Introduction to GKM theory

§1. Equivariant cohomologies

- What is the equivariant cohomology?

$H^*(M)$: invariant of topology of M

i.e., if $M \cong N$ homeomorphic

then $H^*(M) \cong H^*(N)$ isom (algebra)

"equivariant" cohomology $H_G^*(M)$

: invariant of $G \curvearrowright M$ i.e.,

$$(M, G) \cong (N, G) \Rightarrow H_G^*(M) \cong H_G^*(N)$$

Definition Let G be a topological group (or Lie group), M be a topological space (or (smooth) manifold)

G acts on M ($G \curvearrowright M$ or (M, G)) if

$\exists\ \varphi : G \times M \to M$ conti or C^∞ s.t.

(i) $\varphi(e, x) = x \quad \forall x \in M \quad$ (e: identity in G)

(ii) $\varphi(g, \varphi(h, x)) = \varphi(gh, x) \quad \forall g, h \in G, \forall x \in M$,

$(M, G) \cong (N, G) \overset{\text{def}}{\Longleftrightarrow} \exists\ f : M \xrightarrow{\cong} N$ homeo (diffeo)

s.t.

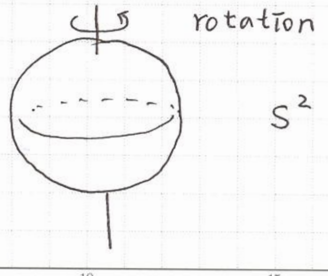
$$\begin{tikzcd} G \times M \arrow[r, "\varphi"] \arrow[d, "1\times f"'] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d, "f"] \\ G \times N \arrow[r, "\varphi"'] & N \end{tikzcd}$$

E.g.) $S^1 \curvearrowright S^2 \qquad S^2 \subset \mathbb{C} \times \mathbb{R}$

$$S^2 = \{ (z, r) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + r^2 = 1 \}$$

$$S^1 = \{ t \in \mathbb{C} \mid |t|^2 = 1 \}$$

$$t \cdot (z, r) := (tz, r)$$



IDEA of definition of $H^*_G(M)$

⊙ Easiest case

stabilizer subgroup

$G \curvearrowright M$ free $\overset{\text{def}}{\Longleftrightarrow}$ $\forall x \in M$, $G_x := \{g \in G \mid g \cdot x = x\}$
$= \{e\}$

$\Rightarrow$ $M/G = \{$ the set of orbits $\}$ has a nice property

$G(x) = \{g \cdot x \mid g \in G\}$

Example M: manifold, G: compact connected Lie group

$\Rightarrow$ M/G a manifold

Eg) ① $S^1 \curvearrowright S^2$ is NOT free

($\because$ at N, S $G_N = G_S = S^1 \neq \{e\}$)

② $S^1 \curvearrowright \mathbb{C}^* (= \mathbb{C} \setminus \{0\})$

$\mathbb{C}^*/S^1 = \mathbb{R}_{>0}$.

Consequently, when $G \curvearrowright M$ freely, $H^*(M/G)$ has "enough" information.

⊙ In general, $G \curvearrowright M$ may NOT free.

$H^*(M/G)$ does NOT have enough information

Eg. $S^2/S^1 = I^1 \simeq \{pt\}$

$\Rightarrow$ $H^*(S^2/S^1) \cong H^*(\{pt\})$.

Borel's idea (1959)

THM [Milnor, 1956] Let G be a topological group.

Then $\exists$ a contractible space EG s.t. $G \curvearrowright EG$ freely.

- Put $\boxed{BG} := EG/G$ classifying space
- $H \subset G$ topological subgroup

$\rightsquigarrow$ $H \curvearrowright EG$ freely & $EG/H \simeq BH = EH/H$ (homotopy equiv.)

Eg. ① $G = \mathbb{Z}$. $E\mathbb{Z} = \mathbb{R} \circlearrowleft \mathbb{Z}$ additively (free!)

$$B\mathbb{Z} = \mathbb{R}/\mathbb{Z} \cong S^1$$

② $G = S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ $(=: T^1)$

$$ES^1 = S^\infty, \quad BS^1 = \mathbb{C}P^\infty$$

(this is the infinite-dimensional version of the well-known Hopf fibration $S^3 \to \mathbb{C}P^1$)

③ $G = T^n = (S^1)^n$ $\quad ET^n = (S^\infty)^n$, $BT^n = (\mathbb{C}P^\infty)^n$.

