RENORMALIZATION GROUP

23차 KIAS-APCTP 통계물리 겨울학교

박수찬

2026년 1월 5일 ∼ 9일

가톨릭대학교

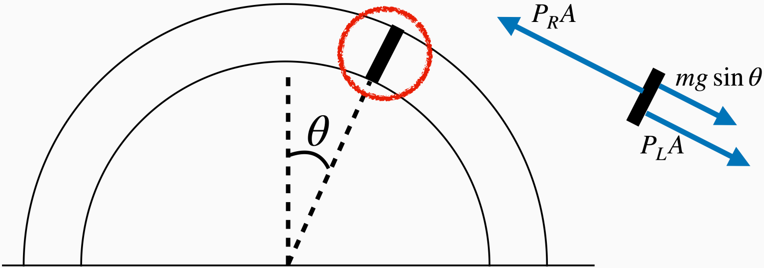
Table of contents

LANDAU THEORY

Table of contents

Callen (1984), Ch 9

- Doughnut shape tube and a movable piston



  - ideal gas of N monatomic molecules on each side : $PV = Nk_B T$
  - T : Temperature of heat reservoir
  - m : piston mass
  - R : radius of the tube.
  - A : cross sectional area of the tube.
  - P_L (P_R) : pressure of the gas on the left (right) hand side

Mechanical solution

- Pressure ($v := V/N$ volume per particle)

$$P_{L,R} = \frac{k_B T}{v_{L,R}} \quad \text{with} \quad v_{L,R} = \frac{AR}{N}\left(\frac{\pi}{2} \pm \theta\right)$$

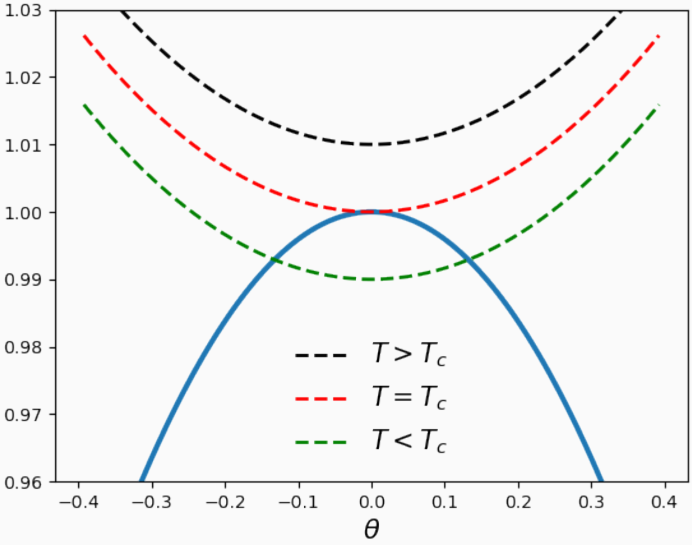
- Equilibrium position θ_e of the piston

$$P_R A = P_L A + mg \sin\theta \Big|_{\theta=\theta_e} \Rightarrow \theta_e = \frac{mgR\pi^2}{8Nk_B T}\left(1 - \frac{4\theta_e^2}{\pi^2}\right)\sin\theta_e$$

- $\theta_e = 0$ is always a (trivial) solution.
- Is nonzero θ_e possible?

$$1 - \frac{4\theta_e^2}{\pi^2} = \frac{T}{T_c}\frac{\theta_e}{\sin\theta_e},$$

where $T_c := mgR\pi^2/(8Nk_B)$. (J/(J/K) = K)


1.03
1.02
1.01
1.00
0.99
0.98
0.97
0.96
−0.4
−0.3
−0.2
−0.1
0.0
0.1
0.2
0.3
0.4
θ
$T > T_c$
$T = T_c$
$T < T_c$

- Condition for the existence of a nontrivial solution

$$1 > 1 - \frac{4\theta^2}{\pi^2} = \frac{T}{T_c}\frac{\theta}{\sin\theta} > \frac{T}{T_c}$$

- $T \geq T_c$: $\theta_e = 0$ is the unique solution
- $T < T_c$: two symmetric nontrivial solutions ($+$ trivial one)
- T_c is the critical point.
- When $T < T_c$, what is the **true** equilibrium solution among three solutions? **stability** analysis with free energy

Thermodynamic solution

- Total free energy for a 'fixed' macroscopic condition θ

$$F(\theta) = -NkT \ln v_R - NkT \ln v_L + mgR\cos\theta + F_0(T, N)$$

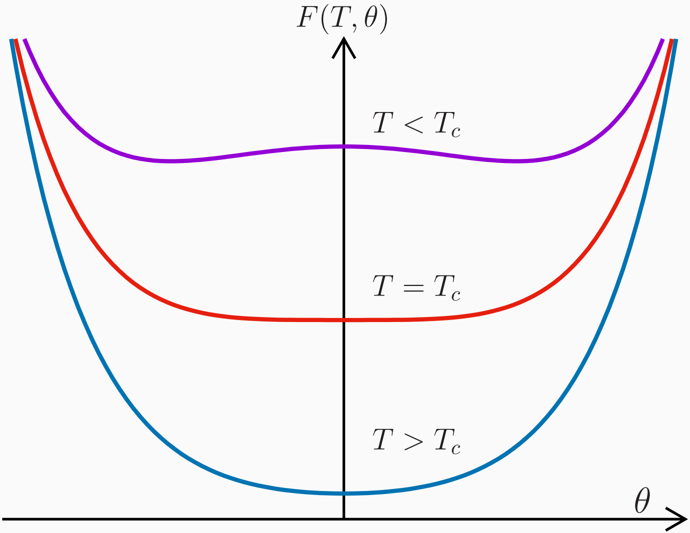
- Free energy minimum condition (freely moving piston)

$$\frac{\partial F}{\partial \theta} = 0 \Rightarrow \frac{Nk_B T}{v_R} AR - \frac{Nk_B T}{v_L} AR - mgR\sin\theta = 0$$

$$\frac{\partial^2 F}{\partial \theta^2} > 0 \Rightarrow \frac{Nk_B T}{(\theta - \pi/2)^2} + \frac{Nk_B T}{(\theta + \pi/2)^2} - mgR\cos\theta > 0$$

- Stability of the trivial solution $\theta_e = 0 \Rightarrow \dfrac{T}{T_c} - 1 > 0$.
- Order parameter $\mathcal{M} := v_L(\theta_e) - v_R(\theta_e) = 2AR\theta_e/N$.
  - $T \geq T_c : \mathcal{M} = 0$
  - $T < T_c : \mathcal{M} \neq 0$

- Free energy as a function of T and θ



At $T = T_c$, minimum is very flat. Large fluctuation.

Landau theory : small θ expansion

- Approximation around T_c ($T := T_c(1+t)$ with $|t| \ll 1$).

$$\sin\theta_e = \frac{T}{T_c}\frac{\theta_e}{1-\frac{4\theta_e^2}{\pi^2}} \Rightarrow \theta_e - \frac{1}{6}\theta_e^3 \approx (1+t)\left(\theta_e + \frac{4}{\pi^2}\theta_e^3\right)$$

$$-t\theta_e \approx \underbrace{\left[(1+t)\frac{4}{\pi^2} + \frac{1}{6}\right]}_{=:C}\theta_e^3$$

  - $t \geq 0 : \theta_e = 0$
  - $t < 0$ and $|t| \ll 1 : |\theta_e| \approx \sqrt{-t/C} \sim \pm|t|^{1/2}$

- t is also called a **reduced temperature**.
- Can we formulate a theory to predict which will happen, $\theta_e = \sqrt{-t}$ or $-\sqrt{-t}$?
- **No! Spontaneous Symmetry Breaking (SSB)**

- If we write $|\mathcal{M}| \sim |t|^{\beta}$, then $\beta = \frac{1}{2}$.
- β is called the **order parameter (critical) exponent**.
- Mathematically speaking,

$$\beta := \lim_{t \uparrow 0} \frac{\ln |\mathcal{M}|}{\ln(-t)}.$$

- Should not be confused with inverse temperature β
- In general, $\mathcal{M}$ has **corrections to scaling**
- Even if there is a **logarithmic corrections**, β is defined in the same way
- Example

$$\mathcal{M} = |t|^{1/2} \times \underbrace{|\log |t||^{\alpha}}_{\text{log correction}} \times \underbrace{(1 + at^{0.2})}_{\text{corrections to scaling}} \Rightarrow \lim_{|t| \to 0} \frac{\ln \mathcal{M}}{\ln |t|} = \frac{1}{2}$$

- Observation: When $T \approx T_c$, the minimum of the free energy is found for small $|\theta|$.
- Taylor expansion

$$
\begin{aligned}
f(\theta, T) &:= \frac{1}{N} F(\theta, T) \\
&= -k_B T \ln\left(1 - \frac{4}{\pi^2}\theta^2\right) + \frac{mgR}{N}(\cos\theta - 1) + f_0(T) \\
&\approx k_B \frac{8}{\pi^2 T_c}\left[\frac{1}{2} t\theta^2 + \frac{1}{4} C\theta^4\right] + f_{\text{reg}}(T)
\end{aligned}
$$

- In terms of $M := v_L(\theta) - v_R(\theta)$

$$
f_{\text{sing}}(M, T) = \frac{1}{2} a t M^2 + \frac{\lambda}{4!} M^4,
$$

where $a > 0$ and $\lambda > 0$.
- The above f_{sing} is also known as **Landau free energy**.

- Free energy at equilibrium
- $T \geq T_c$

$$f_e(T) = f_{\text{reg}}(T)$$

- $T < T_c$

$$f_e(T) = f_{\text{sing}}(\mathcal{M} \sim \sqrt{-t}, T) + f_{\text{reg}}(T) = b|t|^{3/2} + f_{\text{reg}}(T)$$

- f_e는 f의 최솟값
- Ehrenfest classification은 f_e를 이용한다.
- Landau theory는 f를 이용한다.
- f_e, f 모두 free energy (density)라고 부르니 익숙해지자.
- 결정되지 않은 θ (M) 역시 order parameter라 부르니 이 역시 익숙해지자.

Idea of the Landau theory

- Singular and regular parts are separated. Only the singular part is under consideration.
- Free energy (density) f is an analytic function of (macroscopic) θ.
- Approximate free energy should have the same symmetry as the original f.

  $$f(-\theta, T) = f(\theta, T)$$

  (even function of θ)
- Coefficient of $\mathcal{M}^2$ is proportional to $T - T_c$.
- Coefficient of $\mathcal{M}^4$ is positive (due to stability).

Exercise

What if the table is tilted by small angle h? In particular, check $(|t| \ll 1)$

$$\left.\frac{\partial \mathcal{M}(h)}{\partial h}\right|_{h=0} \sim |t|^{-1}.$$

Since h breaks the symmetry, it is called a symmetry-breaking field.

