

Domain-wall statistics and mixed order transition in the random bond Ising model

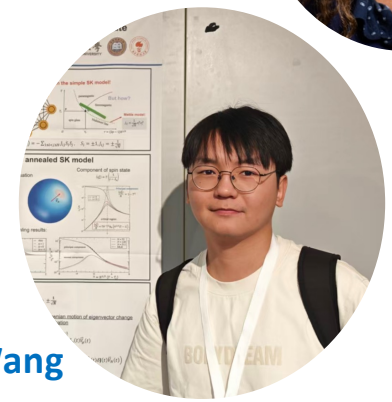
Lei-Han Tang

Center for Interdisciplinary Studies

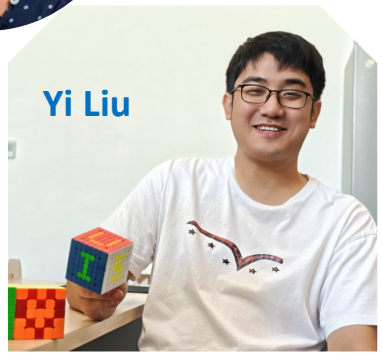
Westlake University



Leticia Cugliandolo



Ding Wang

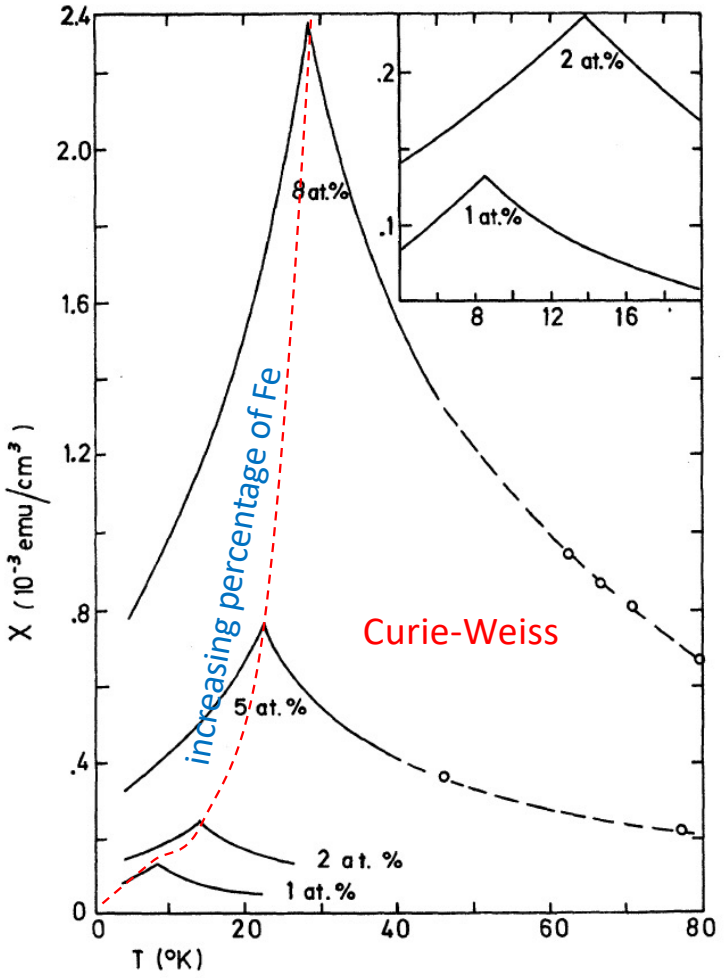


Yi Liu

Phys Rev B **6**, 4220 (1972)

Magnetic Ordering in Gold-Iron Alloys*

V. Cannella†
Physics Department, Wayne State University, Detroit, Michigan 48202
and
J. A. Mydosh†
Institut für Festkörperforschung, Kernforschungsanlage, 517 Jülich, Germany
(Received 18 April 1972)



J. Phys. F: Met. Phys. **5,** 965 (1975)

Theory of spin glasses

S F Edwards† and P W Anderson‡
Cavendish Laboratory, Cambridge, UK

The argument is that because the interaction between the spins dissolved in the matrix oscillates in sign according to distance. there will be no mean ferro- or antiferromagnetism, but there will be a ground state with the spins aligned in definite directions. even if these directions appear to be at random.

