INVOLUTIVITY OF CONSTRAINED PFAFF EQUATIONS

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Introduction

In this lecture series we shall discuss the involutivity and integrability of exterior differential equations of the form $(\rho, \theta) = 0$, for a non-degenerate system of smooth functions ρ and a linearly independent set of 1-forms θ. In particular, we present techniques of finding additional constraints from the torsion tensor so that the resulting constrained Pfaffian system is closed. We do not prove any of our propositions and theorems but leave them as exercises. In fact, under the non-degeneracy assumption (1.6) most of the algebraic expressions are simply the dual arguments of obvious geometric facts. We present some open problems that arise from the geometry of complex, almost complex and Hamiltonian structures, control theory and quasi-linear partial differential equations of first order.

1. Exterior differential equations

Let M be a smooth manifold of dimension n and $\Omega^*(M)$ be the exterior algebra of smooth differential forms. For an ideal $I \subset \Omega^*(M)$

$$I = 0 \tag{1.1}$$

is called an exterior differential equation (or an exterior differential system), where the problem is to find integral manifolds of desired dimensions. The notions of ideal and integral manifold are defined in §2. For the concepts and notations that are not defined in this paper we refer the readers to [1].

1.1. **Pfaff equations (Pfaffian system).**

A Pfaff equation on a domain $U \subset \mathbb{R}^n$ is

$$a_1(x)dx^1 + \cdots + a_n(x)dx^n = 0, \quad x = (x^1, \ldots, x^n) \in U. \tag{1.2}$$

The problem is to find a submanifold Σ^k (an integral manifold of dimension k), $1 \leqslant k \leqslant n-1$, on which (1.2) holds, where dx^j is the infinitesimal change in x^j along Σ^k. Now consider a system of Pfaff equations

$$a_1^j(x)dx^1 + \cdots + a_n^j(x)dx^n = 0, \quad j = 1, \ldots, s. \tag{1.3}$$

Denoting the left-hand-side of (1.3) by θ^j we rewrite (1.3) as

$$\theta := (\theta^1, \ldots, \theta^s) = 0 \quad \text{(Pfaff equations).} \tag{1.4}$$

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1.2. **Constrained Pfaff equations.** In this lecture series we discuss the involutivity and integrability of exterior differential equations of type

$$(\rho,\theta) := (\underbrace{\rho^1,\dots,\rho^d}_{\rho},\underbrace{\theta^1,\dots,\theta^s}_{\theta}) = 0, \tag{1.5}$$

where ρ's are smooth (C^∞) real-valued functions and θ's are 1-forms on a smooth manifold M^n. Our argument is local and stays in the C^∞category assuming the following non-degeneracy conditions:

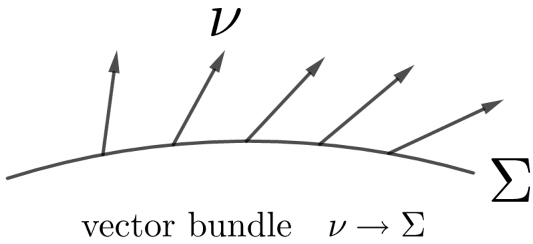
$$d\rho^1\wedge\cdots\wedge d\rho^d\neq 0 \quad \text{and } \theta^1\wedge\cdots\wedge\theta^s\neq 0. \tag{1.6}$$

Geometrically, (1.5) with the condition (1.6) corresponds in one-to-one manner to a vector bundle

$$\mathcal{V}\to\Sigma, \quad \text{where } \Sigma=\{\rho=0\}, \mathcal{V}=\langle\theta\rangle^\perp \tag{1.7}$$

as in the following diagram:

$$\begin{array}{ccc} \mathcal{V} & \hookrightarrow & TM \\ \downarrow & & \downarrow \\ \Sigma & \hookrightarrow & M \end{array}$$



In particular, $(\rho, d\rho)$ corresponds to the tangent bundle $T\Sigma$. Let $\Omega^*(M)$ be the exterior algebra of differential forms of M with smooth coefficiens. By (ρ,θ) we denote an ideal of $\Omega^*(M)$ generated by ρ and θ. If ρ and θ satisfy (1.6) we call (ρ,θ) a regular ideal.

Consider the regular ideals (ρ,θ) and $(\tilde\rho,\tilde\theta)$ that correspond to vector bundles $\mathcal{V}\to N$ and $\tilde{\mathcal{V}}\to\tilde N$, respectively. We define the inclusion $(\mathcal{V}\to N)\subset(\tilde{\mathcal{V}}\to\tilde N)$ by

$$N\subset\tilde N \quad \text{and} \quad \mathcal{V}_x\subset\tilde{\mathcal{V}}_x, \quad \forall x\in N.$$

Proposition 1.1. $(\mathcal{V}\to N)\subset(\tilde{\mathcal{V}}\to\tilde N)$ *if and only if* $(\rho,\theta)\supset(\tilde\rho,\tilde\theta)$.

The problem of finding a submanifold on which (1.5) holds may be called a constrained Pfaff equations. When solving the Pfaff equation without constraints

$$(\theta)=0 \tag{1.8}$$

we first compute the torsion $d\theta$ mod θ. If the torsion vanishes on $\rho=0$ we impose the constraint ρ to (1.8) and solve (1.5).

Problem 1.2. By weakening or dispensing with the conditions (1.6), generalize our theory to fit to the cases with singularities and to the global problems.

1.3. **EDS and PDEs.** An exterior differential system (EDS) is equivalent to a system of partial differential equations (PDEs). Conversely, a system of partial differential equations is equivalent to an EDS on a jet space. For example, given a second order PDE for $u(x, y)$

$$F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0, \tag{1.9}$$

where F is a differentiable function in eight variables (x, y, u, p, q, r, s, t) where $p = u_x$, $q = u_y$, $r = u_{xx}$, $s = u_{xy}$,, $t = u_{yy}$, with the assumption $dF \neq 0$. For any C^2 function $u(x, y)$, the 2-jet graph of u:

$$(x, y, u(x, y), u_x(x, y), u_y(x, y), u_{xx}(x, y), u_{xy}(x, y), u_{yy}(x, y)) \tag{1.10}$$

is an integral manifold of dimension 2 of the Pfaffian system

$$\begin{aligned} \theta^0 &:= du - pdx - qdy \\ \theta^1 &:= dp - rdx - sdy \\ \theta^2 &:= dq - sdx - tdy, \end{aligned} \tag{1.11}$$

on which

$$dx \wedge dy \neq 0 \text{ (independence condition)}. \tag{1.12}$$

Let $M^7 \subset \mathbb{R}^8$ be the submanifold defined by $F = 0$. Then solving (1.9) is equivalent to finding an integral manifold $\Sigma^2 \subset M^7$ of the Pfaffian system

$$\theta := (\theta^0, \theta^1, \theta^2) = 0$$

with the independence condition (1.12).

Given a system of PDEs (or equivalently an EDS), by 'prolongation' we mean applying to the system algebraic operations and d(differentiation) to obtain a new equations of desired form.