Table of contents

Ising model

- Hamiltonian with periodic boundary conditions

$$H_{\mathsf{Ising}}(s) = -J \sum_{\langle i,j \rangle} s_i s_j - H \sum_i s_i$$

- Remarks
  - s : (ordered) set of all N spins s_i. $s_i = \pm 1$
  - We only consider the ferromagnetic Ising model $(J > 0)$
  - $\langle i, j \rangle$: sum over nearest neighbor pairs
  - H : external magnetic field (symmetry breaking field)
  - When $H = 0$, $H_{\mathsf{Ising}}(s) = H_{\mathsf{Ising}}(-s)$
  - When $H = 0$, symmetric two ground states : $s = 1$ or $s = -1$

Partition function and thermodynamics

- "Landau" free energy

$$
\begin{aligned}
Z = \sum_s e^{-\beta H_{\mathsf{Ising}}} &= \sum_s \sum_{-1\le M\le 1} \delta\left(\frac{1}{N}\sum_i s_i - M\right) e^{-\beta H_{\mathsf{Ising}}} \\
&= \sum_M \left[\sum_s \delta\left(\frac{1}{N}\sum_i s_i - M\right) e^{-\beta H_{\mathsf{Ising}}}\right] =: \sum_M e^{-N\beta f_{\mathsf{Lan}}(M,T,H)},
\end{aligned}
$$

where $f_{\mathsf{Lan}}(M, T, H) = -\frac{k_B T}{N} \ln \sum_s \delta\left(\frac{1}{N}\sum_i s_i - M\right) e^{-\beta H_{\mathsf{Ising}}(s)}$.

- A random variable M is also called an "order parameter"
- Note that $\mathbb{P}(M = m) = e^{-N\beta f_{\mathsf{Lan}}(m,T,H)}/Z$
- "The" order paramter $\mathcal{M} = \lim_{N\to\infty} \langle M\rangle = \lim_{N\to\infty} \sum_m m\mathbb{P}(M = m)$

- If $f_{\text{Lan}}(M, T, H)$ has (the unique) minimum at $M = m_0(T, H)$, then

$$e^{-N\beta f(m_0,T,H)} \leq Z \leq Ne^{-N\beta f(m_0,T,H)}$$
$$\Rightarrow f_{\text{Lan}}(m_0, T, H) - k_B T\frac{\ln N}{N} \leq -\frac{k_B T}{N}\ln Z \leq f_{\text{Lan}}(m_0, T, H)$$

  Therefore,

$$f(T, H) := -\lim_{N\to\infty}\frac{k_B T}{N}\ln Z = f(m_0, T, H)$$

- Note that $m_0 = \lim_{N\to\infty}\langle M\rangle = \mathcal{M}$.
- 위의 계산이 free energy minimum이 평형상태가 된다는 의미이다.
- m_0 is actually not unique for $T < T_c$ (cf. the Toy model).

Landau theory for the Ising model on a complete graph

- What is f_{Lan}? Let us first resort to the Landau theory for $H = 0$.
  - $f_{\mathrm{Lan}}(-M, T) = f_{\mathrm{Lan}}(M, T)$: even function of M
  - Numerical observation : around $T = T_c$, m_0 is small.
- Let's be brave and use the Landau theory! (neglect the regular part)

$$f_{\mathrm{Lan}} \approx \frac{1}{2} a t M^2 + \frac{\lambda}{4!} M^4$$

- Now the repeat of the calculation in the Toy model gives the solution.
- It does work for a certain system: all-to-all interaction

- all-to-all interaction with $J \mapsto J/N$. $Js_i^2 = J$ is included.

$$H_{\mathsf{Ising}} = -\frac{J}{2N}\sum_{i,j=1}^{N} s_i s_j - H\sum_{i=1}^{N} s_i = -N\left(\frac{1}{2}JM^2 + HM\right)$$
$$\Rightarrow e^{-N\beta f_{\mathsf{Lan}}} = \binom{N}{N(1+M)/2} e^{-N\beta(-JM^2/2-HM)}.$$

$$\begin{aligned}
f_{\mathsf{Lan}} &= -\frac{1}{2}JM^2 - HM - \frac{k_B T}{N}\ln\binom{N}{N(1+M)/2} \\
&\approx -\frac{1}{2}JM^2 - HM - Tk_B\left(-\frac{1+M}{2}\ln\frac{1+M}{2} - \frac{1-M}{2}\ln\frac{1-M}{2}\right) \\
&\approx -k_B T\ln 2 - HM + \frac{1}{2}(k_B T - J)M^2 + \frac{k_B T}{12}M^4
\end{aligned}$$

- Landau theory works. We have only to find the minimum of f_{Lan}.

Discussion

- Let's rewrite the Hamiltonian as (assume $H = 0$)

$$H_{\text{Ising}} = -\frac{1}{2} J \sum_i s_i \sum_{j \in \mathcal{N}_i} s_j = -\frac{1}{2} J' \sum_i s_i \underbrace{\frac{1}{|\mathcal{N}|} \sum_{j \in \mathcal{N}_i} s_j}_{=: M'_i} = -\frac{1}{2} J' \sum_i s_i M'_i,$$

  where $\mathcal{N}_i$ is the index set of the nearest neighbors of site i with cardinality $|\mathcal{N}|$ (translationally invariant) and $J' = J|\mathcal{N}|$
- If $M'_i \simeq M$ in **typical configurations**, then $H_{\text{Ising}} \approx -\frac{1}{2} J' N M^2$
- By (abusing) the law of large numbers, one would expect $M'_i \simeq M$ when $|\mathcal{N}|$ is large. Accordingly, when $|\mathcal{N}|$ is small, the above approximation is not likely to work.
- Then how small is small? It turns out that the embedding dimensions are more important than $|\mathcal{N}|$ itself (see Sec. 3).

Exercise (Project 1-1)

- Consider the following 1-D "Ising model" with PBC ($s_{N+i} = s_i$).

$$H_2 = -\frac{J}{2}\sum_{i=1}^{N} s_i(s_{i+1} + s_{i+2} + s_{i-1} + s_{i-2}),$$

  which has $|\mathcal{N}| = 4$ just like the 2D Ising model.
- Its free energy density is as follows. ($K := \beta J$)

$$\begin{aligned} f(T) &:= -k_B T \lim_{N\to\infty} \frac{1}{N} \ln Z \\ &= -k_B T \ln \frac{e^{2K} + 1 + \sqrt{(e^{2K}+1)(e^{2K} - 3 + 4e^{-2K})}}{2} \end{aligned}$$

  which is analytic; hence no phase transition. Check it. (Mathematica would be helpful)
- If $H \neq 0$, the calculation would be a nightmare, but try it.

Further discussion

If

$$H_n = -J \sum_{i=1}^{N} s_i \sum_{j=i+1}^{j+n} s_j$$

with large but fixed n (for example, $n = 10^{10}$), do we expect a phase transition? Consider the Ferron-Probenius theorem.

Beyond the Landau theory

- How good is the Landau theory.
  - Landau theory predicts $\beta = \frac{1}{2}$.
  - Onsager solution for the 2D Ising model gives $\beta = \frac{1}{8}$.
  - Numerical studies for the 3D Ising model have concluded $\beta < \frac{1}{2}$.
- Then, why do we care about the Landau theory? First because it works sometimes (when?) and second because it is insightful.
- Can we develop a general "approximation" scheme that gives a plausible result? This is all about the renormalization group (RG).
- We want to come up with an effective model.
  - Block spin : similar degrees of freedom with similar Hamiltonian (Sec. 2)
  - Field theory : an effective model with (uncontrained) continuous degrees of freedom (Sec. 3)

진짜 시키고 싶은 Project (?)

- Find the mathematical formual of the “exact” f_{Lan} in one dimension (numerically, maybe?)
- Can you find f_{Lan} for the critical 2-D Ising model (numerically)?

$$K_c = \frac{1}{2} \ln \left(1 + \sqrt{2}\right)$$

- 저도 답은 모릅니다.

REAL SPACE RENORMALIZATION GROUP

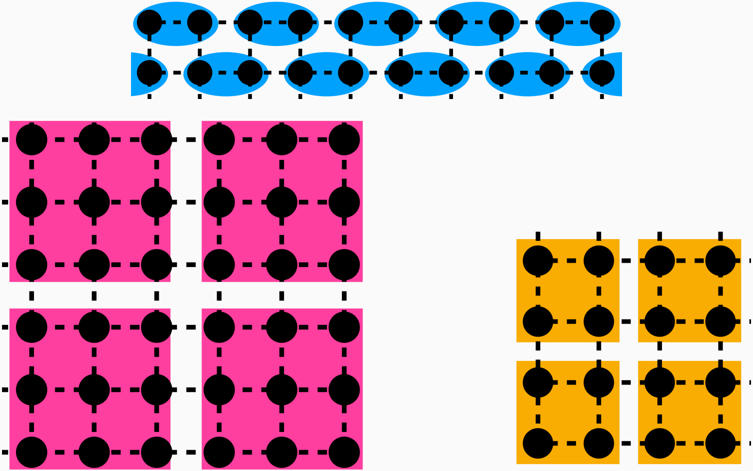
Table of contents

졸보기 (뭉뚱그리기)

source : https://www.scientificamerican.com/article/can-you-drink-saturns-rings/

분배함수 졸보기 : the 2D Ising model on a square lattice

Consider a tiling with regular tiles



B spins in a block

- Fix a tile and define a block spin transformation (i : block index)

$$\mathcal{T}(s'_i : \underbrace{s_{i,1}, \ldots, s_{i,B}}_{=:\{s_i\}}) \quad \text{such that} \quad \sum_{s'_i=\pm} \mathcal{T}(s'_i : \{s_i\}) = 1$$

- Example 1: decimation (blue blocks) $\mathcal{T}(s'_i : \{s_i\}) = \delta(s'_i, s_{i,1})$
- Example 2: majority rule (pink blocks) $\mathcal{T}(s'_i : \{s_i\}) = \Theta(s'_i S)$
- Example 3 : half-and-half rule (yellow blocks)

$$\mathcal{T}(s'_i : \{s_i\}) = \frac{1+s'_i}{2}\left[\Theta(S) + \delta_{S,0}\frac{1}{2}\right] + \frac{1-s'_i}{2}\left[\Theta(-S) + \delta_{S,0}\frac{1}{2}\right],$$

  - $s'_i = \pm 1$ (same as the original spin)
  - $S := \sum_{k=1}^{B} s_{i,k}$ (sum of B original spins in a block)
  - $\Theta(x)$ is the Heaviside step function with $\Theta(0) = 0$.

- new (reduced) Hamiltonian with block variables $\mathscr{H} := \beta H_{\text{Ising}}$

$$e^{-\mathscr{H}'(s')} = \mathrm{Tr}_s \prod_i \mathcal{T}(s'_i : \{s_i\}) e^{-\mathscr{H}(s)}$$

- Partition function and free energy density (N' is also an integer)

$$Z' = \mathrm{Tr}_{s'}\, e^{-\mathscr{H}'(s')} = \mathrm{Tr}_s\, e^{-\mathscr{H}(s)} = Z,$$
$$f = \frac{1}{N} \ln Z = \frac{1}{N} \ln Z' = \frac{1}{b^d N'} \ln Z' = b^{-d} f'$$

  where $b^d (= B)$ is the number of sites in a block and we neglect $-k_B T$ for convenience.