$\boxed{\text{IDEA}}$ If $G \curvearrowright X$ free, $G \curvearrowright Y$ then $G \curvearrowright X \times Y$ free.

For any $G \curvearrowright M$, $G \curvearrowright EG \times M$ free

$\rightsquigarrow$ $(EG \times M)/G =: EG \times_G M$ Borel construction

$$(e, x) \cdot g = (e \cdot g, g^{-1} \cdot x)$$

Definition [Borel, 1959] The G-equivariant cohomology of M is defined by $\boxed{H^*_G(M)} := H^*(EG \times_G M)$.

Note $H^*_{T^n}(\{pt\}) = H^*(BT^n) \cong \mathbb{C}[u_1, \ldots, u_n]$ $\quad \deg u_i = 2$

⓪. General properties of $H^*_G(M)$.

① If $G \curvearrowright M$ free, then

$$\begin{array}{ccc} EG \times_G M & \hookleftarrow & EG \simeq \{*\} \\ \downarrow & & \\ M/G & & \end{array}$$

$\Rightarrow \quad H^*_G(M) = H^*(EG \times_G M) = H^*(M/G)$

(General fact

If $G \curvearrowright X$ free, $G \curvearrowright Y$ then

$$\begin{array}{ccc} X \times_G Y & \hookleftarrow & Y \\ \downarrow & & \\ X/G & & \end{array}$$ fiber bundle)

② $$\begin{array}{ccc} EG \times_G M & \hookleftarrow & M \\ \pi \downarrow & & \\ EG/G = BG & & \end{array}$$ fiber bundle

$\rightsquigarrow \quad \pi^* : H^*(BG) \to H^*_G(M)$

$\rightsquigarrow \quad H^*_G(M)$ has an $H^*(BG)$-algebra structure.

Note $f : (M, G) \xrightarrow{\cong} (N, G)$ (not only a ring isom but also $H^*(BG)$-

$\rightsquigarrow \quad f^* : H^*_G(N) \to H^*_G(M)$: algebra isomorphism.

THM [Masuda, 2008] (M^{2n}, T^n), (N^{2n}, T^n) : smooth projective toric varieties

$(M^{2n}, T^n) \cong (N^{2n}, T^n)$ (weakly equivariantly isom. as var.)

$\Longleftrightarrow \quad H^*_{T^n}(M; \mathbb{Z}) \cong H^*_{T^n}(N; \mathbb{Z})$ weakly isom. as $H^*(BT^n)$-algebras.

(- "weakly equivariantly" $\phi : M \to N$ var. isom with

an automorphism $\gamma : T^n \circlearrowleft$ s.t. $\phi(tx) = \gamma(t)\phi(x)$ $\forall t \in T$, $x \in M$.)

- ring isom $\Phi : H^*_T(N) \to H^*_T(M)$ with an automorphism

$\gamma : T^n \circlearrowleft$ s.t. $\Phi(uw) = \gamma^*(u)\,\Phi(u)$ ($\gamma^* : H^*(BT) \circlearrowleft$)

Q How can we describe / compute $H^*_G(M)$?

§2. GKM manifolds We consider $R = \mathbb{Q}$ or $\mathbb{C}$

◎ Assume $G = T^k = (S^1)^k$

Two morphisms

① $H^*_T(M) \xrightarrow{i^*} H^*_T(M^T)$ induced by $M^T \xhookrightarrow{i} M$

② $H^*_T(M) \xrightarrow{j^*} H^*(M)$ induced by $ET \times_T M \xhookleftarrow{j} M$

$$\begin{array}{c} ET \times_T M \\ \downarrow \\ BT \end{array}$$

Def & Thm We say T-action on M is equivariantly formal if $H^*_T(M)$ is free $H^*(BT)$-module.

(actual definition: Spectral sequence of the fibration ⊛ collapses : $H^p(BT, H^q(M)) \Rightarrow H^{p+q}(ET \times_T M)$

$\Rightarrow H^*_T(M) \cong H^*(M) \otimes H^*(BT)$

$\Rightarrow \operatorname{rank}_{H^*(BT)} H^*_T(M) \cong \operatorname{rank} H^*(M)$.

Thm The following spaces are equivariantly formal.

(1) $H^{odd}(M)$ is trivial.

(e.g. M : toric manifolds,

$M = G/P$ homogeneous manifolds (P : parabolic).)

(2) M : symplectic, $T \circlearrowright M$ Hamiltonian.