Edwards-Anderson (EA) model:

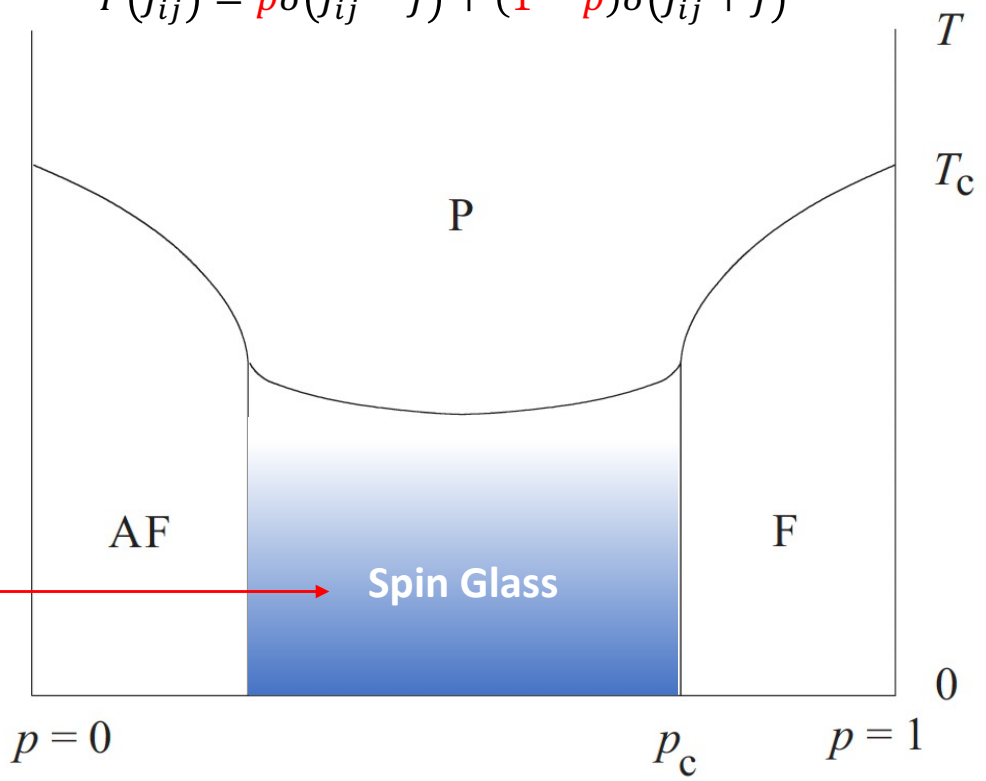
$$H(\{S_i\},\{J_{ij}\}) = -\sum_{\langle ij \rangle} J_{ij} s_i s_j$$

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J)$$

Generic phase diagram

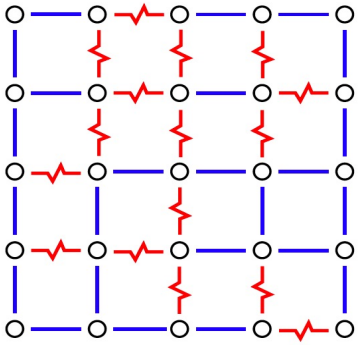
EA: analogy to polymer glass with frozen contacts/spins

How many order parameters?
Ground-state degeneracy?