Definition 1.3. A complete system of order k for an unknown function $u(x)$, $x = (x_1, \dots, x_n)$, is an expression of all the partial derivatives of order k of u in terms of lower order derivatives: for all multi-indices α with $|\alpha| = k$

$$\partial^\alpha u = F^\alpha(x, \partial^\beta u : |\beta| \leqslant k - 1). \tag{1.13}$$

An over-determined PDE system is reduced generically to a complete system of a certain order k, for some large k by prolongation. We may identify (1.13) with a Pfaffian system in the $(k-1)$st jet space. For example, for a complete system of second order for $u(x, y)$

$$\begin{aligned} u_{xx} &= F_1(x, y, u, u_x, u_y) \\ u_{xy} &= F_2(x, y, u, u_x, u_y) \\ u_{yy} &= F_3(x, y, u, u_x, u_y) \end{aligned} \tag{1.14}$$

consider 1-forms on the first jet space of u: $\{(x, y, u, p, q)\}$, where $p = u_x$ and $q = u_y$,

$$\begin{aligned} \theta^0 &:= du - p\ dx - q\ dy \\ \theta^1 &:= dp - F_1\ dx - F_2\ dy \\ \theta^2 &:= dq - F_2\ dx - F_3\ dy. \end{aligned} \tag{1.15}$$

Solving (1.14) is equivalent to finding an integral manifold of (1.15).

2. Involutivity and formal theory of PDEs

By 'formal theory' we mean all the theories of PDEs other than the existence theory of type I - IV down below. Formal theory includes change of variables, transforms, prolongation, involutivity, symmetry and conservation laws and integration. As for the methods of proof of the existence of solutions for PDEs, we classify them as follows;

type 0. direct construction by integration or by formal theory

type I. convergence of iteration of the Picard integral operator $\longrightarrow$ fundamental theorem of ODE $\longrightarrow$ the Frobenius theorem on involutivity

type II. convergence of formal power series, Cauchy-Kowalevski theorem, in analytic (C^ω) category

type III. Hilbert space approach, functional analysis by norm estimates

type IV. other methods if there are any.

Proofs of any type other than type 0 appeared only around the end of the 19th century or in the 20th century. In particular, the Hilbert space theory was developed during the decade after 1903, when Fredholm spoke of his theory on integral equations. Integrability means the existence of general solutions that can be proved by one of the above methods. Involutivity is an algebraic concept. It is a necessary condition for a system to be integrable. A system of PDEs is said to be in involution if all the compatibility conditions are contained in the system. The notion of involutivity depends on the integrability, that is, depends on the integrability types 0 - IV.

Example 2.1. wave equation: Consider the wave equation for $u(x, y)$:

$$u_{xx} - u_{yy} = 0 \tag{2.1}$$

is transformed by the change of variables $s = x + y$, $t = x - y$ to

$$u_{st} = 0.$$

The general solution is

$$\begin{aligned} u &= f(s) + g(t) \\ &= f(x+y) + g(x-y), \end{aligned}$$

for any C^2 functions f and g. Therefore, a general solution depends on two functions in one variable. (2.1) is involutive with the involutivity type 0.

Example 2.2. Complete system of first order: Consider a complete system of first order for $u(x, y)$

$$\begin{cases} u_x &= A(x, y, u) \\ u_y &= B(x, y, u). \end{cases} \tag{2.2}$$

Differentiating the first equation of (2.2) with respect to y and substituting for u_y the second equation, we obtain $u_{xy} = A_y + A_u B$. Similarly, differentiating the second equation with respect to x, we have $u_{yx} = B_x + A_u A$. Equating these two we have the compatibility condition

$$A_y + A_u B = B_x + B_u A. \tag{2.3}$$

Thus (2.2) is not involutive. But the system (2.2)-(2.3) is involutive. To prove this it suffices to prove the existence of general solution. As we have observed in §1 this complete system is equivalent to a Pfaffian system in the 0-th jet space $\{(x, y, u)\}$:

$$\begin{aligned} \theta &:= du - u_x dx - u_y dy \\ &= du - A(x, y, u)dx - B(x, y, u)dy. \end{aligned} \tag{2.4}$$

We compute the Frobenius condition, which we shall discuss again in §3: by applying d to (2.4) and by that $du \equiv Adx + Bdy$, mod θ we have

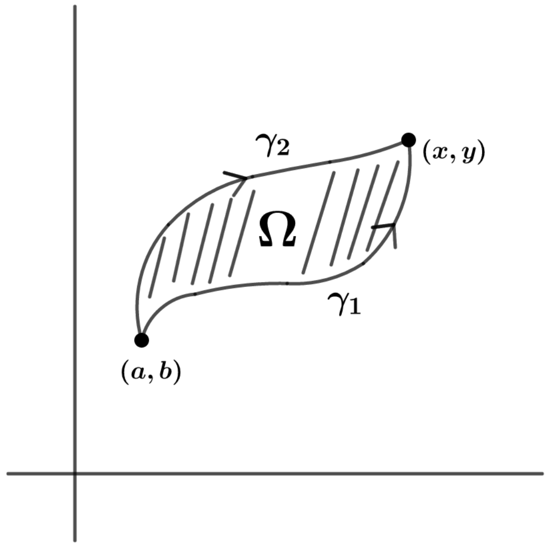
$$d\theta \equiv \underbrace{(A_y - B_x + A_u B - B_u A)}_{\mathcal{T}}\, dx \wedge dy, \quad \text{mod } \theta. \tag{2.5}$$

The coefficient $\mathcal{T}$ is called the torsion of the Pfaffian system (θ). By (2.3) the torsion is zero, and hence by the Frobenius theorem there is a unique integral manifold of maximal dimension 2 at every point (x, y, u). Thus (2.2) is not involutive. But its first prolongation (2.2)-(2.3) is involutive with the involutivity type I. The general solution depends on three constants.

Example 2.3. Construction of solutions by path integral : For (2.2) we construct the general solution by direct integration using the compatibility condition (2.3). If (2.2) has a solution u on a neighborhood of (a, b), then $u(x, y)$ would be obtained by the integral

$$u(x, y) = u(a, b) + \int_\gamma du, \tag{2.6}$$

where γ is a curve connecting (a, b) and (x, y). Let γ_1 and γ_2 be such curves as in the following figure.



Exercise Show that the path integral (2.6) is independent of choice of path. Hence the involutivity of (2.2)-(2.3) belongs to type 0 as well.

2.1. **Frobenius involutivity.** In the early 19th century, the problem of determining the largest dimension of the integral manifold of (1.4) (Pfaff's problem) has been actively studied. Closely related to the Pfaff's problem, a system of PDEs of first order

$$X_j \rho = 0, \quad j = 1, \ldots, p, \tag{2.7}$$

where X_j are smooth vector fields in a domain $U \subset \mathbb{R}^n$ that are linearly independent everywhere, has been studied in the mid-19th century. A function ρ that satisfies (2.7) is called a first integral. There can be at most $s := n - p$ functionally independent first integrals. In 1840 F. Deahna [6] found that what we call today the Frobenius integrability condition

$$[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}, \tag{2.8}$$

where $\mathcal{D}$ is the distribution spanned by X_js, implies the existence of $s := n-p$ (maximal number) first integrals. It was A. Clebsch who gave a rigorous proof [4] (1866) to the Deahna's theorem. Now let

$$\theta := (\theta^1, \ldots, \theta^s) = \mathcal{D}^\perp \tag{2.9}$$

be a system of 1-forms that annihilates $\mathcal{D}$. G. Frobenius [7] (1877) introduced the exterior differential operator d and restated the integrability condition (2.8) as

$$d\theta^\alpha \equiv 0, \quad \mod (\theta), \tag{2.10}$$

which provided the development thereafter of geometry and topology with the new tool. (2.10) is the involutivity of (1.8). There are several proofs for the Frobenius theorem. As in a standard textbook [15] it is proved by using the fundamental theorem of ODE. It can also be proved by constructing first integrals by integration. Therefore, the involutivity (2.8), or equivalently, (2.10) may be classified as type 0, also as type 1. The general solution depends on n constants.

