[A,B]$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \ \left[\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

a canonical example

$$A_\mu = \sum_{a=1,2,3} A^a_\mu \frac{\sigma^a}{2}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \sum_{a=1,2,3} F^a_{\mu\nu} \frac{\sigma^a}{2}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon_{abc} A^b_\mu A^c_\nu$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \, \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

more generally

$$A_\mu = \sum_a A_\mu^a T^a$$

$$F_{\mu\nu} = \sum_a F_{\mu\nu}^a T^a$$

$$\mathrm{tr}_{\mathrm{def}} T^a T^b = \frac{1}{2} \delta_{ab}$$

$$[T^a, T^b] = f_{abc} T^c$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - i f_{abc} A_\mu^b A_\nu^c$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \; \left[\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad A_\mu \quad \to \quad U^\dagger A_\mu U + iU^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad F_{\mu\nu} \quad \to \quad U^\dagger F_{\mu\nu} U$$

$$U \in \mathcal{U}(N)$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad A_\mu \quad \to \quad U^\dagger A_\mu U + i U^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad F_{\mu\nu} \quad \to \quad U^\dagger F_{\mu\nu} U$$

$$\mathrm{tr} F_{\mu\nu} = 0 = \mathrm{tr} A_\mu \qquad U \in \mathcal{SU}(N)$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \;\left[\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu}\right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad\qquad A_\mu \quad \rightarrow \quad U^\dagger A_\mu U + iU^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad\qquad F_{\mu\nu} \quad \rightarrow \quad U^\dagger F_{\mu\nu} U$$

$$A_\mu^T I = I A_\mu \qquad\qquad U \in \mathcal{O}(N)$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad A_\mu \quad \rightarrow \quad U^\dagger A_\mu U + i U^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad F_{\mu\nu} \quad \rightarrow \quad U^\dagger F_{\mu\nu} U$$

$$A_\mu^T I = I A_\mu \qquad U \in \mathcal{SO}(N)$$

$$\mathrm{tr} F_{\mu\nu} = 0 = \mathrm{tr} A_\mu$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

what types of matrices are preserved under the commutator ?

$$A_\mu = A_\mu^\dagger \qquad A_\mu \quad \rightarrow \quad U^\dagger A_\mu U + iU^\dagger \partial_\mu U$$

$$F_{\mu\nu} = F_{\mu\nu}^\dagger \qquad F_{\mu\nu} \quad \rightarrow \quad U^\dagger F_{\mu\nu} U$$

$$A_\mu^T J = J A_\mu \qquad U \in \mathcal{SP}(N/2) = USp(N)$$

$$J^T = -J, \quad J^2 = -1$$

Yang-Mills theory

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

$$A_\mu = \sum_a A_\mu^a T^a$$

$$F_{\mu\nu} = \sum_a F_{\mu\nu}^a T^a$$

$$[T^a, T^b] = f_{abc} T^c$$

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - i f_{abc} A_\mu^b A_\nu^c$$

Yang-Mills theory with matter fields

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi^* D^\mu\Phi + m^2|\Phi|^2 + \cdots\right]$$

$$D_\mu\Phi^k = \partial_\mu\Phi^k - iA^a_\mu (t^a)^k{}_l \Phi^l$$

$$(t^a)^k{}_l, \quad k,l = 1,\ldots,n$$

$$[t^a, t^b] = f_{abc}t^c$$

for example $\mathcal{SU}(2) = \mathcal{SP}(1)$

$$t^a_{(s)} = J^a_{spin=s}$$

$$s = 0, 1/2, 1, 3/2, \ldots$$

$$n = 1, 2, 3, 4, \ldots$$

Yang-Mills theory with matter fields

$$\int [dA][d\Phi][d\Phi^*]\; e^{-\int \left[\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}+D_\mu\Phi^* D^\mu\Phi+m^2|\Phi|^2\right]}$$

$$= \int [dA] e^{-\int \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}} \times \int [d\Phi][d\Phi^*]\; e^{-\int\left[D_\mu\Phi^* D^\mu\Phi+m^2|\Phi|^2\right]}$$

$$= \int [dA] e^{-\int \frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}} \times \frac{1}{\mathrm{Det}(-(\partial - iA)^2 + m^2)}$$

$$= \int [dA] e^{-\int\left[\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}\right] - \Delta\mathcal{W}(F;m,\Lambda)}$$

$$\Delta\mathcal{W}(F;m,\Lambda) = \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