Thm (Borel) (M, T) : equivariantly formal.

$\Rightarrow$ i^* is injective & j^* is surjective.

$H^*_T(M) \otimes_{H^*(BT)} \mathbb{Q} \to H^*(M)$ is an isomorphism

Q What is $\mathrm{im}(i^*) \cong H^*_T(M)$?

Def M : complex T-manifold of $\dim_{\mathbb{C}} = n$.

M is a GKM manifold if

① $T \curvearrowright M$ equivariantly formal.

② $0 < |M^T| < \infty$

③ At every $p \in M^T$,

$$T_pM \underset{\uparrow}{\cong} \mathbb{C}(\alpha_{1,p}) \oplus \mathbb{C}(\alpha_{2,p}) \oplus \cdots \oplus \mathbb{C}(\alpha_{n,p})$$

as T-representations

$\{\alpha_{i,p}\}_{i=1}^{n}$ are pairwise linearly independent.

Rmk $\boxed{M^{(1)}} := \{ x \in M \mid \dim_{\mathbb{R}}(T\cdot x) \leq 1 \}$: equivariant 1-skeleton.

③ $\Rightarrow$ $M^{(1)}$: union of S^2's.

Example $M = \mathbb{C}P^2 = (\mathbb{C}^3 \setminus 0)/\mathbb{C}^*$ $(x_0, x_1, x_2) \sim (tx_0, tx_1, tx_2)$, $t \in \mathbb{C}^*$

$T = T^2 \curvearrowright M$, $(t_1, t_2)\cdot[x_0, x_1, x_2] = [x_0, t_1x_1, t_2x_2]$

$M^T = \{[1,0,0], [0,1,0], [0,0,1]\}$

① at $p = [1,0,0]$ $(t_1,t_2)\cdot[1, x_1, x_2] = [1, \boxed{t_1}x_1, \boxed{t_2}x_2]$

$\rightsquigarrow$ $T_pM \cong \mathbb{C}(L_1) \oplus \mathbb{C}(L_2)$

$L_i : \mathrm{Lie}(T) \cong \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(a_1, a_2) \mapsto a_i$.

② at $p = [0,1,0]$ $(t_1,t_2)\cdot[x_0, 1, x_2] = [\boxed{t_1^{-1}}x_0, 1, \boxed{t_1^{-1}t_2}x_2]$

$\rightsquigarrow$ $T_pM \cong \mathbb{C}(-L_1) \oplus \mathbb{C}(-L_1+L_2)$

③ at $p = [0,0,1]$ $(t_1,t_2)\cdot[x_0, x_1, 1] = [\boxed{t_2^{-1}}x_0, \boxed{t_1t_2^{-1}}x_1, 1]$

$\therefore$ $(\mathbb{C}P^2, T^2)$ is a GKM manifold.

Example $M = Fl(\mathbb{C}^3) \cong GL_3(\mathbb{C})/B$

$$T = (S^1)^3 = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} : t_i \in S^1 \right\} \subset GL_3(\mathbb{C})$$

$$t \cdot gB = tgB \qquad t \in T.$$

$$M^T = \{ wB \mid w \in \mathfrak{S}_3 \} \cong \mathfrak{S}_3$$

eg.

$$p = eB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B$$

$$\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ g_{21} & 1 & 0 \\ g_{31} & g_{32} & 1 \end{pmatrix} B = \begin{pmatrix} t_1 & 0 & 0 \\ t_2 g_{21} & t_2 & 0 \\ t_3 g_{31} & t_3 g_{32} & t_3 \end{pmatrix} B$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ \boxed{t_1^{-1} t_2} g_{21} & 1 & 0 \\ \boxed{t_1^{-1} t_3} g_{31} & \boxed{t_2^{-1} t_3} g_{32} & 1 \end{pmatrix} B.$$

$$T_p M \cong \mathbb{C}(-L_1 + L_2) \oplus \mathbb{C}(-L_1 + L_3) \oplus \mathbb{C}(-L_2 + L_3)$$

By similar computation, $(G/B, T)$ is a GKM manifold.

Def (M, T) : GKM manifold.