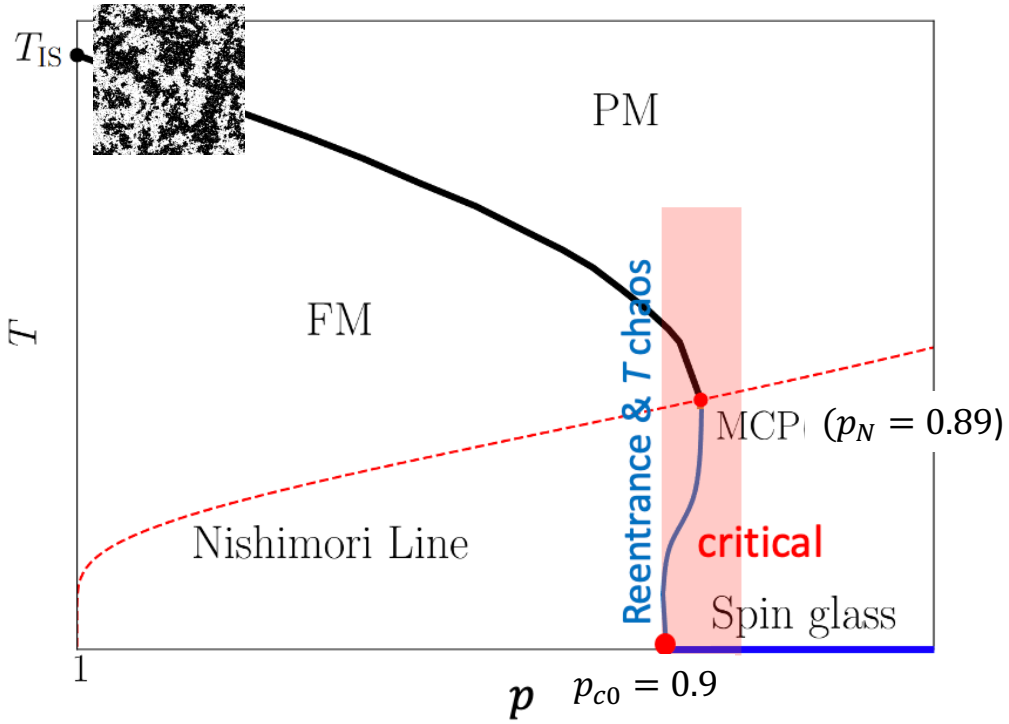


H. Nishimori, *Statistical Physics of Spin Glasses and Information Processing, An Introduction*. Oxford, 2001

2D $\pm J$ Random Bond Ising Model



p = % of ferromagnetic bond

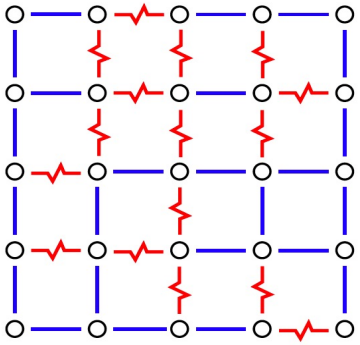


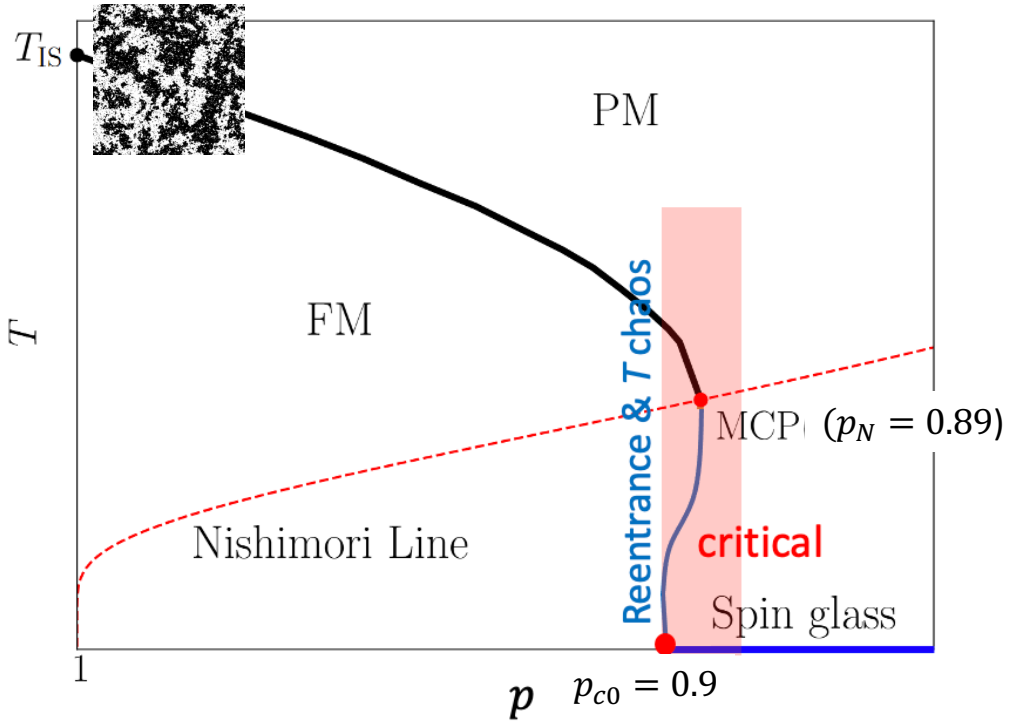
M. Picco, A. Honecker, and P. Pujol, *J. Stat. Mech.* P09006 (2006)
A. Honecker, M. Picco, and P. Pujol, *Phys Rev Lett* **87**, 047201 (2001)

$1 - p_c$	Method	Reference
Nishimori point in the $\pm J$ model		
0.111 ± 0.002	Transfer matrix	[30]
0.114 ± 0.003	Series expansion	[17]
0.1128 ± 0.0008	Non-equilibrium	[31]
0.1095 ± 0.0005	Transfer matrix	[18]
0.1094 ± 0.0002	Transfer matrix	[15]
0.1093 ± 0.0002	Fermionic transfer matrix	[19]
0.110 028	Duality	[21]
$\leq$0.178 203	Rigorous upper bound	[32]
$T = 0$ critical point in the $\pm J$ model		
$\sim$0.099	Series expansion	[22]