Yang-Mills theory with matter fields

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \mathrm{Tr}_\Lambda \log(-(\partial - iA)^2 + m^2)$$

$$= -\mathrm{Tr} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \exp\left[-s(-(\partial - iA)^2 + m^2)\right]$$

$$= -\int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \mathrm{tr} \int dx^4 \langle x| \exp\left[-s(-(\partial - iA)^2 + m^2)\right] |x\rangle$$

$$= -\frac{1}{16\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^3} e^{-m^2 s} \times n$$

$$+ \frac{1}{192\pi^2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} e^{-m^2 s} \times F_{\mu\nu}F^{\mu\nu} \times \mathcal{T}_2(t) \quad + \cdots$$

heat kernel expansion for U(1) R.G.

$$Q_{scalar} \equiv -(\partial - iA)^2 + m^2$$

$$= \underbrace{(-\partial^2 + m^2)}_{Q^{(0)}_{scalar}} + \underbrace{2iA \cdot \partial + i(\partial \cdot A) + A^2}_{Q^{(1)}_{scalar}}$$

heat kernel expansion for Yang-Mills R.G.

$$Q_{scalar} \equiv -(1_{n\times n}\partial - iA)^2 + m^2 1_{n\times n}$$

$$= \underbrace{1_{n\times n}(-\partial^2 + m^2)}_{Q^{(0)}_{scalar}} + \underbrace{2iA\cdot\partial + i(\partial\cdot A) + A^2}_{Q^{(1)}_{scalar}}$$

heat kernel expansion

$$-\frac{\partial}{\partial\beta}G_\beta(x;y) = QG_\beta(x;y) \;= \left(Q^{(0)} + Q^{(1)}\right) G_\beta(x;y)$$

$$G_\beta = G_\beta^{(0)} + G_\beta^{(1)} + G_\beta^{(2)} + \cdots$$

$$G_\beta^{(n)}(x;y) = -\int_0^\beta ds \int_z \; G_{\beta-s}^{(0)}(x;z)Q^{(1)}G_s^{(n-1)}(z;y)$$

$$= (-1)^n \int_0^\beta ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n \int_{z_1} \cdots \int_{z_n}$$

$$G_{\beta-s_1}^{(0)}(x;z_1)Q^{(1)}G_{s_1-s_2}(z_1;z_2)\cdots Q^{(1)}G_{s_n}^{(0)}(z_n;y)$$

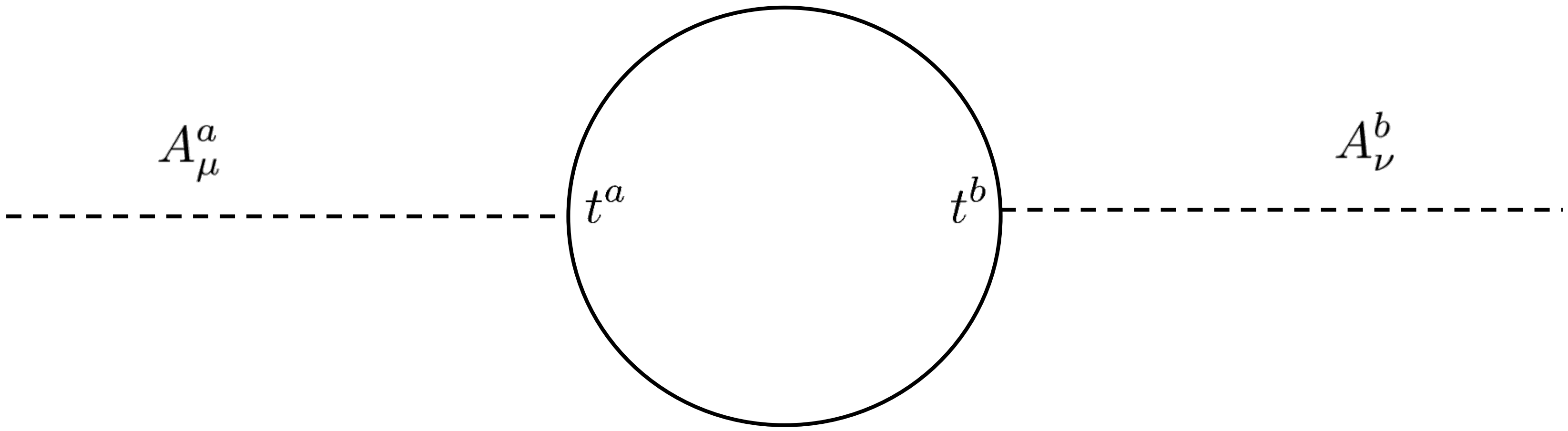
Yang-Mills theory with matter fields

$$W_{eff} = \cdots + \int \frac{1}{2g_{ren}^2} \mathrm{tr}^{\mathrm{defining}} F_{\mu\nu} F^{\mu\nu} + \cdots$$