The GKM graph $\Gamma = (V, E, \alpha)$ is a labeled graph

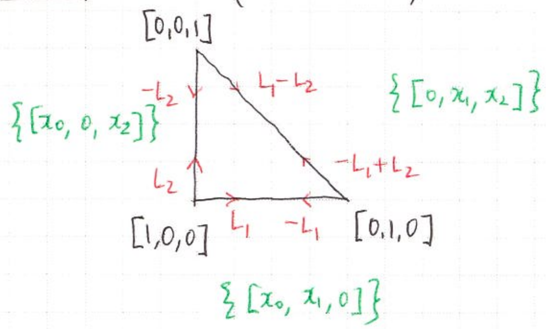
(1) $V = M^T$ vertices

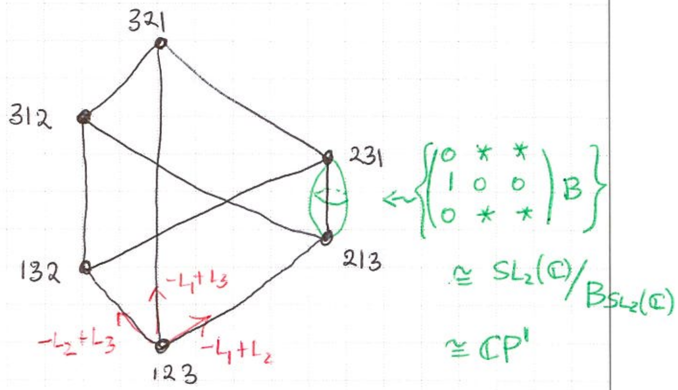
(2) $E = \{ (p, q) \in M^T \times M^T \mid \exists$ an embedded $S^2 \subset M^{(1)}$ s.t. $\{p, q\} \subset S^2 \}$

oriented !

(3) We label each edge (p, q) with the weight $\alpha = \alpha_{(p,q)}$ of the T-action corresponding to the embedded $S^2 \supset \{p, q\}$.

Example $(\mathbb{C}P^2, T^2)$ $(Fl(\mathbb{C}^3), T^3)$





- $i^*: H^*_T(M) \to H^*_T(M^T) \cong \bigoplus_{p\in M^T} H^*(BT)$

If (M,T) is equivariantly formal, then i^* is injective.

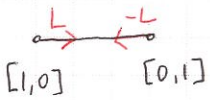
Thm (Goresky–Kottwitz–MacPherson, 1998)

(M,T) : GKM manifold

$$H^*_T(M) \cong \Big\{ (f_p) \in \bigoplus_{p\in M^T} H^*(BT) : \underbrace{\alpha_{(p,q)} \mid f_p - f_q \quad \forall (p,q)\in E}_{\text{"GKM condition"}} \Big\}$$

$=: H^*(\Gamma)$ "graph cohomology".

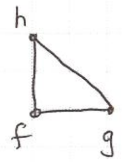
Example ① $\mathbb{C}P^1$. $t\cdot[x_0, x_1] = [x_0, t\cdot x_1]$



$H^*_{S^1}(\mathbb{C}P^1) \cong \{$ $f(L)$ — $g(L)$ $: L \mid f(L)-g(L)$ in $\mathbb{C}[L]\}$

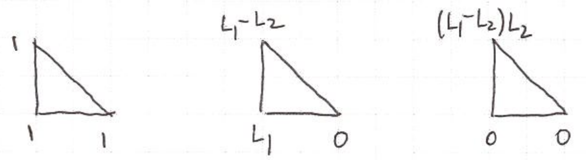
$\rightsquigarrow \{$ 1 — 1 , 0 — L $\}$: a basis of $H^*_{S^1}(\mathbb{C}P^1)$ as $H^*(BS^1)$-module.

② $H^*_{T^2}(\mathbb{C}P^2) \cong \Big\{$



$: f, g, h \in \mathbb{Q}[L_1, L_2]$, $L_1 \mid f-g$, $L_2 \mid f-h$, $-L_1+L_2 \mid g-h \Big\}$

As $H^*(BT)$-module, $H^*_T(\mathbb{C}P^2)$ has generators



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③ $(Fl(\mathbb{C}^3), T^3)$

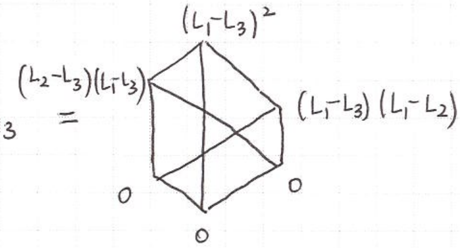
<u>Homework</u> Check the following elements satisfy GKM conditions.