2.2. **Cartan involutivity.** This is an involutivity of type II. A system of PDEs or an equivalent EDS is Cartan-involutive if it can be transformed into a finite number of steps of initial value problems of Cauchy-Kowalevski type. An initial value problem of the form

$$(\partial_t)^k u = F(x, t, u, \partial_t^j \partial_x^\alpha u) : j < k, |\alpha| + j \leqslant k, \\ (\partial_t)^j u(x, 0) = \phi_j(x), \quad j = 0, 1, \dots, k-1 \tag{2.11}$$

is said to be of Cauchy-Kowalevski type. The Cauchy-Kowalevski theorem asserts that if F and ϕ_j are analytic on a neighborhood of the origin $x = 0, t = 0$, (2.11) has a unique analytic solution on a neighborhood of the origin. A Cartan-involutive system with analytic data has a unique analytic solution locally, by the Cauchy-Kowalevski theorem. Even in non-analytic categories, the Cartan-involutivity gives informations on the 'arbitrariness', or the degree of freedom, of solutions, that is, the number of constants, the number of functions of how many variables, on which the general solution depends.

2.3. **Spencer's formal integrability.** This is an involutivity of type III. For over-determined systems of linear partial differential equations, D. C. Spencer [13] studied the formal integrability by means of homological algebra. The integrability is based on the Hilbert space norm estimates by Kohn and Nirenberg [21]-[22].

2.4. **Contributors to the theory of involutivity (involutiveness).** The following mathematicians have contributed to the theory of involutivity:

i) Pfaff – Deahna – Clebsch – Frobenius.

ii) Lie – Vessiot – Cartan – Ehresmann.

iii) Spencer – Kohn – Goldschmidt.

iv) Other mathematicians: Liouville, Jacobi, Poisson, Darboux, $\cdots$.

• Pfaff: Johann Friedrich Pfaff, 1765-1825, Holy Roman Empire, teacher of Gauss and Möbius.

• Deahna: Heinrich Wilhelm Feodor Deahna, 1815-1844, German, published in Crelle's journal (1840, at the age of 25) what we call today 'the Frobenius theorem'. He was a student at Göttingen, died at the age of 29.

• Clebsch: Rudolf Friedrich Alfred Clebsch, 1833-1872, German, algebraic geometry and invariant theory, teacher of G. Frege and A. von Brill.

• Frobenius: Ferdinand Gerog Frobenius, 1849-1917, German, elliptic functions, differential equations, number theory, group theory, Göttingen, Berlin, student of Weierstrass, Kummer, teacher of Fuchs, Landau, Schur.

• Lie: Sophus Lie, 1842-1899.

- Vessiot: Ernest Vessiot, 1865-1952, French, Ecole Normale Supérieure.
- Cartan: Elie Cartan, 1869-1951.
- Spencer: Donald C. Spencer, 1912-2001.

2.5. **Exterior differential systems and ideals.** Let M be a smooth (C^∞) manifold of dimension n. Let $\Omega^0(M) = C^\infty(M)$ be the ring of smooth real-valued functions of M and for each integer $k = 1, \dots, n$ let $\Omega^k(M)$ be the module over $C^\infty(M)$ of k-forms and let $\Omega^*(M) = \bigoplus_{k=0}^n \Omega^k(M)$ be the exterior algebra of smooth differential forms.

Definition 2.4. A subalgebra I is an (algebraic) ideal if the following conditions hold;
i) $I \wedge \Omega^*(M) \subset I$,
ii) if $\sum_{k=0}^n \phi_k \in I$, $\phi_k \in \Omega^k(M)$, then each $\phi_k \in I$ (homogeneity condition).

Because of the homogeneity condition I is two-sided, that is,

$$\Omega^*(M) \wedge I \subset I.$$

For $\phi, \psi \in \Omega^*(M)$ we write

$$\phi \equiv \psi \mod I$$

if and only if $\phi - \psi \in I$.

An ideal I is said to be closed if

$$dI \subset I.$$

A submanifold $i : \Sigma \hookrightarrow M^m$ is said to be an integral manifold of an ideal I if $i^*I = 0$. Notice that $i^*I = 0$ implies that $i^*(dI) = 0$. A system of C^∞ real-valued functions $\rho^1, \dots, \rho^d$ are said to be *non-degenerate* if $d\rho^1 \wedge \cdots \wedge d\rho^d \neq 0$. Let $\theta = (\theta^1, \dots, \theta^s)$ be a system of smooth 1-forms. By the rank of (θ) at $x \in M$ we mean the dimension of the linear span $\langle \theta^1(x), \dots, \theta^s(x) \rangle \subset T_x^*M$.

Definition 2.5. An ideal I is said to be regular if I is generated by a non-degenerate system of real-valued functions $\rho := (\rho^1, \dots, \rho^d)$ and a finite set of smooth 1-forms $\theta := (\theta^1, \dots, \theta^s)$ of constant rank s. A regular ideal generated by ρ and θ shall be denoted by (ρ, θ).

Remark 2.6. 1) Let $(\rho, \theta) \subset \Omega^*(M)$ be a regular ideal and ω be a k-form for some $k = 1, \dots, n$. Then $\omega \in (\rho, \theta)$ if and only if $\omega(V_1, \dots, V_k) = 0$, for any $V_1, \dots, V_k$ in the same fibre $\mathcal{V}_x$, $\forall x \in \Sigma := \{\rho = 0\}$.

2) If Σ is the common zero locus of a non-degenerate $\rho = (\rho^1, \dots, \rho^d)$, the tangent bundle $T\Sigma$ corresponds to the ideal $(\rho, d\rho)$.

Now coming back to (2.7) - (2.10), We have

Theorem 2.7. *On M^n let $X_1, \dots, X_p$ be smooth independent vector fields and $\theta = (\theta^1, \dots, \theta^s)$, $s = m - p$, be a system of smooth independent* 1*-forms that annihilate $X_j's$ as in* (2.7) *and* (2.9). *Then for a smooth real-valued function ρ with $d\rho \neq 0$ the following are equivalent:*

(i) *ρ is a first integral;*
(ii) *X_j, $j = 1, \ldots, p$, is tangent to the level sets of ρ;*
(iii) *ρ satisfies*

$$d\rho \in (\theta). \tag{2.12}$$

Definition 2.8. A set of real-valued functions $\rho = (\rho^1, \ldots, \rho^d)$ is a weak first integral if

$$X_j\rho = 0 \text{ on } \rho = 0, \quad \text{for all } j = 1, \ldots, p \tag{2.13}$$

or equivalently,

$$d\rho \in (\rho, \theta). \tag{2.14}$$

We fix notations now. On a smooth manifold M^n let

$$X_1, \ldots, X_p \tag{2.15}$$

be smooth vector fields that are linearly independent at every point and let

$$\mathcal{D} =< X_1, \ldots, X_p >\subset TM \tag{2.16}$$

be the linear span of those p vector fields. As introduced previously let

$$\theta := (\theta^1, \ldots, \theta^s), \quad s + p = n, \tag{2.17}$$

be a system of 1-forms that annihilates $\mathcal{D}$. Let us denote the ideal generated by (2.17) by I. To be precise, we denote by $\underline{\mathrm{I}}$ the module over $C^\infty(M)$ generated by (2.17), which is a submodule of $\Omega^1(M)$. By $\langle I \rangle$ we denote the linear span of (2.17), which is a subbundle of T^*M. Many authors mean by I any of these three concepts by abuse of notation. But in this paper we will distinugish three different concepts, namely, an ideal of $\Omega^*(M)$, a submodule of $\Omega^1(M)$ and a subbundle of T^*M.