$$\frac{1}{2g^2(m,\Lambda;\mu)_{ren}} \mathcal{T}_2^{\mathrm{def}} = \frac{1}{2g^2} \mathcal{T}_2^{\mathrm{def}} + \frac{1}{192\pi^2} \tilde{I}_4(m_{scalar};\mu,\Lambda) \mathcal{T}_2(t_{scalar})$$

$$\tilde{I}_4(m;\mu,\Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s}\, e^{-m^2 s}$$

Yang-Mills theory with matter fields



$$\mathrm{tr}\; t^a t^b = \delta^{ab} \mathcal{T}_2(t)$$

Yang-Mills theory with Dirac spinor

$$\int \mathcal{L}_{spnior} = \int dx^4 \ \ [\bar{\Psi} i\gamma^{\mu} D_{\mu}\Psi + \cdots]$$

$$D_{\mu}\Psi^k = \partial_{\mu}\Psi^k - iA^a_{\mu}(t^a)^k{}_l\Psi^l$$

$$(t^a)^k{}_l, \quad k, l = 1, \ldots, n$$

$$[t^a, t^b] = f_{abc}t^c$$

for example $\mathcal{SU}(2) = \mathcal{SP}(1)$

$$t^a_{(s)} = J^a_{spin=s}$$

$$s = 0, 1/2, 1, 3/2, \ldots$$

$$n = 1, 2, 3, 4, \ldots$$

heat kernel expansion for a unit-charged massive dirac spinor

$$\int \Delta\mathcal{L}(F;m,\Lambda) \;=\; \boxed{-\frac{1}{2}}\,\mathrm{Tr}_\Lambda \log Q_{spinor}$$

$$Q_{spinor} \equiv (i\gamma^\mu(\partial_\mu - iA_\mu) + im)^\dagger (i\gamma^\mu(\partial_\mu - iA_\mu) + im)$$

$$= (i\gamma^\mu(\partial_\mu - iA_\mu) - im)(i\gamma^\mu(\partial_\mu - iA_\mu) + im)$$

$$= -\left[\gamma^\mu(\partial_\mu - iA_\mu)\right]^2 + m^2$$

'i'$\times m$ is necessary for the reality of the action in the Lorentzian signature

more generally, complex mass can be introduced as $im + M\gamma^5$ which merely shifts $m^2 \;\to\; m^2 + M^2$

exactly the same thing happens in Yang-Mills
except the spinor now has one more index

$$\int \Delta\mathcal{L}(F;m,\Lambda) = \boxed{-\frac{1}{2}} \mathrm{Tr}_\Lambda \log Q_{spinor}$$

$$Q_{spinor} \equiv (i\gamma^\mu(\partial_\mu - iA_\mu) + im)^\dagger (i\gamma^\mu(\partial_\mu - iA_\mu) + im)$$

$$= (i\gamma^\mu(\partial_\mu - iA_\mu) - im)(i\gamma^\mu(\partial_\mu - iA_\mu) + im)$$

$$= -[\gamma^\mu(\partial_\mu - iA_\mu)]^2 + m^2$$

'i'$\times m$ is necessary for the reality of the action in the Lorentzian signature

more generally, complex mass can be introduced as $im + M\gamma^5$ which merely shifts $m^2 \rightarrow m^2 + M^2$

heat kernel expansion for a unit-charged dirac spinor with a chiral mass

$$\int \Delta\mathcal{L}(F; m, \Lambda) = \boxed{-\frac{1}{2}} \mathrm{Tr}_\Lambda \log Q_{spinor}$$

$$\frac{1}{96\pi^2} I_4(m/\Lambda)\, \mathrm{tr} F_{\mu\nu}F^{\mu\nu}$$

vs

$$-\frac{1}{2} \times \left(4 \cdot \frac{1}{192\pi^2} - \frac{1}{16\pi^2}\right) I_4(m/\Lambda)\, \mathrm{tr} F_{\mu\nu}F^{\mu\nu}$$