0.105 ± 0.01	Matching algorithm	[23]
$0.095 < p_c < 0.108$	Matching algorithm	[24]
0.104 ± 0.001 0.106 ± 0.002	Exact ground states	[25]
0.115	Ground state enumeration	[27]
0.1031 ± 0.0001	Exact ground states	[12]
0.103 ± 0.001	Exact ground states	[28]

2D $\pm J$ Random Bond Ising Model

p = % of ferromagnetic bond





This talk

- entropic sampling and magnetization resolved density of states
- FM to SG transition at $T = 0$
  - mixed-state transition (analogy to explosive percolation)
  - pinned domain walls and mapping to directed polymers, exact finite-size scaling exponents
- Domain-wall statistics at finite T
  - Multi-critical point and crossovers
- Is 3D similar or different from 2D?

Entropic Sampling Monte Carlo Simulations

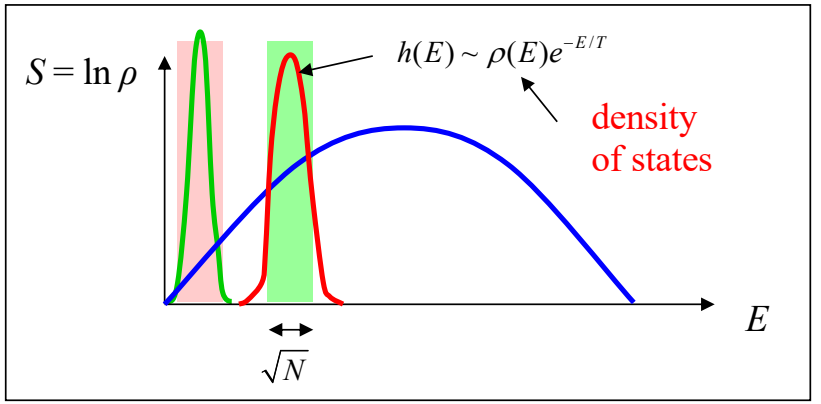
Yi Liu, Ding Wang, Xin Wang, Dao-Xin Yao and LHT., *Commun. Theor. Phys.* **77** 125603 (2025)