- semi-group property : If $\mathscr{H}'$ and $\mathscr{H}$ have the same form, repeating the block transformation gives $\mathscr{H}''$ with the same form as $\mathscr{H}$. Formally, we write $\mathscr{H}' = \mathcal{R}_b \mathscr{H}$.

$$\mathscr{H}'' = \mathcal{R}_b \mathscr{H}' = \mathcal{R}_b^2 \mathscr{H}$$

real space renormalization group (RSRG)

Table of contents

the 1D Ising model : Transfer matrix

- Partition function of the 1D Ising model ($K := \beta J, h = \beta H$)

$$Z = \sum_{s_1=\pm} \cdots \sum_{s_N=\pm} e^{Ks_1 s_2 + hs_1} e^{Ks_2 s_3 + hs_2} \ldots e^{Ks_N s_1 + hs_N}$$

- 2×2 transfer matrix T

$$T_{ss'} = e^{Kss' + h(s+s')/2} \to T = \begin{pmatrix} T_{++} & T_{+-} \\ T_{-+} & T_{--} \end{pmatrix} = \begin{pmatrix} e^{K+h} & e^{-K} \\ e^{-K} & e^{K-h} \end{pmatrix}$$

- Note that (cf : decimation)

$$\sum_{s_2=\pm} e^{Ks_1 s_2 + h(s_1+s_2)/2} e^{Ks_2 s_3 + h(s_2+s_3)/2} = \sum_{s_2=\pm} T_{s_1 s_2} T_{s_2 s_3} = (T^2)_{s_1 s_3}$$

- Partition function and the transfer matrix

$$Z = \sum_{s_1=\pm} \left(T^N\right)_{s_1 s_1} = \mathrm{Tr}\; T^N$$

- Eigenvalue λ of T

$$\begin{aligned}
\lambda^2 - 2\lambda e^K \cosh h + e^{2K} - e^{-2K} = 0 \\
\Rightarrow \lambda_\pm = & e^K \cosh h \pm \sqrt{e^{2K}\cosh^2 h - e^{2K} + e^{-2K}} \\
= & e^K \cosh h \pm \sqrt{e^{2K}\sinh^2 h + e^{-2K}}
\end{aligned}$$

- $Z = \lambda_+^N + \lambda_-^N$
- free energy density

$$\begin{aligned}
-k_B T \lim_{N\to\infty} \frac{\ln Z}{N} &= -k_B T \ln \lambda_+ - k_B T \lim_{N\to\infty} \frac{1}{N} \ln \left[1 + \left(\frac{\lambda_-}{\lambda_+}\right)^N\right] \\
&= -k_B T \ln \left(e^K \cosh h + \sqrt{e^{2K}\sinh^2 h + e^{-2K}}\right)
\end{aligned}$$

Decimation for $h = 0$

- Assume ($A = 1$ and $N' = N/2$)

$$-\mathscr{H} = N \ln A + K \sum_{\langle i,j \rangle} s_i s_j, \quad -\mathscr{H}' = N' \ln A' + K' \sum_{\langle i,j \rangle} s_i' s_j'$$

- decimation

$$\begin{aligned}
\mathrm{Tr} e^{-\mathscr{H}} &= \sum_{s_1, s_3, s_5, \ldots} \sum_{s_2} A^2 e^{K(s_1+s_3)s_2} \sum_{s_4} A^2 e^{K(s_3+s_5)s_4} \cdots \\
&= \sum_{s_1, s_3, s_5, \ldots} \left\{ 2A^2 \cosh\left[K(s_1+s_3)\right] \right\} \left\{ 2A^2 \cosh\left[K(s_3+s_5)\right] \right\} \cdots \\
&= \sum_{s_1, s_3, s_5, \ldots} \left\{ 2A^2 \cosh(2K) \frac{1+s_1 s_3}{2} + 2A^2 \frac{1-s_1 s_3}{2} \right\} \cdots \\
&= \sum_{s_1, s_3, s_5, \ldots} \left(A' e^{K'} \frac{1+s_1 s_3}{2} + A' e^{-K'} \frac{1-s_1 s_3}{2} \right) \cdots = \mathrm{Tr}' \, e^{-\mathscr{H}'}
\end{aligned}$$

- Another approach

$$T' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A' \begin{pmatrix} e^{K'} & e^{-K'} \\ e^{-K'} & e^{K'} \end{pmatrix}$$

$$= T^2 = A^2 \begin{pmatrix} e^{2K} + e^{-2K} & 2 \\ 2 & e^{2K} + e^{-2K} \end{pmatrix}$$

- $ac = (A')^2 = 4A^4 \cosh(2K) = bd$.
- $a/b = e^{2K'} = \cosh(2K) = d/c$,
- RG transformation

$$K' = \mathcal{R}_2(K) = \frac{1}{2} \ln \cosh(2K), \quad A' = 2A^2 \sqrt{\cosh(2K)} = 2A^2 e^{K'}$$

- Free energy density $(b = 2, d = 1)$

$$\ln A + f(K) = \underbrace{\ln A + \frac{1}{2}(\ln 2 + K')}_{=\frac{1}{2}\ln A'} + \frac{1}{2} f(K')$$

RG transformation

- One more decimation

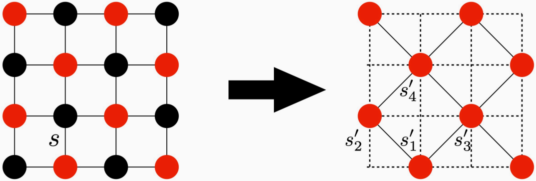
$$\ln A + f(K) = \frac{1}{2}\left(\ln A' + \frac{1}{2}(\ln 2 + K'') + \frac{1}{2} f(K'')\right)$$

- What do we expect if we perform m decimations. ($\mathcal{R}_b^m$)
  - It is convenient to write $\tanh K' = \tanh^2 K$.
  - Fixed point K^* : $\tanh K^* = \tanh^2 K^*$
  - $\tanh K^* = 0$ ($T = \infty$: stable) or 1 ($T = 0$: unstable)
  - Around $K^* = 0$, $K' \approx K^b$ ($b = 2$)
  - $K^{(m)} = R_b^m(K) \approx K^{b^m}$: faster than exponential.

$$\lim_{m\to\infty} \mathcal{R}_b^m(K) = 0$$

Generation of new terms : 2D case

- Decimation in 2D



$$\mathrm{Tr}_s\, e^{Ks(s'_1+s'_2+s'_3+s'_4)+hs} = 2\cosh(K(s'_1+s'_2+s'_3+s'_4)+h)$$

$$\mathscr{H}' = \sum \ln\cosh(K(s'_1+s'_2+s'_3+s'_4)+h)$$

$$= K^{(1)}\sum_i s'_i + \sum_{ij} K^{(2)}_{ij} s'_i s'_j + \sum_{ijk} K^{(3)}_{ijk} s'_i s'_j s'_k + \sum_{ijkl} K^{(4)}_{ijkl} s'_i s'_j s'_k s'_l + \cdots ,$$

where we neglect s independent constant. Note that if $h=0$, only even terms appear.

- From the beginning, we have to consider all terms that are supposed to be generated by the RG process, which makes finding an exact solution by RG infeasible.
- Although 2D Ising model is solved exactly, no exact RSRG transformation is available.
- Even 1D model with nnn interaction is difficult to study the RSRG because longer distance interaction is generated.

Project 1-2

- Majority rule이나 Half-and-half rule을 이용하여 1차원 Ising model의 RSRG를 분석하시오.
- symmetry를 무시한 block transformation을 하면 어떻게 되는지 분석하시오. 예를 들어,

$$\mathcal{T}(s'; s_1, s_2) = \frac{1+s'}{2}\frac{(1+s_1)(1+s_2)}{4} + \frac{1-s'}{2}\left(1 - \frac{(1+s_1)(1+s_2)}{4}\right)$$

- Project 1-1의 문제의 RG 분석을 시도해 보시오.
- 참고로 출제자도 답은 모름

Exercise

다음 reduced Hamitonian을 분석하시오. $(N = 2^n)$

$$
\begin{aligned}
-\mathscr{H} &= K_1 \sum_{i=1,3,5,\dots} s_i s_{i+1} + K_2 \sum_{i=2,4,6,8} s_i s_{i+1} \\
&= K_1 s_1 s_2 + K_2 s_2 s_3 + K_1 s_3 s_4 + K_2 s_4 s_5 + \cdots
\end{aligned}
$$

Exact solution을 먼저 구해보기를 추천드립니다.

RG analysis : general discussion

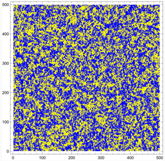
- $\{K\} \equiv (h_1, K_1, \ldots)$: a (ordered) set of all coupling constants that can appear in the RG process. Some (in fact, most of) coupling constants are zero in the original model.
- $\mathscr{H}'$ contains more nonzero coupling constants than $\mathscr{H}$.
- All we have to do is to find $\{K'\} = \mathcal{R}_b(\{K\})$: renormalization-group transformation (practically almost impossible to find exactly).

- If there is a non-trivial fixed point K^*, h^* such that $K_1' - K^* = b^{y_K}(K_1 - K^*)$ and $h_1 - h^* = b^{y_h}(h_1 - h^*)$, then

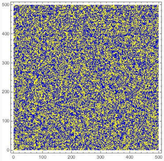
$$f_{\text{sing}}(e_K, e_h) = b^{-d} f_{\text{sing}}(b^{y_K} e_K, b^{y_h} e_h),$$

  where $e_K = K_1 - K^*$ and $e_h := h_1 - h^*$. We assume all the other exponent has negative exponent y.
- 1D Ising model is not a good example $(e_K' = e_K^2)$

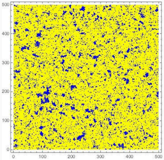
$T > T_c$



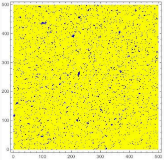
$\xrightarrow{\text{RG flow}}$



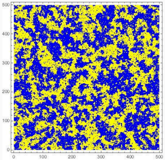
$T < T_c$



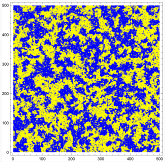
$\xrightarrow{\text{RG flow}}$



$T = T_c$



$\xrightarrow{\text{RG flow}}$



Callan-Symansik-like equation

- f is a generalized homogeneous function

$$f(b^{y_K} e_K, b^{y_h} e_h) = b^d f(e_K, e_h)$$

- cf. Euler homogeneous function

$$U(bS, bV, bN) = b^\alpha U(S, V, N)$$

- If b is a continuous variable,

$$b\frac{d}{db} b^d f(e_K, e_h)\bigg|_{b=1} = df(e_K, e_h) = \left[y_K \frac{\partial}{\partial \ln e_K} + y_h \frac{\partial}{\partial e_h} \right] f(e_K, e_h),$$

  which is a Callan-Symansik(-like) equation or RG equation in the field theory.
- cf. $U = TS - PV + \mu N$ in thermodynamics

MOMENTUM SPACE RENORMALIZATION GROUP

Table of contents

n-vector model

- (reduced) Hamiltonian of the n-vector model

$$-\mathscr{H} = \frac{1}{2}\mathcal{K}_{ij}\boldsymbol{s}_i \cdot \boldsymbol{s}_j + \boldsymbol{h}_i \cdot \boldsymbol{s}_i, \quad \mathcal{K}_{ij} = K(2\mathfrak{p}\delta_{ij} + \delta_{|i-j|,1})$$

  where $\boldsymbol{s}_i$ is an n-dimensional vector with $|\boldsymbol{s}_i|^2 = 1$.

  - $n = 1$: Ising, $n = 2$: XY, $n = 3$: Heisenberg, $\cdots$.
  - $\delta_{|i-j|,1}$는 i, j가 nn이면 1 아니면 0을 의미한다.
  - $\mathfrak{p}$는 $\mathcal{K}$가 positive-definite 행렬이 되도록 하는 임의의 값이다.
  - $|\boldsymbol{s}_i|^2 = 1$ (상수)이므로, **$\mathfrak{p}$의 값은 물리에 영향을 주지 않는다**.