	τ_{123}	τ_{132}	τ_{213}	τ_{231}	τ_{312}	τ_{321}
123	1	0	0	0	0	0
132	1	L_2-L_3	0	0	0	0
213	1	0	L_1-L_2	0	0	0
231	1	L_1-L_3	L_1-L_2	$(L_1-L_2)(L_1-L_3)$	0	0
312	1	L_2-L_3	L_1-L_3	0	$(L_2-L_3)(L_1-L_3)$	0
321	1	L_1-L_3	L_1-L_3	$(L_1-L_2)(L_1-L_3)$	$(L_2-L_3)(L_1-L_3)$	$(L_1-L_2)(L_2-L_3)(L_1-L_3)$

$\{\tau_w \mid w \in \mathfrak{S}_3\}$ forms a basis for $H_T^*(Fl(\mathbb{C}^3))$.

<u>Advantage</u>

Easily compute cup products!

- $\tau_{132} \cdot \tau_{213} =$ 

$= \tau_{231} + \tau_{312}$.

- $\tau_{321}^2 = (L_1-L_2)(L_1-L_3)(L_2-L_3)\,\tau_{321}$

$$H_T^*(G/B) \longrightarrow H^*(G/B) \qquad \rightsquigarrow \qquad \sigma_{132} \cup \sigma_{213} = \sigma_{231} + \sigma_{312}$$

$$\tau_w \longmapsto \sigma_w \qquad\qquad \sigma_{321}^2 = 0.$$

§3. Canonical classes in $H^*(\Gamma)$.

- (M,T) : GKM manifold & Hamiltonian T-action

  $\rightsquigarrow$ $\Gamma = (V, E, \alpha)$ GKM graph.

Note For $(p,q) \in E$, we also have $(q,p) \in E$ and

$$\alpha_{(p,q)} = -\alpha_{(q,p)}.$$

- Take ξ s.t. $\alpha_e(\xi) \neq 0$ $\forall e \in E$.

  We only consider the edges $e \in E$ with $\alpha_e(\xi) > 0$

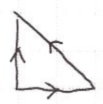
  say $E^\circ \subset E$.

  Write $p \prec q$ if $\exists$ a directed edge $(p,q) \in E^\circ$

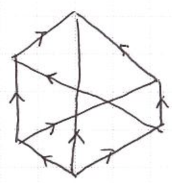
  $\rightsquigarrow$ partial order on V.

  ※ [Thm 1.4.2, Guillemin-Zara 2001] If (Γ, α) is the GKM graph of a Hamiltonian G-manifold, then (V, E°) is acyclic.

Example



$\xi = L_1^* + 2L_2^*$



$\xi = L_1^* + 2L_2^* + 3L_3^*$

Def A Kirwan class $p \in M^T$ is an equivariant class $\nu_p \in H_T^{2\lambda_p}(M)$ satisfying

(i) $\nu_p(p) = \prod_{(q,p) \in E^\circ} (-\alpha_{(q,p)})$

(ii) $\nu_p(q) = 0$ $\forall q \prec p$

$\lambda_p = \# \{ (q,p) \in E^\circ \}$

By convention, $H^*(BT) = \mathbb{Q}[L_1, \cdots, L_n]$, $\deg L_i = 2$.

Thm (Kirwan, 1984) For a Hamiltonian T-manifold, $\exists$ Kirwan class ν_p for all $p \in M^T$. Moreover, $\{\nu_p \mid p \in M^T\}$ forms an $H^*(BT)$-module basis for $H^*_T(M)$.

<u>However</u> Kirwan classes are NOT uniquely determined in general.

Fortunately, in $Fl(\mathbb{C}^n)$, Kirwan basis is determined uniquely

We can check that $\{\tau_w \mid w \in \mathfrak{S}_3\}$ is a Kirwan basis.

Several people have added different conditions that would ensure that this basis is unique, i.e., to be "canonically" associated.

(1) Guillemin – Zara (2002, 2003)

GKM spaces, "equivariant Thom classes"

When $\overline{W^u(p)}$ is smooth, $\nu_p \leftrightarrow [\overline{W^u(p)}]^{PP}_T$

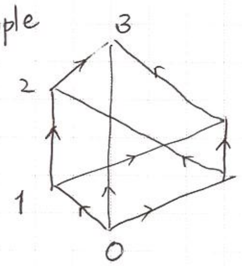
(2) Goldin – Tolman (2009)

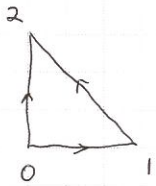
Hamiltonian T-spaces, "canonical classes" — Kirwan class & $\nu_p(q) = 0$ if $p \not\leq q$

existence & uniqueness are proved under the assumption that ξ is "index increasing"

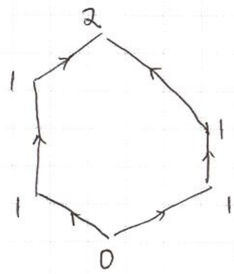
$p < q \Rightarrow \lambda_p < \lambda_q$

Example





: index increasing



: GKM graph of $\mathrm{Hess}(S, (2,3,3)) = X_{\hexagon}$ permutohedral var.