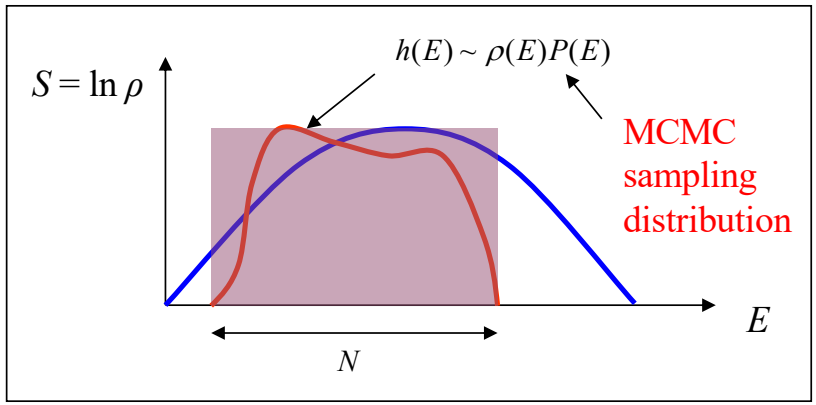
Entropic Sampling

Conventional MC

$P(E) \sim \exp(-E/T)$

Energy histogram sharply peaked





Entropic sampling

$P(E) \sim 1/\rho(E) = e^{-S(E)}$

Energy histogram broadened to cover both low and high T configurations

Density of states:

$$\sum_E \rho(E) = 2^N$$

- Well-suited to explore the equilibrium statistics at low temperatures
- Accurate computation of the magnetization-resolved density of states down to the g.s.

Computation of thermodynamic averages from the extended density of states

Micro-canonical formulation

$$g(E) = \sum_M g(E, M)$$

$$S(E) = \ln g(E) \qquad T^{-1} = \frac{\partial S}{\partial E}$$

Canonical formulation

$$Z = \sum_{E,M} g(E) e^{-E/T} \qquad F(T) = -T \ln Z$$

free energy

$$\langle E \rangle (T) = \frac{\sum_{E,M} E \; g(E, M) e^{-E/T}}{\sum_{E,M} g(E, M) e^{-E/T}}$$

mean energy

$$\langle |M| \rangle (T) = \frac{\sum_{E,M} |M| g(E, M) e^{-E/T}}{\sum_{E,M} g(E, M) e^{-E/T}}$$