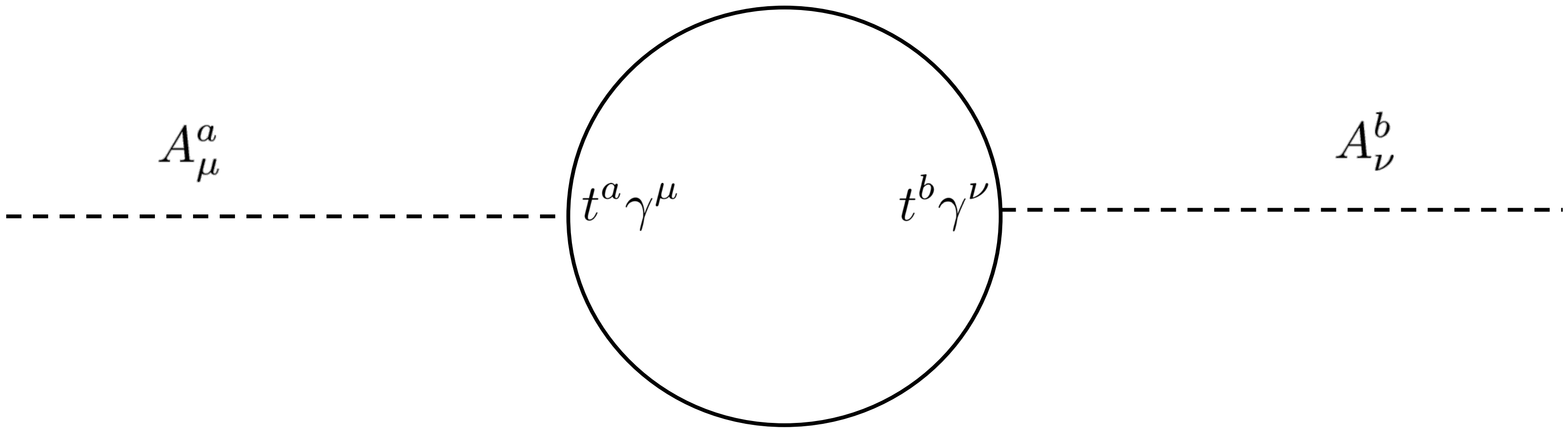
$$= \frac{4}{192\pi^2} I_4(m/\Lambda)\, \mathrm{tr} F_{\mu\nu}F^{\mu\nu}$$

$G_\beta^{(1)}$ $G_\beta^{(2)}$

$$Q_{spinor} = \boxed{1_{2^{d/2}\times 2^{d/2}} Q^{(0)}_{scalar}} + \boxed{1_{2^{d/2}\times 2^{d/2}} Q^{(1)}_{scalar} + \frac{i}{2} F_{\mu\nu}\gamma^{\mu\nu}}$$

$Q^{(0)}_{spinor}$ $Q^{(1)}_{spinor}$

Yang-Mills theory with matter fields



$$\mathrm{tr}\ t^a t^b = \delta^{ab}\mathcal{T}_2(t)$$

Yang-Mills theory with matter fields

$$W_{\text{eff}} = \cdots + \int \frac{1}{2g_{ren}^2} \text{tr}_{\text{defining}} F_{\mu\nu} F^{\mu\nu} + \cdots$$

$$\frac{1}{2g^2(m,\Lambda;\mu)_{ren}} \mathcal{T}_2^{\text{def}} = \frac{1}{2g^2} \mathcal{T}_2^{\text{def}} + \frac{1}{192\pi^2} \tilde{I}_4(m_{scalar};\mu,\Lambda) \mathcal{T}_2(t_{scalar})$$

$$+ \frac{4}{192\pi^2} \tilde{I}_4(m_{spinor};\mu,\Lambda) \mathcal{T}_2(t_{spinor})$$

$$\tilde{I}_4(m;\mu,\Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s} \, e^{-m^2 s}$$

Yang-Mills theory with matter fields

Yang-Mills fields interact among themselves,
so there are additional contribution from the gauge fields

$$\int_{p^2<\mu^2} [dA][d\Phi][d\Phi^*][\cdots] e^{-W_{\mathrm{eff}}(A,\cdots;\mu)} \equiv$$

$$\int_{p^2<\Lambda^2} [dA][d\Phi][d\Phi^*]\ e^{-\int \left[\frac{1}{2g^2}\mathrm{tr} F_{\mu\nu}F^{\mu\nu}+\cdots\right]}$$

Yang-Mills theory

Yang-Mills fields interact among themselves,
so there are additional contribution

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$A_\mu \to A_\mu + a_\mu$$

$$\int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right] + \left[\frac{1}{2g^2} \mathrm{tr} (D_\mu a_\nu - D_\nu a_\mu)^2 - \frac{i}{g^2} \mathrm{tr} F_{\mu\nu} [a^\mu, a^\nu] \right]$$