Definition 2.9. For the system of smooth vector fields (2.15) a submanifold Σ is said to be invariant if Σ is invariant under the flows of X_j's, that is, X_j's are tangent to Σ.

Theorem 2.10. *Let $\rho = (\rho^1, \ldots, \rho^d)$ be a non-degenerate set of real-valued functions. Then $\rho = 0$ is invariant under (2.15) if and only if ρ is a weak first integral of (2.15).*

2.6. **Maximal involutive subsystem.** Let θ, $\mathcal{D}$ and $X_1, \ldots, X_p$ be the same as in (2.15)-(2.17). If $\mathcal{D}$ satisfies (2.8) then $\mathcal{D}$ is integrable by the Frobenius theorem. If (2.8) does not hold we consider the problem of deciding the involutive subbundle of the smallest rank that contains $\mathcal{D}$ as a subbundle. For each integer $j = 0, 1, 2, \ldots,$ let

$$\mathcal{D}^{(j+1)} := \mathcal{D}^{(j)} + [\mathcal{D}^{(j)}, \mathcal{D}^{(j)}], \quad \mathcal{D}^{(0)} := \mathcal{D}.$$

Under the assumption that $\mathcal{D}^{(j)}$ has constant rank for each j, let ν be the first integer such that

$$\mathcal{D}^{(\nu)} = \mathcal{D}^{(\nu+1)}.$$

Then this is the smallest integrable system that includes $\mathcal{D}$ and the number of the functionally independent first integrals is

$$\text{dimension of } M^n - \text{rank of } \mathcal{D}^{(\nu)}.$$

Now we present a dual argument as in [8]. Given a system $\theta = (\theta^1, \ldots, \theta^s)$ of independent 1-forms, we denote by I the ideal generated by θ. Let $d : I \to \Omega^*(M)$ be the exterior differentiation and $\pi : \Omega^*(M) \to \Omega^*(M)/I$ be the projection. Then

$$\delta = \pi \circ d : \underline{\mathrm{I}} \to \Omega^2/I_2, \quad \text{where } I_2 := I \cap \Omega^2(M), \tag{2.18}$$

is a module homomorphism over $C^\infty(M)$. Let $\underline{\mathrm{I}}^{(1)} \subset \underline{\mathrm{I}}$ be the kernel of δ. Then we have the following exact sequence

$$0 \longrightarrow \underline{\mathrm{I}}^{(1)} \longrightarrow \underline{\mathrm{I}} \stackrel{\delta}{\longrightarrow} d\underline{\mathrm{I}}/I_2 \longrightarrow 0.$$

The ideal $I^{(1)}$ corresponding to the module $\underline{\mathrm{I}}^{(1)}$ is called the *first derived system.* Inductively, assuming that $\underline{\mathrm{I}}^{(r-1)}$ has constant rank, we construct the r-th derived system by the exactness of

$$0 \longrightarrow \underline{\mathrm{I}}^{(r)} \longrightarrow \underline{\mathrm{I}}^{(r-1)} \stackrel{\delta}{\longrightarrow} d\underline{\mathrm{I}}^{(r-1)}/I_2^{(r-1)} \longrightarrow 0,$$

where $I_2^{(r-1)} := I^{(r-1)} \cap \Omega^2(M)$. We obtain eventually the smallest integer ν such that $I^{(\nu+1)} = I^{(\nu)}$. We call

$$I = I^{(0)} \supset I^{(1)} \supset I^{(2)} \supset \cdots \supset I^{(\nu-1)} \supset I^{(\nu)} \tag{2.19}$$

the *derived flag* of I and ν the *derived length.* Note that $I^{(\nu)}$ is the largest involutive subsystem of I. A basic observation in [8] is that a function ρ is a first integral if and only if $d\rho \in I^{(\nu)}$. Assuming that each level $I^{(r)}$ of (2.19) has constant rank let q be the rank of $I^{(\nu)}$. Then by the Frobenius theorem there exist q first integrals that are functionally independent. If the rank of $I^{(\nu)}$ is non-zero, the Pfaffian system θ is said to be of type ν. If $I^{(\nu)}$ has rank zero, the Pfaffian system (θ) is said to be of infinite type. The condition that θ has type ν is given as a system of non-linear partial differential equations of order $\nu+1$ for the coefficients of θ (cf. [26]). The notion of the first integral of (1) is naturally defined in the theory of derived flag as in [1] Chapter 1.

Exercise Show that ${\mathcal{D}^{(k)}}^\perp = \langle I^{(k)} \rangle$ for each $k = 0, 1, \cdots, \nu$.

3. Method of prolongation to a Frobenius-involutive system

Given a system of PDEs, or an EDS, we transform the system to an equivalent system of desired form by prolongation. Prolongation to an involutive system is one of the main purposes of the formal theory. In this section we present some techniques of prolongation to Frobenius-involutive systems, which turn out to be useful for overdetermined problems.

3.1. Construction of the derived system.

Given a Pfaffian system $I = (\theta)$ the notions of derived system (2.19) and the involutive subsystem of maximal rank were first introduced in [8]. In this subsection we present a method of construction of the derived system so that we can determine the number of independent first integrals, where the first integrals are constructible. By choosing arbitrary 1-forms $\omega := (\omega^1, \dots, \omega^p)$, we complete $\theta := (\theta^1, \dots, \theta^s)$ to a coframe (θ, ω) of M. For each $\alpha = 1, \dots, s$, let

$$d\theta^\alpha \equiv \sum_{1\leqslant j<k\leqslant p} T^\alpha_{jk}\omega^j \wedge \omega^k, \ \bmod I,$$

or in terms of matrices

$$\begin{bmatrix} d\theta^1 \\ \vdots \\ d\theta^s \end{bmatrix} \equiv \underbrace{\begin{bmatrix} T^1_{12} & T^1_{13} & \cdots & T^1_{p-1,p} \\ \vdots & \vdots & & \vdots \\ T^s_{12} & T^s_{13} & \cdots & T^s_{p-1,p} \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} \omega^1 \wedge \omega^2 \\ \omega^1 \wedge \omega^3 \\ \vdots \\ \omega^{p-1} \wedge \omega^p \end{bmatrix}, \quad \bmod I, \tag{3.1}$$

where the superscripts in $\omega^j \wedge \omega^k$ are arranged in lexicographical order and the matrix $\mathcal{T}$ of size $s \times \binom{p}{2}$ is called the torsion of the Pfaffian system (θ), which is the obstruction to the integrability. To construct $I^{(1)}$ suppose that

$$\phi = \sum_{\alpha=1}^{s} a_\alpha(x)\theta^\alpha \in I. \tag{3.2}$$

We see that $\delta\phi = 0$ is equivalent to

$$\sum_{\alpha=1}^{s} a_\alpha(x) d\theta^\alpha \equiv 0, \quad \bmod I.$$

Substituting (3.1) for $d\theta^a$ we have

$$(a_1, \cdots, a_s) \underbrace{\begin{bmatrix} T^1_{12} & T^1_{13} & \cdots & T^1_{p-1,p} \\ \vdots & \vdots & & \vdots \\ T^s_{12} & T^s_{13} & \cdots & T^s_{p-1,p} \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} \omega^1 \wedge \omega^2 \\ \omega^1 \wedge \omega^3 \\ \vdots \\ \omega^{p-1} \wedge \omega^p \end{bmatrix} \equiv 0, \quad \bmod \mathcal{I}. \tag{3.3}$$