- partition function

$$Z = \operatorname{Tr}\exp\left(\frac{1}{2}\mathcal{K}_{ij}\boldsymbol{s}_i \cdot \boldsymbol{s}_j + \boldsymbol{h}_i \cdot \boldsymbol{s}_i\right), \quad \operatorname{Tr} \equiv \prod_i \int d^n\boldsymbol{s}_i \delta(|\boldsymbol{s}_i| - 1).$$

a few Observables

- Consider uniform $\boldsymbol{h}$, that is, $\boldsymbol{h}_i = \boldsymbol{h} = (h^1, \ldots, h^n)$
- Magnetization $\boldsymbol{m} = (m_1, \ldots, m_n)$

$$Nm_\alpha := \sum_i \langle s_i^\alpha \rangle = \frac{1}{Z} \operatorname{Tr} \left(\sum_i s_i^\alpha \right) \exp \left(\frac{1}{2} \mathcal{K}_{ij} \boldsymbol{s}_i \cdot \boldsymbol{s}_j + \boldsymbol{h}_i \cdot \boldsymbol{s}_i \right) = \frac{\partial \ln Z}{\partial h^\alpha}$$

- Susceptibility $\chi^{\alpha\beta}$

$$\begin{aligned} k_B T \chi^{\alpha\beta} &= \frac{\partial m^\alpha}{\partial h^\beta} = \frac{1}{N} \frac{\partial^2 \ln Z}{\partial h^\beta \partial h^\alpha} = \frac{1}{N} \left(\frac{1}{Z} \frac{\partial^2 Z}{\partial h^\beta \partial h^\alpha} - \frac{1}{Z^2} \frac{\partial Z}{\partial h^\beta} \frac{\partial Z}{\partial h^\alpha} \right) \\ &= \frac{1}{N} \underbrace{\left(\sum_{i,j} \left\langle s_i^\alpha s_j^\beta \right\rangle - \left\langle \sum_i s_i^\alpha \right\rangle \left\langle \sum_j s_i^\beta \right\rangle \right)}_{= O(N)} \end{aligned}$$

Hubbard-Stratonovich transformation

- Gaussian 적분

$$I = \int_{-\infty}^{\infty} \left(\prod_{i=1}^{N} d\phi_i \right) \exp\left[\sum_{ij} \left(-\frac{1}{2}\phi_i (\mathcal{K}^{-1})_{ij} \phi_j + s_i \phi_i \right) \right],$$

where $\mathcal{K}$ is a real symmetric positive-definite matrix.

- 치환 적분 $\phi_i = \mathcal{K}_{ij}(\psi_j + s_j)$ (note 참고)

$$I = \det \mathcal{K} \exp\left(\frac{1}{2} \sum_{ij} s_i \mathcal{K}_{ij} s_j \right) \int \left(\prod_i d\psi_i \right) \exp\left(-\frac{1}{2} \sum_{ij} \psi_i \mathcal{K}_{ij} \psi_j \right)$$

$$= \sqrt{(2\pi)^N \det \mathcal{K}} \exp\left(\frac{1}{2} \sum_{ij} s_i \mathcal{K}_{ij} s_j \right)$$

Hubbard-Stratonovich Transformation

$$\exp\left(\frac{1}{2}s_i\mathcal{K}_{ij}s_j\right) = C\int[d\phi]\exp\left(-\frac{1}{2}\phi_i\mathcal{J}_{ij}\phi_j + s_i\phi_i\right).$$

- $\mathcal{J} := \mathcal{K}^{-1}$, $[d\phi] := \prod_{i=1}^{N} d\phi_i$, $C := 1/\sqrt{\det\mathcal{K}(2\pi)^N}$
- Einstein convention (반복되는 index에 대한 합)
- discrete variable $s \to$ continuous variable ϕ
- s간의 상호작용을 ϕ간의 상호작용으로 변경.
- s는 서로 독립적으로 처리 가능.

- Hubbard-Stratonovich transformation for the n-vector model

$$\begin{aligned}\frac{Z}{C^n} &= \mathrm{Tr}\int [d\phi]\exp\left[-\frac{1}{2}\mathcal{K}_{ij}^{-1}\boldsymbol{\phi}_i\cdot\boldsymbol{\phi}_j + s_i\cdot(\boldsymbol{\phi}_i+\boldsymbol{h}_i)\right] \Leftarrow \boldsymbol{\phi}_i+\boldsymbol{h}_i \mapsto \boldsymbol{\phi}_i \\ &= \mathrm{Tr}\int [d\phi]\exp\left[-\frac{1}{2}\mathcal{J}_{ij}(\boldsymbol{\phi}_i-\boldsymbol{h}_i)\cdot(\boldsymbol{\phi}_j-\boldsymbol{h}_j) + \boldsymbol{s}_i\cdot\boldsymbol{\phi}_i\right] \\ &= \int [d\phi]\exp\left[-\frac{1}{2}\mathcal{J}_{ij}\boldsymbol{\phi}_i\cdot\boldsymbol{\phi}_j + \mathcal{J}_{ij}\boldsymbol{h}_i\cdot\boldsymbol{\phi}_j - \frac{1}{2}\mathcal{J}_{ij}\boldsymbol{h}_i\cdot\boldsymbol{h}_j + A(\phi)\right],\end{aligned}$$

  where $A(\phi) := \ln\mathrm{Tr}\exp(\boldsymbol{s}_i\cdot\boldsymbol{\phi}_i)$.
- $A(\phi)$ is related to the modified Bessel function I.
- For $n=1$,

$$\begin{aligned}A(\phi) &= \ln\prod_i\left(\sum_{s_i=\pm1}e^{s_i\phi_i}\right) = \sum_i \ln(2\cosh(\phi_i)) \\ &\approx N\ln 2 + \sum_i\left(\frac{\phi_i^2}{2} - \frac{\phi_i^4}{12}\right)\end{aligned}$$

- For general n,

$$A(\phi) \approx N \ln S_{n-1} + \sum_i \left(\frac{|\boldsymbol{\phi}_i|^2}{2n} - \frac{(|\boldsymbol{\phi}_i|^2)^2}{4n^2(n+2)} \right),$$

  where S_{n-1} is the area of $(n-1)$-sphere.
  $(S_0 = 2, S_1 = 2\pi, S_2 = 4\pi)$
- Partition function

$$\therefore Z = \int [d\phi] \exp \left[-\widetilde{\mathcal{H}}(\phi) + \mathcal{J}_{ij} \boldsymbol{h}_i \cdot \boldsymbol{\phi}_j - \frac{1}{2} \mathcal{J}_{ij} \boldsymbol{h}_i \cdot \boldsymbol{h}_j \right]$$

  where

$$\begin{aligned} \widetilde{\mathcal{H}}(\phi) &= \frac{1}{2} \mathcal{J}_{ij} \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j + n \ln C - A(\boldsymbol{\phi}) \\ &\approx \frac{1}{2} \left(\mathcal{J}_{ij} - \frac{1}{n} \delta_{ij} \right) \boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j + \frac{(\boldsymbol{\phi_i} \cdot \boldsymbol{\phi_i})^2}{4n^2(n+2)} + C' \end{aligned}$$

  where C, C' are constants.

What is $\mathcal{J} = \mathcal{K}^{-1}$

- d dimensional hypercubic lattice with index $i = \boldsymbol{x} = (x_1, \ldots, x_d)$.
- $1 \le x_i \le L$, $N = L^d$. 편의상 **L은 홀수**라고 하자.
- Dirac notation: orthonormal basis $\{|\boldsymbol{x}\rangle\}$.
- operator $\hat{\mathcal{K}}$: $\langle \mathbf{x}|\hat{\mathcal{K}}|\boldsymbol{y}\rangle = \mathcal{K}_{\mathbf{x},\boldsymbol{y}} = \langle \boldsymbol{y}|\hat{\mathcal{K}}|\mathbf{x}\rangle$
- Translational invariance: Fourier mode

$$|\boldsymbol{k}\rangle \equiv \frac{1}{\sqrt{N}} \sum_{\boldsymbol{x}} e^{i\boldsymbol{k}\cdot\mathbf{x}} |\mathbf{x}\rangle, \quad \boldsymbol{k} = \frac{2\pi}{L}(n_1, \ldots, n_d), n_i = -\frac{L-1}{2}, \ldots, \frac{L-1}{2},$$

$$\hat{\mathcal{K}}|\boldsymbol{k}\rangle = \frac{1}{\sqrt{N}} \sum_{\boldsymbol{xy}} |\boldsymbol{y}\rangle\langle \boldsymbol{y}|\hat{\mathcal{K}}|\mathbf{x}\rangle e^{i\boldsymbol{k}\cdot\mathbf{x}} = \underbrace{2K\left[\mathfrak{p} + \sum_{i=1}^{d} \cos(k_i)\right]}_{\equiv 1/\Delta_0(\boldsymbol{k})>0} |\boldsymbol{k}\rangle,$$

  which gives $\hat{\mathcal{J}}|\boldsymbol{k}\rangle = \Delta_0(\boldsymbol{k})|\boldsymbol{k}\rangle$. $\mathcal{J}_{ij} \sim e^{-\kappa r}$ (note).
- $\mathfrak{p}$ should be larger than d.

- For later purpose, we calculate

$$\sum_{\mathbf{x}} \mathcal{K}_{\mathbf{x},\boldsymbol{y}} = \sum_{\mathbf{x}} \langle \mathbf{x} | \hat{\mathcal{K}} | \boldsymbol{y} \rangle = \sqrt{N} \langle \boldsymbol{k} = 0 | \hat{K} | \boldsymbol{y} \rangle = 2K(\mathfrak{p} + d),$$

which is independent of $\boldsymbol{y}$. Accordingly, we also get

$$\sum_{\mathbf{x}} \mathcal{J}_{\mathbf{x},\boldsymbol{y}} = \frac{1}{2K(\mathfrak{p} + d)} = \frac{T}{nT_0}, \qquad T_0 \equiv \frac{2J(\mathfrak{p} + d)}{nk_B}.$$

- For small k, $\Delta_0(k) \approx \frac{T}{nT_0} + k^2/\tilde{\alpha}$ with $\tilde{\alpha} = 4K(\mathfrak{p} + d)^2$.
- determinant of $\mathcal{K}$

$$\frac{1}{N} \ln \det \mathcal{K} = \ln(2K) + \underbrace{\frac{1}{N} \sum_{\boldsymbol{k}} \ln \left[\mathfrak{p} + \sum_i \cos(k_i) \right]}_{\text{finite and independent of } T}$$

Observables in terms of continuous fields

- We neglect constant.
- “effective” Hamiltonian $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_0 + V_I$ with

$$\widetilde{\mathcal{H}}_0 \equiv \frac{1}{2}\left(\mathcal{J}_{ij} - \frac{1}{n}\delta_{ij}\right)\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_j, \qquad V_I = \frac{\lambda}{4!}(\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_i)^2$$

- partition function

$$Z = \int [d\phi] \exp\left[-\widetilde{\mathcal{H}}_0 - V_I + \mathcal{J}_{ij}\boldsymbol{h}_i \cdot \boldsymbol{\phi}_j - \frac{1}{2}\mathcal{J}_{ij}\boldsymbol{h}_i \cdot \boldsymbol{h}_j\right]$$

- order parameter for uniform external field ($\boldsymbol{h}_i = \boldsymbol{h}$, $\forall_i$)

$$\frac{1}{N}\sum_i \langle s_i^\alpha \rangle = \frac{1}{N}\frac{\partial \ln Z}{\partial h^\alpha} = \frac{1}{N}\sum_{ij}\left(\mathcal{J}_{ij}\left\langle \phi_j^\alpha \right\rangle - \mathcal{J}_{ij}h^\alpha\right)$$
$$= \Delta_0(0)\frac{1}{N}\sum_j \left\langle \phi_j^\alpha \right\rangle - \Delta_0(0)h^\alpha$$

  where s_i^α is the α-th component of $\boldsymbol{s}_i$ and so on.