Yang-Mills theory

cubic and higher pieces of a_μ ignored and
linear pieces removed by equation of motion

$$\int \mathcal{L} = \int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right]$$

$$A_\mu \;\to\; A_\mu + a_\mu$$

$$D_\mu a_\nu \equiv \partial_\mu a_\nu - i[A_\mu, a^\nu]$$

$$\int dx^4 \left[\frac{1}{2g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu} \right] + \left[\frac{1}{2g^2} \mathrm{tr}(D_\mu a_\nu - D_\nu a_\mu)^2 - \frac{i}{g^2} \mathrm{tr} F_{\mu\nu} [a^\mu, a^\nu] \right]$$

Yang-Mills theory

Fadeev-Popov gauge fixing in the background gauge adds a term

$$\int \mathcal{L} = \cdots + \int dx^4 \, \frac{1}{g^2} \mathrm{tr}(\partial_\mu a^\mu - i[A_\mu, a^\mu])^2$$

$$- \int dx^4 \, \frac{1}{g^2} \mathrm{tr}\,(a^\nu D_\mu D^\mu a_\nu \boxed{- a^\mu D_\nu D_\mu a^\nu - i a^\mu [F_{\mu\nu}, a^\nu] + \boxed{a^\mu D_\mu D_\nu a^\nu})}$$

$$= \; - \int dx^4 \, \frac{1}{g^2} \mathrm{tr}\,(a^\nu D_\mu D^\mu a_\nu \boxed{- 2i a^\mu [F_{\mu\nu}, a^\nu])}$$

heat kernel expansion for the gauge field fluctuation

$$\int \Delta\mathcal{L}(F;m,\Lambda) \;=\; \boxed{+\frac{1}{2}}\,\mathrm{Tr}_\Lambda \log Q_{vector}$$

$$Q_{vector} \;\equiv\; -\delta_{\mu\nu}D^\mu D_\mu + 2iF^{\text{adjoint}}_{\mu\nu}$$

$$= \boxed{1_{4\times4}Q^{(0)}_{scalar}} + \boxed{1_{4\times4}Q^{(1)}_{scalar} + 2iF^{\text{adjoint}}_{4\times4}}$$

$$Q^{(0)}_{vector} \qquad\qquad Q^{(1)}_{vector}$$

heat kernel expansion for the gauge field fluctuation

$$\int \Delta\mathcal{L}(F;m,\Lambda) = \boxed{+\frac{1}{2}} \operatorname{Tr}_\Lambda \log Q_{vector}$$

$$\frac{1}{192\pi^2} I_4(m/\Lambda)\, \mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} \quad \text{vs} \quad \frac{1}{2} \times \left(4 \cdot \frac{1}{192\pi^2} - 2 \cdot \frac{1}{16\pi^2}\right) I_4(m/\Lambda)\, \mathrm{tr}_{\mathrm{adj}}\, F_{\mu\nu}F^{\mu\nu}$$

$$= -\frac{10}{192\pi^2} I_4(m/\Lambda)\, \mathrm{tr}_{\mathrm{adj}}\, F_{\mu\nu}F^{\mu\nu}$$