Since $\omega^j \wedge \omega^k$ are independent (3.3) holds if and only if the column vector $\vec{a}(x) := (a_1, \cdots, a_s)^t$ is in the null space of $\mathcal{T}^t$. Our constant rank assumption for $I^{(1)}$ implies that the null space of $\mathcal{T}(x)^t$ has constant dimension, say s_1. Then we choose s_1 linealy independent 1-forms $\phi := (\phi^1, \dots, \phi^{s_1})$ of the form (3.2), with $(a_1, \dots, a_s)^t$ in the null space of $\mathcal{T}^t$, to obtain a basis of $I^{(1)}$. For the next step, we extend the system ϕ to a coframe $(\phi^1, \dots, \phi^{s_1}, \pi^1, \dots, \pi^{p_1})$. The torsion matrix $\mathcal{T}^{(1)}$ for $I^{(1)}$ is obtained from

$$\begin{bmatrix} d\phi^1 \\ \vdots \\ d\phi^{s_1} \end{bmatrix} \equiv \underbrace{\begin{bmatrix} T^{(1)}{}^1_{12} & T^{(1)}{}^1_{13} & \cdots & T^{(1)}{}^1_{p_1-1,p_1} \\ \vdots & \vdots & & \vdots \\ T^{(1)}{}^{s_1}_{12} & T^{(1)}{}^{s_1}_{13} & \cdots & T^{(1)}{}^{s_1}_{p_1-1,p_1} \end{bmatrix}}_{\mathcal{T}^{(1)}} \begin{bmatrix} \pi^1 \wedge \pi^2 \\ \pi^1 \wedge \pi^3 \\ \vdots \\ \pi^{p_1-1} \wedge \pi^{p_1} \end{bmatrix}, \quad \bmod I^{(1)}, \tag{3.4}$$

The torsion matrix $\mathcal{T}^{(1)}$ has dimension $s_1 \times \binom{p_1}{2}$. Then in the same way as in $I^{(1)}$ we obtain a basis of $I^{(2)}$, and by repeating the same to obtain $I^{(\nu)}$, where ν is the type of the Pfaffian system (θ).

Example 3.1. On $\mathbb{R}^5 = \{(x, y, z, u, v)\}$ let I be generated by 1-forms

$$\begin{aligned} \theta^1 &= dz + ydu \\ \theta^2 &= ydx + du \\ \theta^3 &= e^x dv. \end{aligned}$$

Then with respect to the coframe $(dx, dy, \theta^1, \theta^2, \theta^3)$ we have

$$\begin{bmatrix} d\theta^1 \\ d\theta^2 \\ d\theta^3 \end{bmatrix} \equiv \underbrace{\begin{bmatrix} y \\ -1 \\ 0 \end{bmatrix}}_{\mathcal{T}} dx \wedge dy, \qquad \mathrm{mod}\ (\theta).$$

Hence, $I^{(1)}$ is generated by

$$\begin{aligned} \phi^1 &:= \theta^1 + y\theta^2 && = y^2dx + dz + 2ydu \\ \phi^2 &:= \theta^3 && = e^x dv. \end{aligned}$$

Now take $(dx, dy, du, \phi^1, \phi^2)$ as coframe. Then

$$\begin{bmatrix} d\phi^1 \\ d\phi^2 \end{bmatrix} \equiv \begin{bmatrix} -2y & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx \wedge dy \\ dx \wedge du \\ dy \wedge du \end{bmatrix}, \quad \mathrm{mod}\ (\phi^1, \phi^2).$$

Thus $I^{(2)}$ is generated by ϕ^2 and the number of independent first integrals is 1. The function $\rho(x, y, z, u, v) = v$ is a first integral.

Notice that the generators of $I^{(1)}$ is obtained from the torsion of I, which is obtained by differentiating I. Finding the generators of $I^{(k)}$ is a 'k-th prolongation' of I.

3.2. **Closedness of constrained Pfaffian systems.** Coming back to the constrained Pfaffian system (1.5) that satisfies (1.6), we shall present a method of prolongation of (ρ, θ) to a closed system (ρ, r, θ) by inserting additional constraints $r = (r^1, r^2, \ldots)$. We shall also see that (ρ, θ) is involutive if and only if it is closed. Our method of prolongation to involutive system is based on the following theorems.

Theorem 3.2. *For a constrained Pfaffian system (ρ, θ) on M^n as given in (1.5) let $\mathcal{V} \to \Sigma$ be the associated vector bundle (1.7). Then*

i) $d\rho \in (\rho, \theta) \Longleftrightarrow \mathcal{V}$ is tangent to Σ.
ii) $d\theta \in (\rho, \theta) \Longleftrightarrow$ Torsion matrix $\mathcal{T}$ is identically zero on Σ.
iii) If (ρ, θ) is closed, then $\mathcal{V}$ is tangent to Σ and integrable, and therefore, Σ is foliated by integral manifolds of maximal dimension $p := n - s$.

i) is also equivalent to that Σ is invariant under any set of vector fields $X_1, \ldots, X_p$ that generates $\mathcal{V}$.

Let (ρ, θ), $\rho = (\rho^1, \ldots, \rho^d)$, $\theta = (\theta^1, \ldots, \theta^s)$ be a constrained Pfaffian system as in (1.5). A submanifold $i : \mathcal{S} \hookrightarrow M$ is an integral manifold of (1.5) if the following two conditions hold: firstly, $\mathcal{S}$ is a submanifold of Σ, that is, $\rho \circ i = 0$, and secondly, $i^*\theta = 0$; in a word, if

$$i^*(\rho, \theta) = 0. \tag{3.5}$$

For any ideal $I \subset \Omega^*(M)$, $i^*I = 0$ implies that

$$i^*(dI) = d(i^*I) = 0.$$

Therefore, (3.5) implies that on an integral submanifold $\mathcal{S}$

$$d\rho \in (\rho, \theta) \quad \text{and } d\theta \in (\rho, \theta).$$

Theorem 3.3. *(Generalized Frobenius theorem) If a constrained Pfaffian system $I = (\rho, \theta)$ as given in (1.5) is closed then the Pfaffian system restricted on $\rho = 0$ is Frobenius integrable.*

Recall that a constrained Pfaffian system (ρ, θ) is closed iff

$$\begin{aligned} d\rho &\in (\rho, \theta) \\ d\theta &\in (\rho, \theta). \end{aligned} \tag{3.6}$$

Now let us consider the simplest case that $d = s = 1$. Let $[T_{jk}]$ be the torsion of θ, that is,

$$d\theta \equiv \sum_{1 \leqslant j < k \leqslant p} T_{jk} \omega^j \wedge \omega^k, \text{ mod } \theta. \tag{3.7}$$

The second equation of (3.6) implies

$$T_{jk} \in (\rho), \text{ for each } j, k \text{ with } j < k, \tag{3.8}$$

which means that the torsion vanishes on $\rho = 0$. The first equation of (3.6) implies that $\rho = 0$ is invariant. Therefore, $\rho = 0$ is an integral manifold of codimension 1.