- Susceptibility

$$k_B T\chi^{\alpha\beta} = \frac{1}{N}\frac{\partial^2 \ln Z}{\partial h^\alpha \partial h^\beta}$$
$$= \frac{\Delta_0(0)^2}{N}\sum_{ij}\left(\langle \phi_i^\alpha \phi_j^\beta \rangle - \langle \phi_i^\alpha \rangle \langle \phi_j^\beta \rangle\right) - \Delta_0(0)\delta_{\alpha\beta}$$

Discussion

- Since $\Delta_0(0) = T/(nT_0)$, divergence of χ originates only from fluctuations of ϕ. Hence we will regard $\boldsymbol{\phi}$ as a continuous spin vector.
- Since $\mathcal{J}_{ij}\boldsymbol{h}_i \cdot \boldsymbol{h}_j = Nh^2\Delta_0(0)$ for uniform $\boldsymbol{h}_i$ and we are mostly interested in the case with $\boldsymbol{h} = 0$, we will drop this term from now on. We rewrite the partition function as

$$Z = \int [d\phi] \exp\left[-\widetilde{\mathcal{H}}_0 - V_I + \boldsymbol{h}_i \cdot \boldsymbol{\phi}_i\right].$$

- $\widetilde{\mathcal{H}}$에서 모든 field ϕ_i가 값이 같은 아주 작은 상수라면,

$$\frac{\widetilde{\mathcal{H}}}{N} \approx \frac{1}{2}a\boldsymbol{\phi}^2 + \frac{\lambda}{4!}|\boldsymbol{\phi}|^4.$$

Landau free energy를 얻는다. 이 때, $a \propto (T - T_0)$이고 $\lambda > 0$이다.

'Steepest descent method' and the mean field theory

- saddle point analysis: to find the minimum of $\widetilde{\mathcal{H}} - \boldsymbol{h}_i \cdot \boldsymbol{\phi}_i$.

$$\frac{\partial \widetilde{\mathcal{H}}}{\partial \boldsymbol{\phi}_i} = \sum_j \left(\mathcal{J}_{ij} - \frac{1}{n}\delta_{ij} \right) \boldsymbol{\phi}_j + \frac{\lambda}{3!}(\boldsymbol{\phi}_i \cdot \boldsymbol{\phi}_i)^2 \boldsymbol{\phi}_{\boldsymbol{i}} = \boldsymbol{h}_{\boldsymbol{i}}.$$

- homogeneous solution for $h = 0$ (set $\boldsymbol{\phi}_i = (\phi_0, 0, \ldots, 0)$)

$$0 = \left(\frac{T}{nT_0} - \frac{1}{n} \right) \phi_0 + \frac{\lambda \phi_0^3}{3!}$$

$$\phi_0 \propto \begin{cases} (T_0 - T)^{1/2}, & T < T_0 \\ 0, & T > T_0 \end{cases}$$

- Same order parameter exponent as the Landau theory.
- But, critical point T_0 depends on $\mathfrak{p}$? Something is wrong!

Failure of the ‘steepest descent’ method

- Fully connected graph에서 본 것처럼 “최솟값”이 free energy가 된다는 아이디어는 틀린 것은 아니다. 문제는 잘못된 함수의 최솟값을 찾았기 때문에 발생한다.
- Partition function을 다시 보자.

$$
\begin{aligned}
Z &= \sum_C e^{-\beta H(C)} = \sum_C \sum_E e^{-\beta E}\delta[E - H(C)] \\
&= \sum_E e^{-\beta E} \underbrace{\sum_C \delta[E - H(C)]}_{=\Omega(E)=e^{S(E)/k_B}} = \sum_E e^{-\beta[E - TS(E)]} = \sum_E e^{-\beta F(E)} \\
&= e^{-\beta F_{\text{min}}} \times O(N) \Rightarrow \lim_{N\to\infty} \frac{-k_B T \ln Z}{N} = \lim_{N\to\infty} \frac{F_{\text{min}}}{N} = f(T)
\end{aligned}
$$

- 즉, (적분) 변수의 개수가 N이면 변수 개수의 효과, 즉, 엔트로피를 반드시 고려해야 한다.
- 앞의 ‘saddple point’ 방식, 즉, Landau theory는 entropy를 무시했기 때문에 틀린 결과를 준다.
- 이 현상을 ‘fluctuation’을 무시해서 발생한 문제라고 이야기한다.
- 하지만, Landau theory가 맞는 예측을 하는 경우도 있다 (upper critical dimension). 이에 대한 이해는 RG가 제공한다.

momentum space representation of $\widetilde{\mathcal{H}}_0$

- Fourier transform

$$\phi_{\boldsymbol{x}} \equiv \frac{1}{N}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{\varphi}_{\boldsymbol{k}}, \quad \boldsymbol{\varphi}_{\boldsymbol{k}} = \sum_{\boldsymbol{x}} e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{\phi}_{\boldsymbol{x}} = \boldsymbol{\varphi}^*_{-\boldsymbol{k}}$$

- Although $\Re\boldsymbol{\varphi}_{\boldsymbol{k}}$, $\Im\boldsymbol{\varphi}_{\boldsymbol{k}}$ are actually independent (integral) variables, we can formally treat $\boldsymbol{\varphi}_{\boldsymbol{k}}$ and $\boldsymbol{\varphi}_{-\boldsymbol{k}}$ as independent.
- $\widetilde{\mathcal{H}}_0$ in the momentum space ($\hbar = 1$)

$$\begin{aligned}\frac{1}{2}\sum_{\boldsymbol{x},\boldsymbol{y}} \mathcal{J}_{\boldsymbol{x},\boldsymbol{y}}\boldsymbol{\phi}_{\boldsymbol{x}}\cdot\boldsymbol{\phi}_{\boldsymbol{y}} &= \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}'}\varphi_{\boldsymbol{k}}\varphi_{\boldsymbol{k}'}\frac{1}{N^2}\sum_{\boldsymbol{x},\boldsymbol{y}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}+\boldsymbol{k}'\cdot\boldsymbol{y}}\mathcal{J}_{\boldsymbol{x},\boldsymbol{y}} \\ &= \frac{1}{2N}\sum_{\boldsymbol{k},\boldsymbol{k}'}\varphi_{\boldsymbol{k}}\varphi_{\boldsymbol{k}'}\langle -\boldsymbol{k}'|\hat{\mathcal{J}}|\boldsymbol{k}\rangle = \frac{1}{2N}\sum_{\boldsymbol{k}}|\boldsymbol{\varphi}_{\boldsymbol{k}}|^2\Delta_0(k)\end{aligned}$$

(naive) continuum limit of $\mathcal{H}_0$

- $N \to \infty$ (thermodynamic limit) : $\boldsymbol{k}$ is 'almost' continuous.
- 만약 $\varphi_{\boldsymbol{k}}$가 적분가능한 함수(Lesbegue measurable function)라 **믿으면**,

$$2\widetilde{\mathcal{H}}_0 = \frac{1}{N}\sum_{\boldsymbol{k}} \Delta_0(\boldsymbol{k})|\varphi_{\boldsymbol{k}}|^2 \overset{?}{\approx} \int_{-\pi}^{\pi} \frac{d\boldsymbol{k}}{(2\pi)^d}\Delta_0(\boldsymbol{k})|\varphi_{\boldsymbol{k}}|^2$$

$$= \int_{-\Lambda}^{\Lambda} \frac{d^d\boldsymbol{\kappa}}{(2\pi)^d}\frac{\Delta_0(a_0\boldsymbol{\kappa})}{a_0^2}\,|a_0\varphi_{a_0\boldsymbol{\kappa}}|^2 = \int^{\Lambda} \frac{d^d\boldsymbol{k}}{(2\pi)^d}\Delta(\boldsymbol{k})|\boldsymbol{\varphi}(\boldsymbol{k})|^2$$

- a_0 : lattice constant, $\Lambda = \pi/a_0$: cutoff, rename: $\boldsymbol{\kappa} = \boldsymbol{k}/a_0 \mapsto \boldsymbol{k}$

$$\Delta(\boldsymbol{k}) \equiv \frac{\Delta_0(a_0\boldsymbol{k})}{a_0^2}, \quad \boldsymbol{\varphi}(\boldsymbol{k}) \equiv a_0^{1+d/2}\varphi_{a_0\boldsymbol{k}}$$

- $\Delta(\boldsymbol{k}) \approx r + \alpha k^2$

- continuum fields in real space

$$\phi_{\mathbf{x}} = \frac{1}{N}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\mathbf{x}}\varphi_{\boldsymbol{k}} \approx \int_{-\pi}^{\pi}\frac{d^d\boldsymbol{k}}{(2\pi)^d}e^{i\boldsymbol{k}\cdot\mathbf{x}}\varphi_{\boldsymbol{k}}$$
$$= a_0^{d/2}\int_{-\Lambda}^{\Lambda}\frac{d^d\boldsymbol{k}}{(2\pi)^d}e^{i\boldsymbol{k}\cdot\boldsymbol{z}}\varphi_{a_0\boldsymbol{k}} = a_0^{-1+d/2}\int_{-\Lambda}^{\Lambda}\frac{d^d\boldsymbol{k}}{(2\pi)^d}e^{i\boldsymbol{k}\cdot\boldsymbol{z}}\varphi(\boldsymbol{k})$$

- Defining $\phi(\mathbf{x}) = a_0^{1-d/2}\phi_{\mathbf{x}/a_0}$ $(\mathbf{x}\mapsto\mathbf{x}/a_0)$, we get

$$\phi(\mathbf{x}) = \int_{-\Lambda}^{\Lambda}\frac{d^dk}{(2\pi)^d}e^{i\boldsymbol{k}\cdot\mathbf{x}}\varphi(\boldsymbol{k}),\quad \varphi(\boldsymbol{k}) = \int_{-\infty}^{\infty}d^dx e^{-i\boldsymbol{k}\cdot\mathbf{x}}\phi(\mathbf{x})$$

- $\varphi(\boldsymbol{k})$의 차원 $[\varphi(\boldsymbol{k})] = [a_0^{1+d/2}] = [\Lambda^{-1-d/2}] \doteq -1-\frac{d}{2}$
- $\phi(\mathbf{x})$의 차원 $[\phi(\mathbf{x})] = [a_0^{1-d/2}] = [\Lambda^{-1+d/2}] \doteq -1+\frac{d}{2}.$

- $\Delta(k) = r + \alpha^2 k^2$ 인 경우,

$$\widetilde{\mathcal{H}}_0 = \int \frac{d\boldsymbol{k}}{(2\pi)^d} \frac{1}{2} \left(\alpha^2 k^2 + r\right) |\boldsymbol{\varphi}(\boldsymbol{k})|^2 = \frac{1}{2} \int r\phi^2 + \alpha^2 |\nabla \boldsymbol{\phi}|^2 \, d\boldsymbol{x},$$

  where we have assumed $\phi(\boldsymbol{x}) \to 0$ as $|\mathbf{x}| \to \infty$ and
  $|\nabla \boldsymbol{\phi}|^2 \equiv \sum_{\ell=1}^{n} \left(\nabla \phi^{(\alpha)}\right)^2.$

- 미분을 진짜 미분으로 생각하지 말고 discrete difference를 형식적으로 쓴 기호라고 생각하는 것이 더 정확하다.