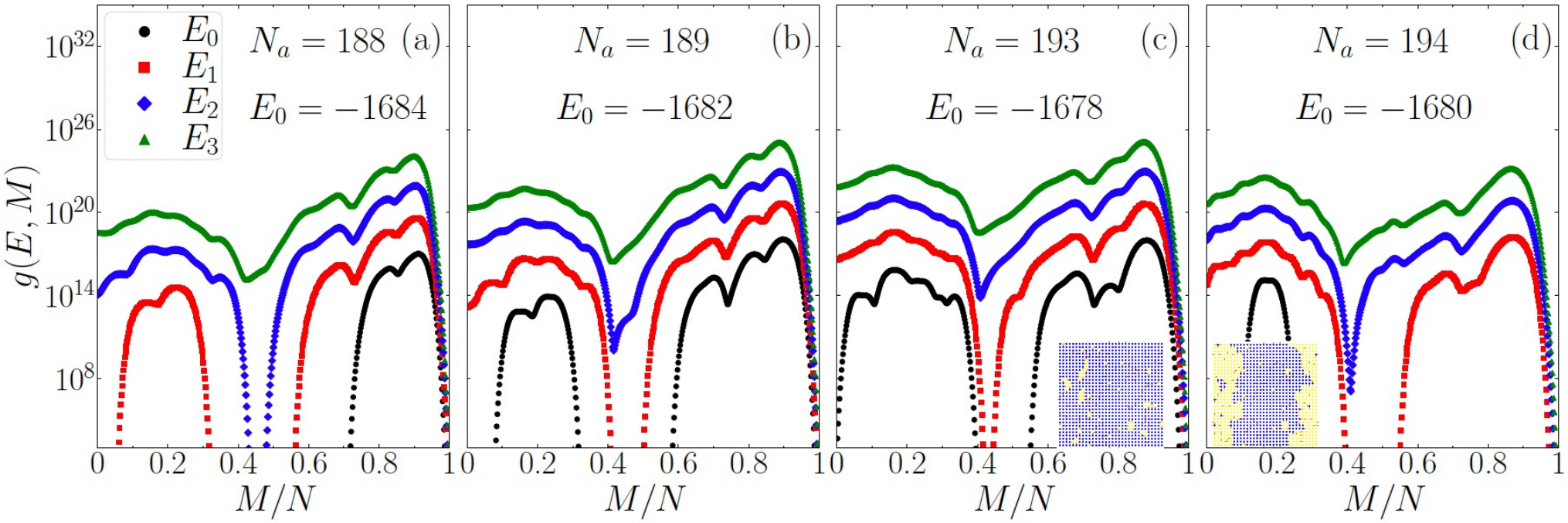
mean magnetization

Magnetization-resolved density of states at low energies

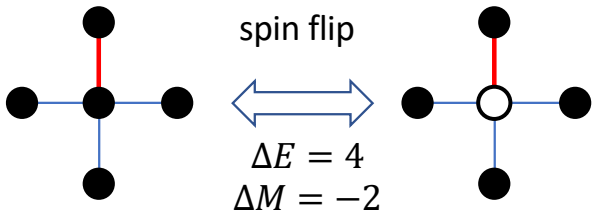
One specific series of disorder configurations

$N = 32\times32$

$N_a = (1-p)N$ = # of antiferromagnetic bonds



Isolated AFM bond



$$g(E_i, M-2) \approx 2N_a\, g(E_{i-1}, M)$$

(right side of the curves)