Example 3.4. In $\mathbb{R}^3 = \{(x, y, z)\}$ let $\theta = x^2 e^y z dy + dz$. Then

$$\begin{aligned} d\theta &= 2xe^y z dx \wedge dy - x^2 e^y dy \wedge dz \\ &\equiv \underbrace{2xe^y z}_{\mathcal{T}} dx \wedge dy, \quad \text{mod } \theta. \end{aligned}$$

Now the torsion vanishes on $xz = 0$. The constrained system (z, θ) satisfies (3.6) and therefore, $z = 0$ is an integral manifold. The other constrained systems (x, θ) satisfies the second equation but not the first of (3.6).

Theorem 3.5. *For $I = (\rho, \theta)$ with $dI \subset I$, we have*

i) In cases $0 \leqslant d \leqslant s$, $\rho = 0$ is invariant and the induced Pfaffian system is integrable.

ii) In cases $d \geqslant s$, $\rho = 0$ is an integral nanifold.

Now we turn to the general constrained system (ρ, θ) with $d\rho \notin (\rho, \theta)$. Since any integral manifold of (ρ, θ) is also an integral manifold of $(\rho, d\rho, \theta)$, we apply Theorem 3.3 to

$$(\rho, d\rho, \theta). \tag{3.9}$$

Observe that the set of tangent vectors that are annihilated by (3.9) is

$$(\rho, d\rho, \theta)^{\perp} = \{T_x\Sigma \cap \mathcal{D}_x : \ x \in \Sigma\}. \tag{3.10}$$

Since the first equation of (3.6) is already satisfied we check whether or not the second equation

$$d\theta \in (\rho, d\rho, \theta) \tag{3.11}$$

holds.

Theorem 3.6. *(Generalized Frobenius Theorem II) Given a constrained Pfaff equation (1.5) that satisfies (1.6), suppose that $(d\rho(x), \theta(x))$ has constant rank s', $s \leqslant s' \leqslant s + d$ for all $x \in \Sigma$ and that*

$$d\theta \in (\rho, d\rho, \theta). \tag{3.12}$$

Then Σ admits a foliation by integral manifolds of θ of dimension $n - d - s'$.

In case that (3.12) does not hold, we find additional constraints $r = (r^1, \dots, r^\delta)$ so that the torsion

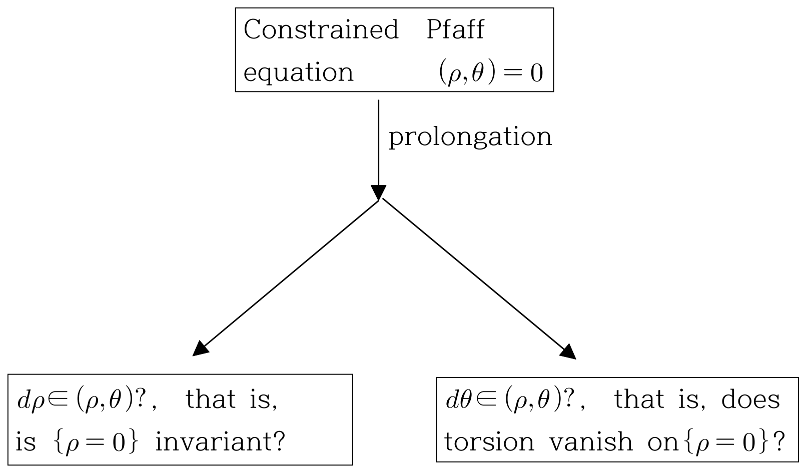
$$d\theta \mod (\rho, d\rho, \theta) \tag{3.13}$$

vanishes on $\{r = 0\}$. Then

$$(\rho, r, d\rho, dr, \theta) \tag{3.14}$$

is involutive by the generalized Frobenius theorem (Theorem 3.3).

Now we summarize our discussions as flowcharts of prolongations to the involutivity.


Constrained Pfaff equation $(\rho,\theta)=0$
prolongation
$d\rho\in(\rho,\theta)$?, that is, is $\{\rho=0\}$ invariant?
$d\theta\in(\rho,\theta)$?, that is, does torsion vanish on $\{\rho=0\}$?

Further prolongations towards the involutivity (closedness):

Case 1: $d\rho \in (\rho, \theta)$ and $d\theta \in (\rho, \theta) \longrightarrow$ no further prolongation and (ρ, θ) is closed.

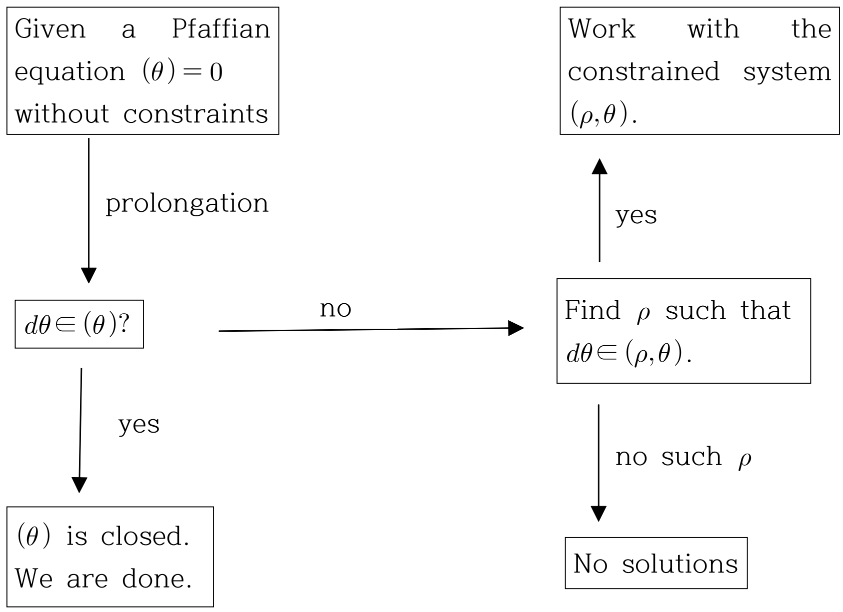
Case 2: $d\rho \in (\rho, \theta)$ and $d\theta \notin (\rho, \theta) \longrightarrow$ Find r such that $T \in (r)$, where $T := d\theta \mod (\rho, \theta)$, is the torsion $\longrightarrow$ prolongation and tell whether or not $dr \in (\rho, r, \theta)$?

If yes, (ρ, r, θ) is closed.
If no, start again with $(\rho, r, \theta) = 0$.

Case 3: $d\rho \notin (\rho, \theta)$ and $d\theta \in (\rho, \theta) \longrightarrow$ assuming $(d\rho, \theta)$ has constant rank on $\{\rho = 0\}$, (ρ, r, θ) is closed.

Case 4: $d\rho \notin (\rho, \theta)$ and $d\theta \notin (\rho, \theta) \longrightarrow$ Compute the torsion $d\theta \mod (\rho, \theta)$ and the torsion vanishing set $\{r = 0\} \longrightarrow$ prolongation $(\rho, r, d\rho, dr, \theta)$ is involutive if we assume $(d\rho, dr, \theta)$ has constant rank on $\{\rho = r = 0\}$.