Field theory

- effective Hamiltonian space-continuous fields

$$\mathcal{H} = \mathcal{H}_G + V_I,$$
$$\mathcal{H}_G = \frac{1}{2}\int d\mathbf{x}\left[\alpha^2 \left|\nabla\boldsymbol{\phi}(\mathbf{x})\right|^2 + \mu^2|\boldsymbol{\phi}(\mathbf{x})|^2\right], \quad V_I = \frac{\lambda}{4!}\int d\mathbf{x}(|\boldsymbol{\phi}(\mathbf{x})|^2)^2,$$

- $\mu^2 \propto T - T_0$. μ^2은 양수, 음수 모두 될 수 있지만, 양자장론의 mass에 해당하는 부분($E^2 = p^2 + m^2$)이라 제곱을 포함하고 쓰고 있다.
- $\lambda = 0$인 경우를 Gaussian theory라고 부른다. (subscript G)
- 나중에 보겠지만, $\phi^6, \phi^8, \cdots$는 고비행동에 영향을 주지 않는다. (물론, 이런 항들이 들어가면 critical point는 달라진다.)
- effective Hamiltonian $\mathcal{H}$를 Ginzburg-Landau-Wilson free energy functional이라고도 부른다.

Generating functional $Z[J]$

- partition function

$$Z[J] \equiv \int \mathcal{D}\phi \exp\left[-\mathcal{H} + \int \boldsymbol{J}(\mathbf{x}) \cdot \boldsymbol{\phi}(\mathbf{x})\, d^d x\right],$$

$$\mathcal{H} = \int d^d x \left[\frac{1}{2}\alpha^2 \left|\nabla\boldsymbol{\phi}\right|^2 + \frac{1}{2}\mu^2|\boldsymbol{\phi}|^2 + \frac{\lambda}{4!}(|\boldsymbol{\phi}|^2)^2\right]$$

- $\boldsymbol{J}$ (source라고도 부른다)는 external magnetic field라는 물리적인 의미도 있지만, 계산을 용이하게 하도록 도입되기도 한다.
- $Z[J]$를 generating functional이라고도 부른다.
- $\mathcal{D}\boldsymbol{\phi}$는 Nn개 변수적분 후 $N \to \infty$를 취할 것이라는 기호로 간주하는 것이 마음이 편하다. (아니면 잘 정의된 measure가 필요)
- α, μ, λ를 **bare parameter**라 부른다.

Discussion about cutoff

- 고비행동은 large scale (or small momentum) physics이므로, Λ가 영향을 주지않는 이론이 필요하다.
- $\Lambda_2 = \sqrt{d}\Lambda$라고 하면, 원래 적분구간은 반지름이 각각 Λ, $\sqrt{d}\Lambda$인 두 구 사이에 있는데, 만약 우리의 이론이 Λ-independent하다면, hypercube에서의 적분이나 구에서의 적분이나 같은 결과를 줄 것이다. 따라서, 이후에는 $\boldsymbol{k}$의 적분구간은 반지름 Λ인 d 차원 구로 변경하여 계산할 것이다.
- 최종적으로 $\Lambda \to \infty$ 극한을 취할 것이다. 이 극한이 말이 되는 결과를 주면, **"continuum theory"가 존재한다**고 이야기한다 (RG).

차원 분석 (Dimensional analysis): canonical (engineering) dimension

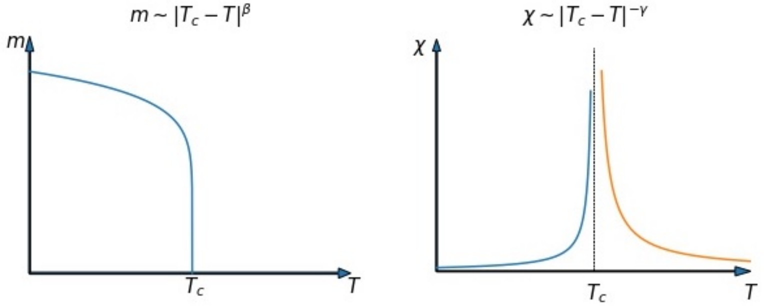
- $\mathcal{H}$는 차원이 없는 수이어야만 한다. $[\mathcal{H}]=0$
- α는 적분변수의 치환으로 항상 차원이 없는 수로 둘 수 있다. $[\alpha]=0$
- $[\mu^2]=[\nabla^2]=2.$
- $\phi(x)$의 차원: $0=\left[\int d^dx|\nabla\phi|^2\right]=-d+2+2[\phi]\Rightarrow[\phi]=\frac{d}{2}-1$
- source $J(x)$의 차원: $0=-d+[\phi]+[J(x)]\rightarrow[J(x)]=1+\frac{d}{2}$
- λ의 차원 : $0=-d+[\lambda]+4[\phi]\rightarrow[\lambda]=\mathbf{4-d}.$
- momentum space field $\varphi(\boldsymbol{k})$의 차원

$$[\varphi(\boldsymbol{k})]=\left[\int d^dxe^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\phi(\boldsymbol{x})\right]=-d-1+\frac{d}{2}=-\frac{d}{2}-1$$

- $\lambda_{s,r}\int d^dx(\nabla)^s\phi^r$의 항이 있는 경우, $[\lambda_{s,r}]=d+r-\frac{1}{2}rd-2s$

Critical behavior and critical exponents

- critical exponents β, γ $(h = 0)$



- Behavior of correlation functions $G_c(r)$
  - At the critical point, $G_c(r) \sim r^{-(d-2+\eta)}$ (scale invariance)
  - η is called the anomalous dimension.
  - For $T \neq T_c$, $G_c(r) \sim \exp(-r/\xi)$ with $\xi \sim |T - T_c|^{-\nu}$.

Critical behavior and critical exponents (cont.)

- exponent α (diverging energy fluctuation)

$$c_v \sim |T - T_c|^{-\alpha}$$

- exponent δ (at the critical point)

$$m \sim |h|^{\delta} \text{sign}(h)$$

comparison of exponents

	α	β	γ	δ	η	ν
mean field	0	$\frac{1}{2}$	1	3	0	$\frac{1}{2}$
2D Ising	0(log)	$\frac{1}{8}$	$\frac{7}{4}$	5	$\frac{1}{4}$	1

- mean field가 틀린 이유는? RG will explain.

차원분석과 critical exponents

- mean magnetization $\langle\phi\rangle$
  - $\langle\phi\rangle$은 μ와 λ의 함수.
  - ϕ의 차원은 $-1+d/2$이므로, $\langle\phi\rangle \propto 1/\sqrt{\lambda}$.
  - $\sqrt{\lambda}\phi$의 차원은 1이므로 μ와 비례
  - $\mu^2 = T - T_0$이므로 $\langle\phi\rangle \propto \sqrt{|T-T_0|} \to \beta = \frac{1}{2}$
- correlation length ξ
  - ξ의 차원은 -1이므로 이 차원을 만들 수 있는 조합은 μ^{-1}.
  - $\xi \propto |T-T_0|^{-1/2} \to \nu = \frac{1}{2}$
- equation of state at the critical point
  - J는 λ와 ϕ의 함수.
  - $[J/\phi] = 2$이고 $[\lambda\phi^2] = 2$.
  - 따라서 가능한 조합은 $J \propto \phi^3 \to \delta = 3$

차원분석과 critical exponents (cont.)

- susceptibility $\chi \propto \dfrac{d\phi}{dJ}$
  - χ의 차원은 $[\phi] - [J] = -2$.
  - μ, λ로 차원 -2를 만들 수 있는 조합은 $\mu^{-2} \to \chi \propto |T - T_0|^{-1}, \gamma = 1$.
- correlation function $G_c(x) = \langle \phi(x)\phi(0) \rangle$ at the critical point
  - G_0는 x, λ, $\langle\phi\rangle$의 함수 ($\langle\phi\rangle = 0$ at the critical point).
  - λ는 $\langle\phi\rangle$와 곱의 형태로만 등장하므로 G_0는 x만의 함수
  - $[G_0] = d - 2$이므로 $G_0 \propto 1/r^{d-2} \to \eta = 0$
  - $\eta \neq 0$은 어떻게 가능한가? (혹은 effective Hamiltonian은 critical behavior를 제대로 설명할 수 없나?)
- mean field theory는 단순 차원비교와 consistent하다.
- specific heat c_v는 차원분석만으로는 알 수 없다 (discontinuity).
- 굳이 하면, c_v는 $\mathcal{H}$의 fluctuation이고 $[\mathcal{H}] = 0$이므로, $\alpha = 0$.

이후에는

- $n = 1$인 경우만 논의한다 (Ising model)
- momentum space만 고려하겠다.

$$\mathcal{H}_G = \int_k \frac{1}{2}(\alpha^2 k^2 + \mu^2)\varphi_{-k}\varphi_k =: \int_k \frac{1}{2}\frac{|\varphi_k|^2}{\Delta(k)^2}$$

$$V_I = \frac{\lambda}{4!}\int_{1,2,3,4} \varphi_1\varphi_2\varphi_3\varphi_4 \delta(1+2+3+4)$$

- notation

$$\int_x := \int d^d\mathbf{x}, \quad \varphi_k := \int_x e^{i\boldsymbol{k}\cdot\mathbf{x}}\phi(\mathbf{x}), \quad \phi(x) := \phi(\mathbf{x}), \quad \varphi_i := \varphi_{k_i},$$

$$\int_k := \int^\Lambda \frac{d^d\boldsymbol{k}}{(2\pi)^d}, \quad \delta(1+2+\cdots) = (2\pi)^d\delta(k_1+k_2+\cdots)$$

Table of contents

Gaussian theory with source J

- Gaussian theory

$$Z_G[J] = \int \mathcal{D}\phi \exp\left(-\mathcal{H}_G + \int_k J_{-k}\varphi_k\right), \; \mathcal{H}_G = \int_k \frac{1}{2}\frac{|\varphi_k|^2}{\Delta(k)^2}$$

- Gaussian theory에서는 $\mu^2 > 0$ 이어야만 됨.
- Average

$$\langle O[\varphi]\rangle = \frac{1}{Z_G[0]}\int \mathcal{D}\phi\, O[\varphi] \exp\left(-\mathcal{H}_G\right)$$

- In particular,

$$\langle \varphi_{k_1}\cdots\varphi_{k_m}\rangle = \frac{1}{Z_G[0]}(2\pi)^{dm}\,\frac{\delta}{\delta J_{-k_1}}\cdots\frac{\delta}{\delta J_{-k_m}} Z_G[J]\bigg|_{J=0}$$

- 위 평균계산 방법은 일반적인 경우에도 적용된다.

- 치환 적분: $\varphi_k \mapsto \varphi_k + J_k \Delta(k)$

$$
\begin{aligned}
Z_G[J] &= \int \mathcal{D}\phi \exp\left\{-\frac{1}{2}\int_k \frac{\phi(-k)\phi(k)}{\Delta(k)}\right\} \exp\left[\frac{1}{2}\int_k \Delta(k)|J_k|^2\right] \\
&= Z_G[0] \exp\left[\frac{1}{2}\int_{12} \Delta(k_1) J_{-1} J_{-2} \delta(1+2)\right].
\end{aligned}
$$

- Correlation function

$$
\langle \varphi_1 \varphi_2 \rangle := \frac{(2\pi)^{2d}\delta^2}{\delta J_{-1}\delta J_{-2}} \exp\left[\frac{1}{2}\int_{34} \Delta(k_3) J_{-3} J_{-4} \delta(1+2)\right] = \Delta(k_1)\delta(1+2)
$$

보통 $\langle \varphi_k \varphi_{-k} \rangle = \Delta(k)$라 쓴다.