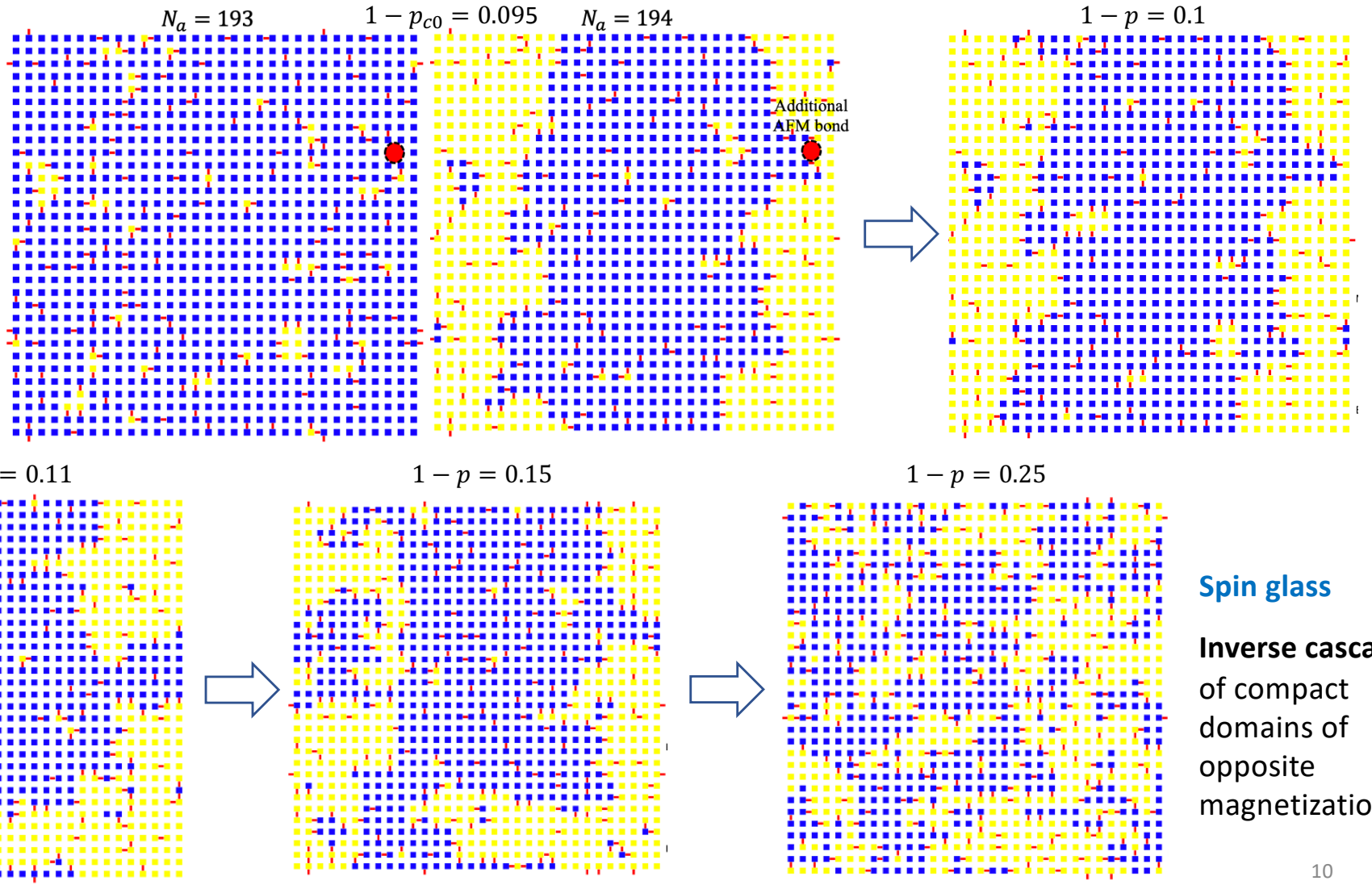
Ground state spin configurations of a particular sequence of bond flips

32 x 32 lattice

N_a= # of AFM bonds (red)

Ferromagnet

spin "+"
spin "−"