When we start with a Pfaffian system without constraints:



Example 3.7. In $\mathbb{R}^4 = \{(x, y, z, u)\}$ let $\theta = xzdy + dz$. Then

$$d\theta = z\ dx \wedge dy + x\ dz \wedge dy \equiv z\ dx \wedge dy, \quad \text{mod } (\theta). \tag{3.15}$$

Consider each of the following constraints:

i) Constraint z: We check whether the ideal (z, dz, θ) is closed. From (3.15) we see that $d\theta \in (z)$. Hence $z = 0$ is an integral manifold of dimension 3.

ii) Constraint u: We check whether the ideal (u, du, θ) is closed. Since

$$d\theta \text{ mod } (u, du, \theta) = zdx \wedge dy \neq 0,$$

we need an additional constraint z. Now we ask whether the system with two constraints

$$(u, z, du, dz, \theta)$$

is closed. It is closed. In fact, $\theta = 0$ on $z = 0$. Hence, $u = z = 0$ is an integral manifold of dimension 2.

iii) Constraint $\rho(x, y, z, u) = z + 1/2(x^2 + y^2 + u^2)$. Then $d\rho = dz + xdx + ydy + udu$. We check whether the ideal $(\rho, d\rho, \theta)$ is closed. Then we have to see whether $d\theta$ is in $I := (\rho, d\rho, \theta)$. Since $d\theta \in I$ if and only if $z \in (\rho)$, which occurs only when $x = y = u = 0$. There is no integral manifold of positive dimension.

Example 3.8. In $\mathbb{R}^5 = \{(x, y, z, u, v)\}$ consider $\theta = (\theta^1, \theta^2)$, where

$$\begin{aligned} \theta^1 &= x^2 e^y z dy + dz \\ \theta^2 &= dv + z du. \end{aligned}$$

Then

$$\begin{aligned} d\theta^1 &= 2xe^y z dx \wedge dy + x^2 e^y dz \wedge dy \equiv 2xe^y z dx \wedge dy, \quad \text{mod } (\theta) \\ d\theta^2 &= dz \wedge du \equiv -x^2 e^y z dy \wedge du, \quad \text{mod } (\theta). \end{aligned} \tag{3.16}$$

with respect to the coframe $(dx, dy, du, \theta^1, \theta^2)$ we restate (3.16) in matrices as

$$\begin{bmatrix} d\theta^1 \\ d\theta^2 \end{bmatrix} \equiv \underbrace{\begin{bmatrix} 2xe^y z & 0 & 0 \\ 0 & 0 & -x^2 e^y z \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} dx \wedge dy \\ dx \wedge du \\ dy \wedge du \end{bmatrix}, \text{mod}(\theta).$$

Now consider a constrained Pfaff equation

$$(v, \theta) = 0. \tag{3.17}$$

We see that

$$\begin{aligned} d\theta^1 &\equiv 2xe^y z dx \wedge dy, \text{ mod } (v, dv, \theta^1, \theta^2) \\ d\theta^2 &\equiv -x^2 e^y z dy \wedge du, \text{ mod } (v, dv, \theta^1, \theta^2). \end{aligned}$$

Thus input another constraint z and see that the constrained system (v, z, θ) is closed. In fact, on $\{v = z = 0\}$ $\theta^1 = 0$ and $\theta^2 = dv$, and hence $\{v = z = 0\}$ is an integral manifold of dimension 3.

4. APPLICATIONS AND OPEN PROBLEMS

4.1. **Boundary values for overdetermined PDE systems.** For an overdetermined system of PDEs the boundary values cannot be free. Let Ω be a domain smoothly bounded by $\rho = 0$.

Problem 4.1. Find an algorithm for the equations for boundary values of overdetermined systems. There are also global invariants. Find the local and global conditions for the boundary values.

Problem 4.2. Given a system of first order PDEs, tell whether the boundary admits a characteristic curve. Discuss the maximum principle and the propagation along the boundary of the singularity of solutions.

4.2. **Discontinuous solutions, global solutions.** Consider the Cauchy problem for a quasi-linear PDE of first order for $u(x, t)$:

$$\begin{aligned} a(x,t,u)\frac{\partial u}{\partial t} + b(x,t,u)\frac{\partial u}{\partial x} &= c(x,t,u) \\ u(x,0) &= \phi(x). \end{aligned} \tag{4.1}$$

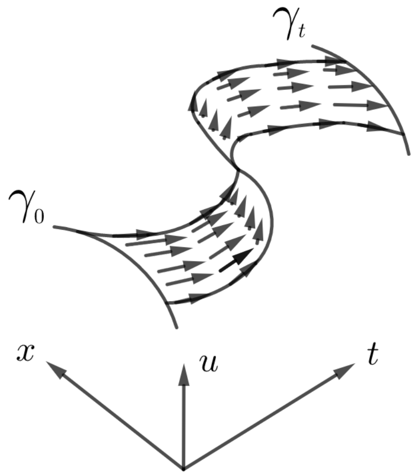
Then $\rho(x, t, u) = 0$ is an implicit solution of (4.1) if and only if ρ is a generalized first integral of the characteristic vector field

$$V = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x} + c\frac{\partial}{\partial u}$$

that satisfies

$$\rho(x, 0, \phi(x)) = 0.$$

Let γ_0 be the curve in $\mathbb{R}^3 = \{(x, t, u)\}$ given by the initial data $(x, 0, \phi(x))$. Then as in the following figure the surface swept over by γ_0 as the curve γ_0 moves on the flow of V. Then as an implicit function $u(x, t)$ may be multiple-valued.



characteristic flow

Problem 4.3. For the Cauchy problem (4.1) define the notions of global solution. Is there any natural branchcut as in Burgers' equation for discontinuous solutions?

Example 4.4. Consider a Cauchy problem for a quasi-linear PDE for $u(x,t)$

$$\begin{aligned} uu_t &= -t \\ u(x,0) &= \sqrt{1-x^2}, \quad -\epsilon < x < \epsilon. \end{aligned} \tag{4.2}$$

In $\mathbb{R}^3 = \{(x,t,u)\}$ the characteristic vector field is

$$(0, u, -t).$$

Along the integral curve of this vector field $dx : dt : du = 0 : u : (-t)$, thus by solving the total differential equation

$$\frac{dx}{0} = \frac{dt}{u} = \frac{du}{-t}$$

we obtain first integrals

$$\rho^1(x,t,u) = u^2 + t^2, \quad \rho^2(x,t,u) = x.$$

Now we find a first integral $F(\rho^1, \rho^2)$ that vanishes on the initial curve $(x, 0, \sqrt{1-x^2})$. Since $\rho^1 = 1 - x^2$ and $\rho^2 = x$ on the initial curve, we see that $\rho^1 + (\rho^2)^2 = 1$. Thus

$$F(\rho^1, \rho^2) = \rho^1 + (\rho^2)^2 - 1 = 0,$$

that is,

$$u^2 + t^2 + x^2 - 1 = 0 \tag{4.3}$$

is the solution. Observe that even though the Cauchy datum is given on a small interval $-\epsilon < x < \epsilon$ the solution is global. Notice that the global solution (4.3) is a double-valued function.

4.3. **Integrability of Hamiltonian systems.** Let (M^{2n}, ω, H) be a Hamiltonian system on M^{2n} with a symplectic form ω. A first integral is a function ρ that satisfies $X_H\rho = 0$, where X_H is the Hamiltonian vector field. We are concerned with the existence of a non-degenerate set of first integrals $\rho_1, \rho_2, \ldots,$ that are mutually independent, that is, $\{\rho_j, \rho_k\} = 0$. There can be at most n independent first integrals. When it attains the maximal number of first integrals we say that the Hamiltonian system is completely integrable. Even locally the existence of independent first integrals is not trivial. In the cases of $n = 1$ and that of $n = 2$ are completely integrable locally.