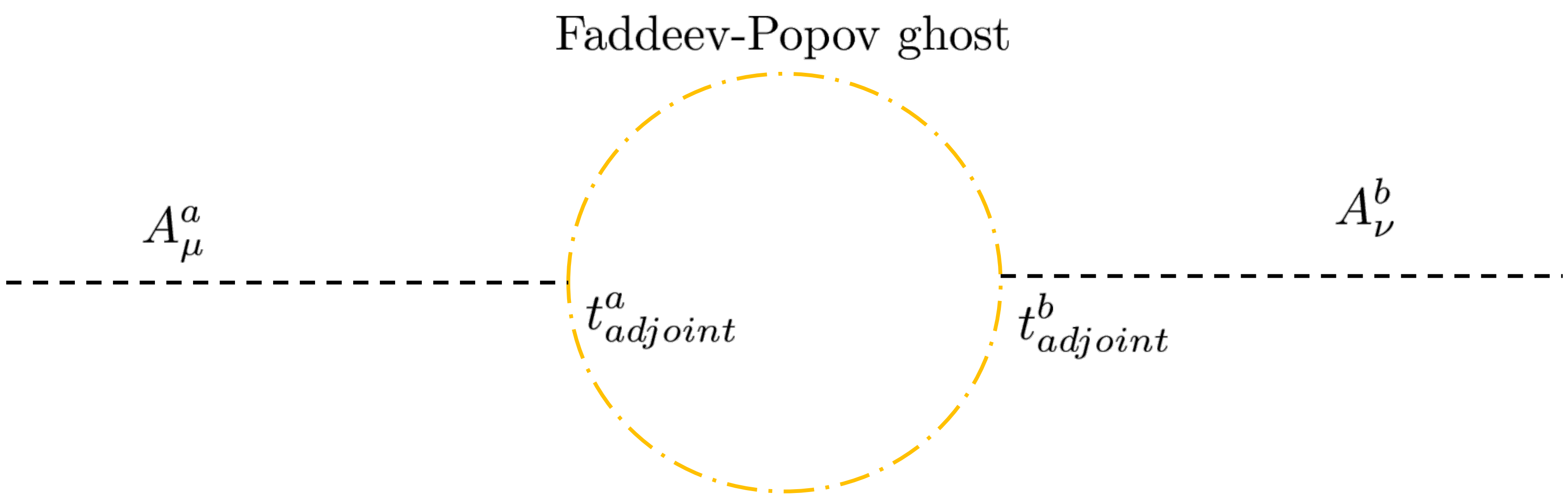
$G_\beta^{(1)}$ $\quad$ $G_\beta^{(2)}$

$$Q_{vector} = \boxed{1_{4\times 4} Q_{scalar}^{(0)}} + \boxed{1_{4\times 4} Q_{scalar}^{(1)} + 2iF_{4\times 4}^{\mathrm{adjoint}}}$$

$Q_{vector}^{(0)}$ $\quad$ $Q_{vector}^{(1)}$

Yang-Mills theory with matter fields

gauge-fixing introduces further contribution from Faddeev-Popov ghost, which is like minus of a single complex scalar in the adjoint representation



$$\delta^{ab}\mathcal{T}_2(t_{adjoint})$$

the ghost is a complex scalar with wrong statistics, contributing that of one complex adjoint scalar with the sign flipped

$$\int \Delta\mathcal{L}(F; m, \Lambda) \; = \boxed{-}\, \mathrm{Tr}_\Lambda \log Q_{scalar}$$

$$\frac{1}{192\pi^2} I_4(m/\Lambda)\,\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}$$

a complex scalar contribution

vs

$$-\frac{1}{192\pi^2} I_4(m/\Lambda)\,\mathrm{tr}_{\mathrm{adj}}\, F_{\mu\nu}F^{\mu\nu}$$

Fadeev-Popov ghost contribution

Yang-Mills theory with matter fields

$$W_{eff} = \cdots + \int \frac{1}{2g_{ren}^2} \mathrm{tr}_{\text{defining}} F_{\mu\nu} F^{\mu\nu} + \cdots$$

$$\frac{1}{2g^2(m,\Lambda;\mu)_{ren}} = \frac{1}{2g(\Lambda)^2} + \frac{1}{192\pi^2} \tilde{I}_4(m_{scalar};\mu,\Lambda) \mathcal{T}_2(t_{scalar}) / \mathcal{T}_2^{\text{def}}$$

$$+ \frac{4}{192\pi^2} \tilde{I}_4(m_{spinor};\mu,\Lambda) \mathcal{T}_2(t_{spinor}) / \mathcal{T}_2^{\text{def}}$$

$$- \frac{10}{192\pi^2} \tilde{I}_4(0;\mu;\Lambda) \mathcal{T}_2(t_{adjoint}) / \mathcal{T}_2^{\text{def}}$$

$$- \frac{1}{192\pi^2} \tilde{I}_4(0;\mu;\Lambda) \mathcal{T}_2(t_{adjoint}) / \mathcal{T}_2^{\text{def}}$$

$$\tilde{I}_4(m;\mu,\Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s}\, e^{-m^2 s}$$

Yang-Mills theory with matter fields

$$W_{eff} = \cdots + \int \frac{1}{2g_{ren}^2} \mathrm{tr}_{\mathrm{defining}} F_{\mu\nu} F^{\mu\nu} + \cdots$$

$$\frac{1}{2g^2(m,\Lambda;\mu)_{ren}} = \frac{1}{2g(\Lambda)^2} + \frac{1}{192\pi^2}\tilde{I}_4(m_{scalar};\mu,\Lambda)\mathcal{T}_2(t_{scalar})\,/\mathcal{T}_2^{\mathrm{def}}$$