- Fourier transform of Δ (note 참고)

$$
\Delta(x, y) = \int_k \frac{e^{ik(x-y)}}{\alpha^2 k^2 + \mu^2} \sim \frac{e^{-\mu|\mathbf{x}-\mathbf{y}|/\alpha}}{|\mathbf{x}-\mathbf{y}|^{(d-1)/2}}
$$

- Gaussian theory는 공간이 discrete한 경우에도 똑같이 적용된다. (continuum limit가 잘 정의된다.)

- Perturbative expansion ($J\varphi$는 적분을 간단하게 쓴 것임)

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi e^{-\mathcal{H}_G - V_I(\varphi) + J\varphi} = \int \mathcal{D}\phi e^{-V_I(\varphi)} e^{-\mathcal{H}_G + J\varphi} \\
&= \int \mathcal{D}\phi \sum_{m=0}^{\infty} \frac{1}{m!} (-V_I)^m e^{-\mathcal{H}_G + J\varphi} \\
&\stackrel{?}{=} \sum_{m=0}^{\infty} \frac{1}{m!} \int \mathcal{D}\phi (-V_I)^m e^{-\mathcal{H}_G + J\varphi} \\
&= \sum_{m=0}^{\infty} \frac{1}{m!} \int \mathcal{D}\phi \left(-V_I[\frac{\delta}{\delta J}] \right)^m e^{-\mathcal{H}_G + J\varphi} \\
&= \exp\left[-V_I \left(\frac{\delta}{\delta J} \right) \right] Z_G[J]
\end{aligned}
$$

- $\mathcal{D}\phi$의 문제를 피하기 위하여 위 식을 field theory의 정의라고 간주하기도 한다.

Does the series converge?

- for example, zero dimensional case

$$Z(\lambda) \equiv \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2-\lambda x^4/4} \stackrel{?}{=} \sum_{n=0}^{\infty} (-\lambda)^n Z_n,$$

$$Z_n = \frac{1}{4^n} \int \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} x^{4n} = \frac{(4n)!}{n! 16^n (2n)!} \sim \frac{1}{\sqrt{n\pi}} \left(\frac{4n}{e}\right)^n$$

수렴반경은 0. The series for nonzero λ diverges!?

> *The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes.* Niels Henrik Abel

- Do you know $1 + 2 + 3 + \cdots \stackrel{!}{=} -\frac{1}{12}$?

perturbation 이론은 틀렸나?

- 부분합과의 오차 R_M 고려

$$R_M \equiv \left| Z(\lambda) - \sum_{n=0}^{M} (-\lambda)^n Z_n \right|$$

$$= \int \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \left| e^{-\lambda x^4/4} - \sum_{n=0}^{M} \frac{1}{n!} \left(-\frac{\lambda x^4}{4} \right)^n \right|$$

- Taylor 전개의 성질

$$\left| f(x) - \sum_{n=0}^{M} \frac{f^{(n)}(0)}{n!} x^n \right| = \left| \int_0^x \frac{(x-t)^M}{M!} f^{(M+1)}(t)\, dt \right| \le K_M \frac{x^{M+1}}{(M+1)!}.$$

$$K_M = \sup_{0 \le t \le x} \left| f^{(M+1)}(t) \right|.$$

perturbation 이론은 틀렸나? (cont.)

- $e^{-x} \leq 1$이므로,

$$
\begin{aligned}
R_M &\leq \int \frac{dx}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{(M+1)!} \left(\frac{\lambda x^4}{4}\right)^{M+1} \\
&= \lambda^{M+1} Z_{M+1} \sim \frac{1}{\sqrt{M\pi}} \left(\frac{4\lambda M}{e}\right)^M
\end{aligned}
$$

- $4\lambda M \approx 1$일 때 $R_M \approx e^{-1/(4\lambda)}$ (minimum).
- M이 엄청 크지 않으면 괜찮다. (asymptotic series가 convergent series보다 많은 경우 적은 노력으로 더 좋은 결과를 준다.)
- Stirling's formula, steepest descent method, Rayleigh-Schrödinger 섭동론 등 물리의 대부분의 근사는 convergent series가 아니고 asymptotic series를 다룬다.

● 수치계산

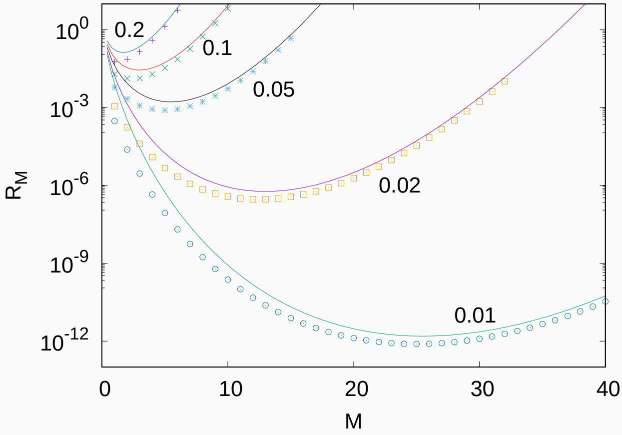
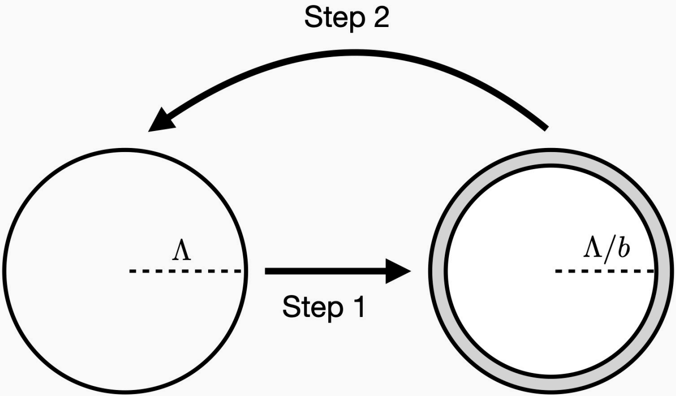


Table of contents

Momentum Shell RG: idea

- integration over φ_k in a momentum shell (cf: decimation)
- rescale and compare to the original theory



decimation (shell integration)

- Partition function

$$
\begin{aligned}
Z = \int \mathcal{D}\varphi e^{-\mathcal{H}} &= \int^{<} \mathcal{D}\varphi \left[\int^{>} \mathcal{D}\varphi e^{-\mathcal{H}} \right] \\
&= \int^{<} \mathcal{D}\varphi e^{-\mathcal{H}_G^{<}} \underbrace{\left[\int^{>} \mathcal{D}\varphi e^{-\mathcal{H}_G^{>} - V_I} \right]}_{\text{decimation}} .
\end{aligned}
$$

  - $\int^{>}$: integral over the field in the shell
  - $\mathcal{H}_G^{>}$: terms with large momentum $(k > \Lambda/b)$
  - $\mathcal{H}_G^{<}$: terms with small momentum $(k < \Lambda/b)$

Perturbative Expansion

- Shell integration

$$\int^{>} \mathcal{D}\varphi e^{-\mathcal{H}_G^{>} - V_I} = Z_0 \langle e^{-V_I} \rangle,$$

where $\langle \mathcal{O} \rangle = \frac{1}{Z_0} \int^{>} \mathcal{D}\varphi \mathcal{O} e^{-\mathcal{H}_G^{>}}, Z_0 = \int^{>} \mathcal{D}\varphi e^{-\mathcal{H}_G^{>}}$

$$\begin{aligned}\frac{Z}{Z_0} &= \int^{<} \mathcal{D}\varphi \exp\left(-\mathcal{H}_G^{<} + \ln\langle e^{-V_I}\rangle\right) \\ &\approx \int^{<} \mathcal{D}\varphi \exp\left[-\mathcal{H}_G^{<} - \langle V_I \rangle + \frac{1}{2}\left(\langle V_I^2 \rangle - \langle V_I \rangle^2\right) + \cdots\right]\end{aligned}$$

- cf: cumulant expansion
- All terms like ϕ^6, ϕ^8 will appear.

calculation of $\langle V_I \rangle$

- 이제부터 $\alpha = 1$, $\lambda/4! = u$라고 하자.

$$\mathcal{H}_G = \int_k \frac{1}{2}(k^2 + \mu^2)|\varphi_k|^2,$$

$$V_I = u \int_{1234} \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{k_4}\delta(1+2+3+4)$$

- $\varphi_k = \varphi_k^> + \varphi_k^<$라 쓰면 ($\varphi_k^> \equiv \varphi_k \Theta(k - \Lambda/b)$)

$$u^{-1}\langle V_I \rangle = \int^< \prod_{i=1}^{4} \varphi_{k_i}^< \delta(1+2+3+4) + 6 \underbrace{\int_{12}^< \varphi_1^< \varphi_2^< \int_{34}^> \left\langle \varphi_3^> \varphi_4^> \right\rangle \delta(1+2+3+4)}_{?=\int_k^< |\varphi_k|^2 \int_p^> \langle|\varphi_p|^2\rangle} + \mathfrak{A}(b, \Lambda)$$

- 평균 계산하기 ($b = e^{\delta} \approx 1 + \delta, \delta \ll 1$)

$$\int_p^{>} \langle |\varphi_p^{>}|^2 \rangle = \frac{S_{d-1}}{(2\pi)^d} \int_{\Lambda/b}^{\Lambda} \frac{p^{d-1}}{p^2 + \mu^2} dp = \frac{S_{d-1}}{(2\pi)^d} \Lambda^{d-2} \int_{1/b}^{1} \frac{x^{d-1}}{x^2 + \mu^2/\Lambda^2} dx$$
$$\approx \Lambda^2 \frac{S_{d-1}}{(2\pi)^d} \Lambda^{d-4} \frac{\delta}{1+r} \equiv \Lambda^2 \frac{C\delta}{1+r}$$

where $r = \mu^2/\Lambda^2$, S_{d-1} is the surface area of $(d-1)$ shpere.

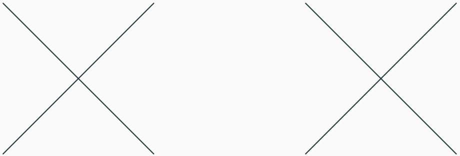
- 정리하면

$$\mathcal{H}_0^{<} + \langle V_I \rangle = \frac{1}{2} \int_k^{<} \left(k^2 + \Lambda^2 r + \Lambda^2 \frac{12uC\delta}{1+r} \right) |\varphi_k|^2 + u \int^{<} \phi^4 + \mathfrak{A}$$

- $\mathfrak{A}$는 free energy에 영향을 주지만, order parameter에는 영향을 주지 않는다.

calculation of $\langle V_I^2 \rangle_c$: Feynman diagram

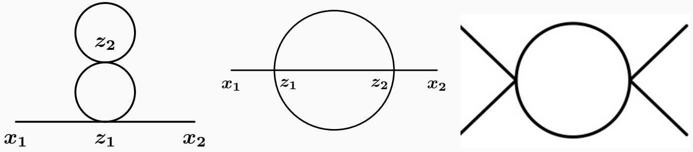
- second cumulant $\langle V_I^2 \rangle - \langle V_I \rangle^2$.
- Diagramatic representation
  - V_I을 1개의 꼭짓점과 4개의 선분이 있는 diagram으로 표현한다.