Problem 4.5. For the cases $n \geqslant 3$ find the conditions on H for the existence of first integrals. Determine the number of independent first integrals.

4.4. **Affine control systems.** Given an affine control system

$$\begin{cases} \dot{x} &= \vec{f}(x) + \sum_{j=1}^{k} \vec{g}_j(x)u^j \\ y &= h(x) \end{cases} \tag{4.4}$$

where $\vec{f}$ and $\vec{g}_j$ are smooth(C^∞) vector fields on an open subset $U \subset \mathbb{R}^n$ and h is a smooth real-valued function on U. A smooth distribution $\mathcal{D}$ is a controlled invariant distribution if the following conditions hold:

i) $L_{\vec{f}}\mathcal{D} \subset \mathcal{D}$, that is, for any section X of $\mathcal{D}$, the Lie derivative $L_{\vec{f}}X$ is a section of $\mathcal{D}$,
ii) $L_{\vec{g}_j}\mathcal{D} \subset \mathcal{D}$, for each $j = 1, \ldots, k$,
iii) $\mathcal{D} \subset (dh)^{-1}(0)$,
iv) $\mathcal{D}$ is integrable, that is, $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$.

Problem 4.6. Given an affine control system (4.4) determine the existence of an invariant distribution.

4.5. **Almost complex structures.** On an open neighborhood $M \subset \mathbb{C}^n$ of the origin consider a system of smooth complex-valued functions a_α^β vanishing at the origin. Let

$$\theta^\alpha := dz^\alpha + \sum_{\beta=1}^{n} a_\beta^\alpha d\bar{z}^\beta, \quad \text{for each } \alpha = 1, \ldots, n. \tag{4.5}$$

Then there is a unique almost complex structure J with respect to which $\theta := (\theta^1, \ldots, \theta^n)$ are $(1,0)$-forms. Consider the complexified exterior algebra

$$\Omega^*_{\mathbb{C}}(M) = \bigoplus_{k=0}^{2n} \Omega^k_{\mathbb{C}}(M), \quad \text{where } \Omega^k_{\mathbb{C}}(M) = \Gamma\left(\mathbb{C} \otimes \Omega^k(M)\right).$$

A complex-valued function f is said to be holomorphic if

$$df \in (\theta) \qquad \text{Cauchy-Riemann equations.} \tag{4.6}$$

We express the torsion of (θ) with respect to the coframe

$$(\theta, \bar{\theta}) := (\theta^1, \ldots, \theta^n, \bar{\theta}^1, \ldots, \bar{\theta}^n): \tag{4.7}$$

$$\begin{bmatrix} d\theta^1 \\ \vdots \\ d\theta^n \end{bmatrix} \equiv \underbrace{\begin{bmatrix} T^1_{12} & T^1_{13} & \cdots & T^1_{n-1,n} \\ \vdots & \vdots & \ddots & \vdots \\ T^n_{12} & T^n_{13} & \cdots & T^n_{n-1,n} \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} \bar\theta^1 \wedge \bar\theta^2 \\ \bar\theta^1 \wedge \bar\theta^3 \\ \vdots \\ \bar\theta^{n-1} \wedge \bar\theta^n \end{bmatrix}, \quad \mathrm{mod}\ (\theta). \tag{4.8}$$

The $n \times \binom{n}{2}$ matrix $\mathcal{T}$ is the torsion of (θ). The torsion $\mathcal{T}$ measures the non-integrability of J: Analogously to the Frobenius theorem for real 1-forms the Newlander-Nirenberg theorem [28] asserts that if $\mathcal{T}$ has rank zero, that is, all the entries of $\mathcal{T}$ are zeros, then the almost complex structure J is integrable, that is, there exist n independent holomorphic functions.

Exercise The Nijenhuis tensor N is a $(2,1)$-type tensor defined by

$$N(X,Y) := [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \quad \forall X,Y \in \Gamma(TM).$$

Show that

$$N(\bar Z_j, \bar Z_k) = -4\sum_{\ell=1}^{n} T^\ell_{jk} Z_\ell,$$

where

$$Z_1, \ldots, Z_n, \bar Z_1, \ldots, \bar Z_n$$

are basis of the complexified tangent bundle $T_{\mathbb{C}}(M)$ which is dual to (4.7). In [26] the derived system of (4.5) has been constructed and the problem of determining the number of independent holomorphic functions has been studied. This is a complex analogue of §3.1. The existence of integrable submanifolds of dimension p, $(p \leqslant n)$, has been studied in [27].

Problem 4.7. Does S^6 admit an integrable almost complex structure?

This is a prominent long-standing problem. It is known that the spheres $S^{4k}, k \geqslant 1$, and $S^{2n}, n \geqslant 4$, cannot admit an almost complex structure (Borel-Serre 1951). Any almost complex structure on a real 2-dimensional manifold is integrable, hence S^2 admits a complex structure. A natural almost complex structure on S^6 is that induced from its embedding into the space of octonions, which is not integrable. G. Etesi [25] claims he proved the affirmative answer.

Problem 4.8. Given an almost complex structure (M^{2n}, J) can you tell whether there exists a J-invariant submanifold?

The first non-trivial case is that of $n = 3$ and J-invariant submanifolds of real codimension 2. Let

$$L_j := \frac{\partial}{\partial z^j} + \sum_{\lambda=1}^{3} a^\lambda_j(z,\bar z)\frac{\partial}{\partial \bar z^\lambda}, \qquad j = 1,2,3$$

be $(1,0)$-vectors that are dual to (4.5). Then the problem is equivalent to the existence of real-valued functions u and v that satisfy

$$\left(\frac{\partial u}{\partial z^j} + \sum_{\lambda=1}^{3} a^\lambda_j(z,\bar z)\frac{\partial u}{\partial \bar z^\lambda}\right)\left(\frac{\partial v}{\partial z^k} + \sum_{\mu=1}^{3} a^\mu_k(z,\bar z)\frac{\partial v}{\partial \bar z^\mu}\right) \in (u,v), \tag{4.9}$$

for each $j, k = 1, 2, 3$. The complex conjugate of (4.9) can be written as

$$\bar{\partial} u \wedge \bar{\partial} v \in (u, v). \tag{4.10}$$

4.6. **CR geometry of real hypersurfaces.** On a neighborhood of the origin of

$$\mathbb{C}^{n+1} = \{(\underbrace{z_1, \ldots, z_n}_{z}, w)\}$$

consider a real-valued function $\rho(z, \bar{z}, w, \bar{w})$ with $\rho_w \neq 0$. Let $\theta = \sqrt{-1}\partial\rho$ be a 1-form of type $(1, 0)$.

Exercise Show that θ is a real 1-form on the real hypersurface $\rho = 0$.

For a basis of the complexified tangent bundle of the real hypersurface $\rho = 0$ we take

$$dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_n, \theta$$

and express the torsion

$$d\theta = \sqrt{-1} \sum_{j,k=1}^{n} T_{jk} dz_j \wedge d\bar{z}_k, \quad \mod (\theta, d\rho), \qquad \text{(Levi-form)} \tag{4.11}$$

where $[T_{jk}]$ is hermitian. The existence of integral manifolds and invariant manifolds has been studied in [33] extrinsically and locally and in [24] intrinsically and globally.

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