NOT index increasing.

(3) Zara (2007) non-index increasing case, "canonical classes".

(4) Pabiniak – Sabatini (2018)

symplectic toric manifold. "canonical classes"

Ⓐ They provide a Kirwan class corr. to $\left[\overline{W^u(p)}\right]_T^{PD}$

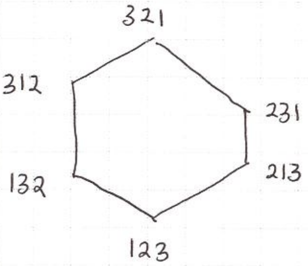
Ⓑ When ξ is index increasing, then it coincides with GT class.

<u>Rmk</u> GKM theory can be applied to ...

- G : compact connected Lie group
- GKM variety ($M^{(1)}$ consists of finitely many S^2's.)

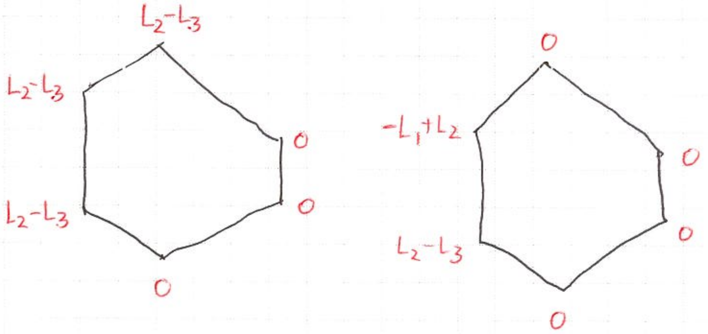
  : NOT necessarily smooth
- equivariant Chow rings, equivariant algebraic K-theory

⊙ Let us go back to the example.



This is the GKM graph of the smooth projective toric variety $X_{⬡}$ whose fan is obtained by Weyl chambers of type A_2.

There are two distinct Kirwan classes :



canonical class in Pabiniak – Sabatini

- $W = \mathfrak{S}_3 \curvearrowright H_T^*(G/B)$.

$\sigma \in H_T^*(G/B)$, $w, v \in W$

$\rightsquigarrow \quad (w \circ \sigma)(v) := w \cdot \underbrace{\sigma(w^{-1}v)}_{}$.

in $H^*(BT) = \mathbb{Q}[L_1, \ldots, L_k]$.

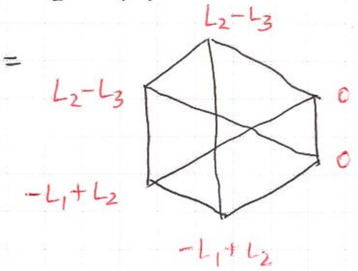
Eg. $(s_1 \circ \tau_{213})(132) = s_1 \circ \tau_{213}(s_1 \cdot 132)$

$= s_1 \circ \tau_{213}(231)$

$= s_1 \circ (L_1 - L_2)$

$= L_2 - L_1$.

In fact, $s_1 \circ \tau_{213} =$



$= (-L_1+L_2)\,\tau_{123} + \tau_{213}$

$\rightsquigarrow \quad s_1 \cdot \sigma_{213} = \sigma_{213}$ in $H^*(G/B)$

Furthermore, $W \curvearrowright H^*(G/B)$ trivial.

Homework ① $H^*(G/B) \cong \mathbb{1}^{\oplus 6}$ as W-representations.

② $H^*(X_{\hexagon}) \cong \mathbb{1}^{\oplus 4} \oplus V_{\boxplus}$ " .

Rmk (Chang–Skjelbred, 1974) (cf Lemma 2.3)

$0 \to H_K^*(X; A) \xrightarrow{\gamma} H_K^*(X^K; A) \xrightarrow{\delta} H_K^*(X^{(1)}, X^K; A)$ is exact.