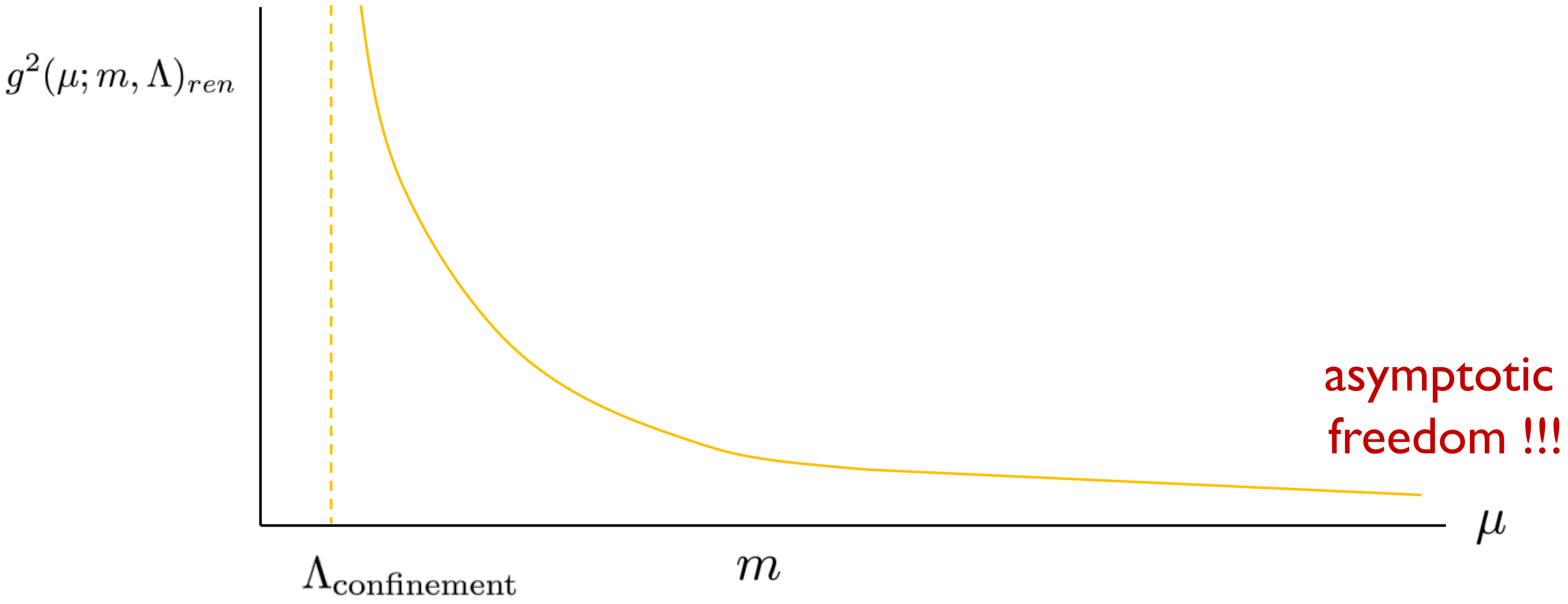
$$+\frac{4}{192\pi^2}\tilde{I}_4(m_{spinor};\mu,\Lambda)\mathcal{T}_2(t_{spinor})\,/\mathcal{T}_2^{\mathrm{def}}$$

$$-\frac{11}{192\pi^2}\tilde{I}_4(0;\mu;\Lambda)\mathcal{T}_2(t_{adjoint})\,/\mathcal{T}_2^{\mathrm{def}}$$

$$\tilde{I}_4(m;\mu,\Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s}\, e^{-m^2 s}$$