  - 큰 momentum에 해당하는 선분끼리 연결하되, 두 꼭짓점은 연결되는 diagram만 그린다.
  - 경우의 수를 잘 계산한다.
  - 연결된 선분에 $\Delta(k)$를 부여하고, loop이 생기면 $\int_k^>$ 를 한다.

calculation of $\langle V_I^2 \rangle_c$: Feynman diagram (cont.)

- 만들어지는 diagram들



- 왼쪽 두 개는 r에 영향을 주지만, higher order이다. (two-loop order)
- 맨 오른쪽에 있는 diagram이 u에 영향을 준다.

Step 2: rescale

- shell integration의 결과

$$Z \propto \int^{<} \mathcal{D}\phi e^{-\mathcal{H}_0' - V_I' - \cdots},$$

$$\tilde{r} = r + \frac{12uC}{1+r}\delta, \quad \tilde{u} = u - \frac{36u^2 C}{(1+r)^2}\delta$$

- rescale
  - $k = \kappa/b$라고 치환하면 적분구간이 Λ까지로 다시 돌아간다.

$$\int_k^{<} = b^{-d} \int_\kappa, \quad k^2 = b^{-2}\kappa^2$$

  - $\varphi_k = b^{1+d/2}\phi_\kappa$ 치환하면 원래 형태로 돌아간다.

$$\int_k^{<} k^2 |\varphi_k|^2 = \int_\kappa \kappa^2 |\phi_\kappa|^2$$

Step 2: rescale (cont.)

- 이 경우 $\tilde{r}$ $\tilde{u}$는 각각 b^2, b^{4-d}가 곱해진다. ($[r]=2$, $[u]=4-d$)
- 이제 $\mathcal{H}$의 형태는 유지되고 r, u자리에 r', u' 으로만 바뀌어 있다.

$$r' = b^2\tilde{r} \approx (1+2\delta)\left(r + \frac{12Cu}{1+r}\delta\right) = r + \delta\left[2r + \frac{12Cu}{1+r}\right],$$

$$u' = b^{\varepsilon}\tilde{u} = (1+\varepsilon\delta)\left(u - \frac{36Cu^2}{(1+r)^2}\delta\right) = u + u\left[\varepsilon - \frac{36U}{(1+r)^2}\right]\delta,$$

where $U = Cu$ and $\varepsilon = 4 - d$.

RG flow equation

- RG flow equation and fixed points

$$\frac{dr}{d\delta} = 2r + \frac{12U}{1+r} = 0, \quad \frac{dU}{d\delta} = \varepsilon U - \frac{36U^2}{(1+r)^2} = 0.$$

- There are two fixed points. $(r^*, U^*) = (0,0)$ and $(-\varepsilon/6, \varepsilon/36)$, where we assume $\varepsilon \ll 1$.
- (linear) stability analysis around the fixed points

$$\begin{aligned}
\frac{\partial r'}{\partial r} &= (1+2\delta) - \frac{12U^*}{(1+r^*)^2}\delta \approx b^{2-12U^*}, \\
\frac{\partial r'}{\partial U} &= 12\delta \approx 0, \quad \frac{\partial U'}{\partial r} = O(U^*)^2, \\
\frac{\partial U'}{\partial U} &= 1 + \varepsilon\delta - 72U^*\delta \approx b^{\varepsilon - 72U^*}.
\end{aligned}$$

RG flow equation (cont.)

- bare parameter가 r_0, u_0 이었다면,

$$r(b) - r^* = b^{2-12\,U^*}(r_0 - r^*), \quad u(b) - u^* = b^{\varepsilon-72\,U^*}(u_0 - u^*)$$

  - For $(0,0)$ (Gaussian fixed point)

$$r(b) - r^* = b^2(r_0 - r^*), \quad u(b) - u^* = b^{\varepsilon}(u_0 - u^*)$$

  - For nontrivial fixed point,

$$r(b) - r^* = b^{2-\varepsilon/3}(r_0 - r^*), \quad u(b) - u^* = b^{-\varepsilon}(u_0 - u^*)$$

Discussion

- r^* is always unstable (강자성 상태, 상자성 상태)
- $U^* = 0$ is stable (unstable) if $d > 4$ ($d < 4$)
- $U^* = \varepsilon/36$ is stable (unstable) if $d < 4$ ($d > 4$)
- For $d > 4$, $U^* = 0$ is the stable fixed point and the theory is identical to the Gaussian theory (or mean field theory).
- For $d < 4$, we have nontrivial fixed point and, accordingly, non-mean-field critical behavior.

Momentum shell integration is not elegant

- 1은 작고, $2, 3$은 큰 momentum인 경우에 $\int_{23}^{>} \delta(1+2+3)$ 의 적분 구간은 무엇인가?
- KPZ equation을 momentum-shell RG로 분석하면 다음의 식이 나온다.

$$I := \int_{12}^{>} \frac{\boldsymbol{k}\cdot\boldsymbol{p}_1\boldsymbol{p}_1\cdot\boldsymbol{p}_2}{p_1^2(p_1^2+p_2^2)}\delta(k+p_1+p_2) \approx Ak^2 + \cdots$$

- 보통 $p_1 = p - \frac{1}{2}k$, $p_2 = -p - \frac{1}{2}k$라 한 후, k에 대하여 Taylor 전개하여 $|p| > \Lambda/b$에 대하여 적분을 한다

- 그런데, 만약 $\boldsymbol{p}_1 = \boldsymbol{p} - a\boldsymbol{k}$, $\boldsymbol{p}_2 = -\boldsymbol{p} - (1-a)\boldsymbol{k}$라 하면

$$\begin{aligned} I &= -\int_p^{>} \frac{(\boldsymbol{p}\cdot\boldsymbol{k} - ak^2)(p^2 + (1-2a)\boldsymbol{p}\cdot\boldsymbol{k} - a(1-a)k^2)}{(\boldsymbol{p}-a\boldsymbol{k})^2[(\boldsymbol{p}-a\boldsymbol{k})^2 + (\boldsymbol{p}+(1-a)\boldsymbol{k})^2]} \\ &\approx -\frac{a}{2}\left(2\int^{>} \frac{(\boldsymbol{p}\cdot\boldsymbol{k})^2}{p^4} - k^2\int^{>}\frac{1}{p^2}\right) \end{aligned}$$

  이 되어 a depend한 결과를 준다!
- $a = \frac{1}{2}$를 선택하면 맞는 답을 준다. 그런데 왜?
- 섭동의 차수가 올라가면 계산은 훨씬 더 복잡해진다. momentum shell integration은 잘해야 2차 섭동항까지만 가능하다고 사람들이 이야기한다.
- Field theoretical renormalization group은 애초에 적분구간에 cutoff가 없어서 이런 문제가 발생하지 않는다. (물론 infinity를 처리해야하는 문제가 발생한다.)

Epilogue : Fokker-Planck equation

Imaginary time Schrödinger equation

- Imaginary time Schrödinger equation ($\hbar = 1$)

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}(\hat{p}, \hat{q})\Psi \xrightarrow{it/\hbar \mapsto \beta} -\frac{\partial \Psi}{\partial \beta} = \hat{H}(\hat{p}, \hat{q})\Psi$$

- Fokker-Planck equation

$$\frac{\partial P(q,t)}{\partial t} = -\frac{\partial}{\partial q}\left[a(q)P\right] + \frac{1}{2}\frac{\partial^2}{\partial q^2}\left[b(q)P\right] = -\hat{H}(\hat{p}, \hat{q})P$$

$$\hat{H} = \frac{\hat{p}^2}{2} b(\hat{q}) + i\hat{p}a(\hat{q}), \quad \hat{p} = -i\frac{\partial}{\partial q}, \quad [\hat{q}, \hat{p}] = i$$

- formal solution

$$\Psi(q,t) = e^{-\hat{H}t}\Psi(q,0) =: \hat{U}(t,0)\Psi(q,0).$$

  - In statmech, $\hat{H}$ is a Hermitian.
  - In stochastic processes, $\hat{H}$ needs not be a Hermitian.

- Propagator and partition function

$$\langle q'|e^{-\hat{H}t}|q\rangle, \quad Z = \mathrm{Tr}\, e^{-\beta\hat{H}} = \int dq \langle q|e^{-\beta\hat{H}}|q\rangle$$

- Group property of the time evolution operator $\hat{U}$

$$\hat{U}(t,0) = \hat{U}(t_n, t_{n-1})\,\hat{U}(t_{n-1}, t_{n-2}) \cdots \hat{U}(t_1, t_0),$$

where $t_0 = 0$ and $t_i = t_0 + n\varepsilon$ with $\varepsilon = t/n$.

Path Integral for the FP equation

- Assume $\varepsilon \lll 1$ $(n \to \infty)$

$$
\begin{aligned}
&\langle q|\hat{U}(t_i, t_{i-1})|q'\rangle \approx \delta(q-q') - \varepsilon\langle q|\hat{H}|q'\rangle \\
&= \frac{1}{2\pi}\int dp e^{ip(q-q')} - \varepsilon \int dp \langle q|p\rangle\langle p|\hat{H}|q'\rangle \\
&= \frac{1}{2\pi}\int dp \exp ip(q-q') - \frac{1}{2\pi}\varepsilon \int dp e^{iqp}\left(\frac{p^2}{2}b(q') + ipa(q')\right)\langle p|q'\rangle \\
&= \frac{1}{2\pi}\int dp \exp\left[ip(q-q') - \varepsilon\left(\frac{p^2}{2}b(q') + ipa(q')\right)\right]
\end{aligned}
$$

- Full propagtor for finite t

$$
\begin{aligned}
&\langle q_n | U(t,0) | q_0 \rangle = \langle q_n | U(t_n, t_{n-1} U(t_{n-1}, t_{n-2}) \cdots U(t_1, t_0) | q_0 \rangle \\
&= \int dq_{n-1} \cdots dq_1 \langle q_n | U(t_n, t_{n-1} | q_{n-1} \rangle \langle q_{n-1} | \cdots | q_1 \rangle \langle q_1 | U(t_0, t_0) | q_0 \rangle \\
&= \lim_{n\to\infty} \frac{1}{(2\pi)^n} \int dp_n \cdots dp_1 \, dq_{n-1} \cdots dq_1 \times \\
&\quad \times \exp\left[\sum_{j=1}^{n} \left\{ i p_j (q_j - q_{j-1}) - \varepsilon \left(\frac{p_j^2}{2} b(q_{j-1}) + i p_j a(q_{j-1}) \right) \right\} \right]
\end{aligned}
$$

- **naive** continuum limit and path integral

$$
\begin{aligned}
&\sum_{j=1}^{n} \left\{ i\varepsilon p_j \frac{q_j - q_{j-1}}{\varepsilon} - \varepsilon \left(\frac{p_j^2}{2} b(q_{j-1}) + i p_j a(q_{j-1}) \right) \right\} \\
&\mapsto \int_0^d i p(t) \dot{q}(t) - H(p(t), q(t)) \, dt
\end{aligned}
$$

- path probability (density)

$$P(q(\tau)) := \int [Dp] \exp\left[\int_0^t ip(\tau)\dot{q}(\tau) - H(p(\tau), q(\tau))\, d\tau\right]$$

- path probability ratio(?)

$$\frac{P(q(\tau))}{P(q(t-\tau))}$$