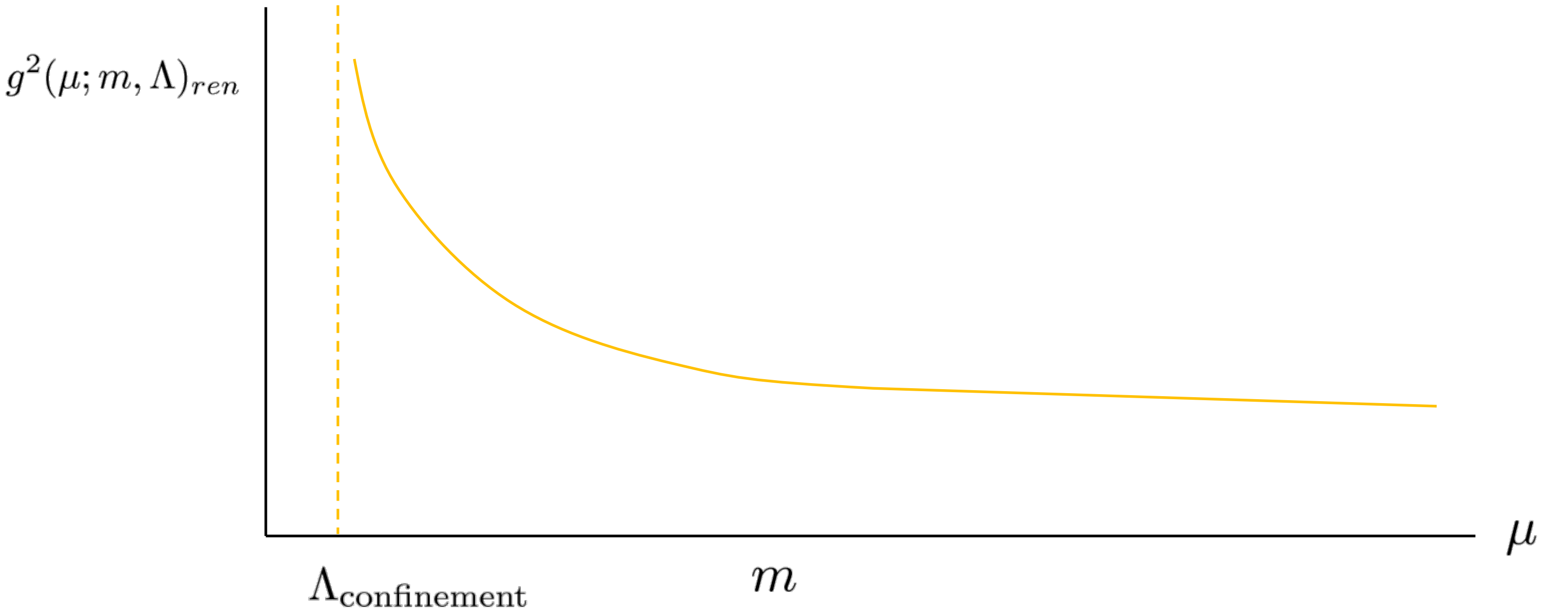
Yang-Mills theory with small number of (massive) matter fields

$$\sum \mathcal{T}_2(t_{scalar}) + 4\sum \mathcal{T}_2(t_{spinor}) < 11\mathcal{T}_2(t_{adjoint})$$



Banks-Zachs fixed points: Yang-Mills-Matter in a conformal window

$$\sum \mathcal{T}_2(t_{scalar}) + 4 \sum \mathcal{T}_2(t_{spinor}) \simeq 11 \mathcal{T}_2(t_{adjoint})$$



N=1 SUSY Yang-Mills theory with matter fields in Chirals

$$W_{eff} = \cdots + \int \frac{1}{2g_{ren}^2} \mathrm{tr}_{\mathrm{defining}} F_{\mu\nu} F^{\mu\nu} + \cdots$$

$$\frac{1}{g^2(m,\Lambda;\mu)_{ren}} = \frac{1}{g(\Lambda)^2} + \frac{3}{3}\frac{1}{32\pi^2}\tilde{I}_4(m_{chiral};\mu,\Lambda)\mathcal{T}_2(t_{chiral})/\mathcal{T}_2^{\mathrm{def}}$$

$$-\frac{9}{3}\frac{1}{32\pi^2}\tilde{I}_4(0;\mu;\Lambda)\mathcal{T}_2(t_{adjoint})/\mathcal{T}_2^{\mathrm{def}}$$

$$\tilde{I}_4(m;\mu,\Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s}\, e^{-m^2 s}$$

N=2 SUSY Yang-Mills theory with matter fields in Hypers

$$W_{eff} = \cdots + \int \frac{1}{2g_{ren}^2} \mathrm{tr}_{\mathrm{defining}} F_{\mu\nu} F^{\mu\nu} + \cdots$$

$$\frac{1}{g^2(m,\Lambda;\mu)_{ren}} = \frac{1}{g(\Lambda)^2} + \frac{6}{3}\frac{1}{32\pi^2}\tilde{I}_4(m_{hyper};\mu,\Lambda)\mathcal{T}_2(t_{chiral})/\mathcal{T}_2^{\mathrm{def}}$$

$$-\frac{6}{3}\frac{1}{32\pi^2}\tilde{I}_4(0;\mu;\Lambda)\mathcal{T}_2(t_{adjoint})/\mathcal{T}_2^{\mathrm{def}}$$

$$\tilde{I}_4(m;\mu,\Lambda) \equiv \int_{1/\Lambda^2}^{1/\mu^2} \frac{ds}{s}\, e^{-m^2 s}$$

N=2 Seiberg-Witten with massive matters

$$\sum \mathcal{T}_2(t_{hyper}) < \mathcal{T}_2(t_{adjoint})$$