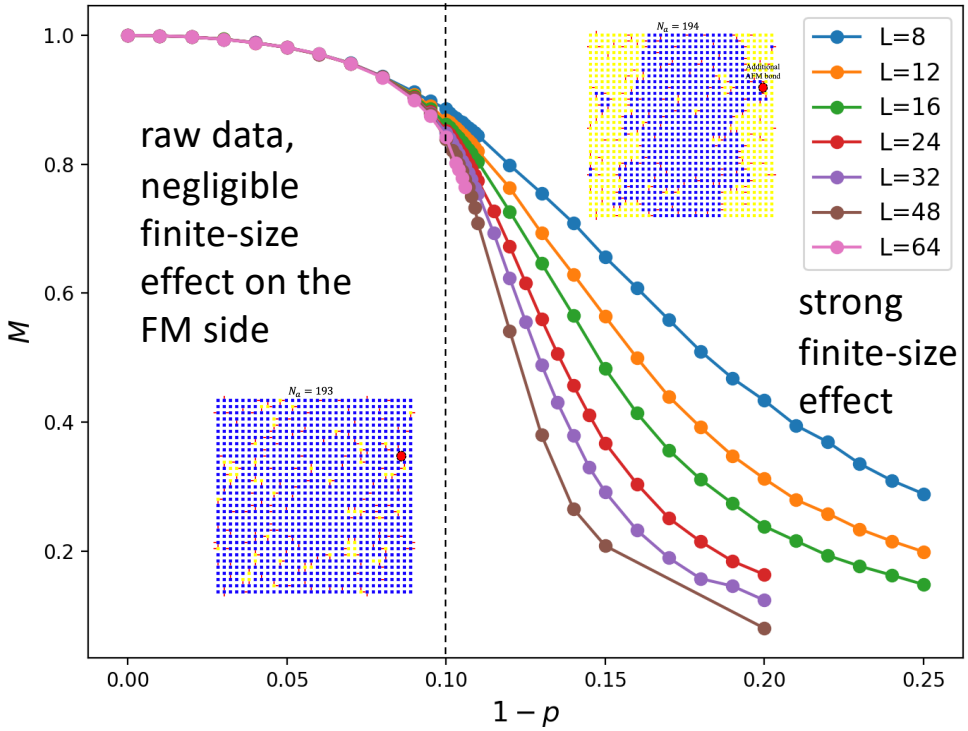


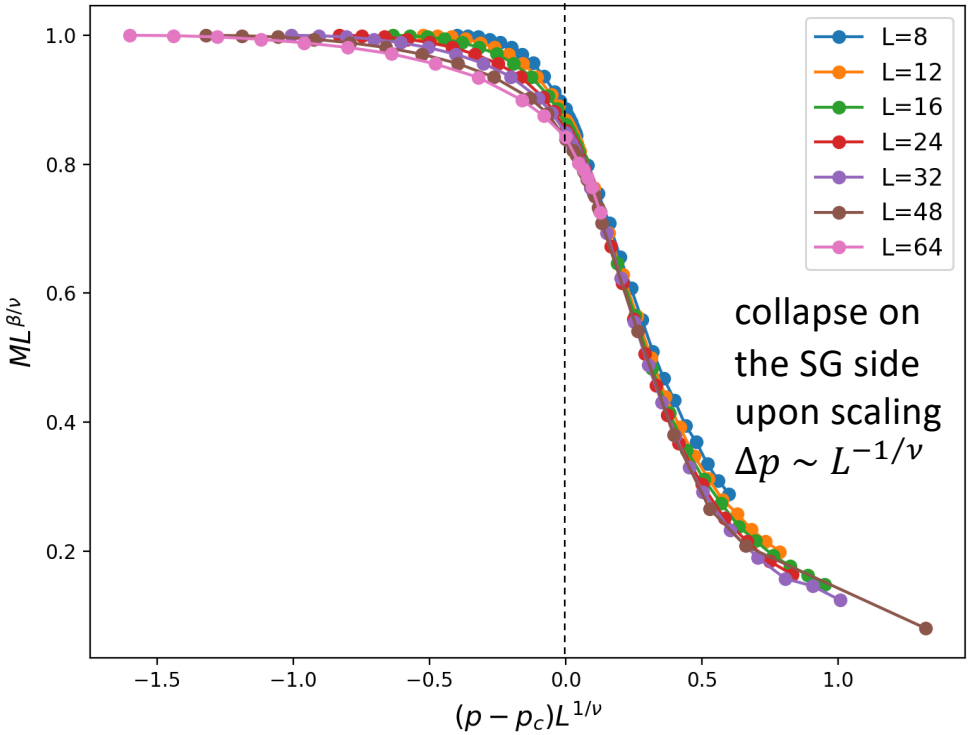
Spin glass

Inverse cascade of compact domains of opposite magnetization

Zero-temperature magnetization for different sizes

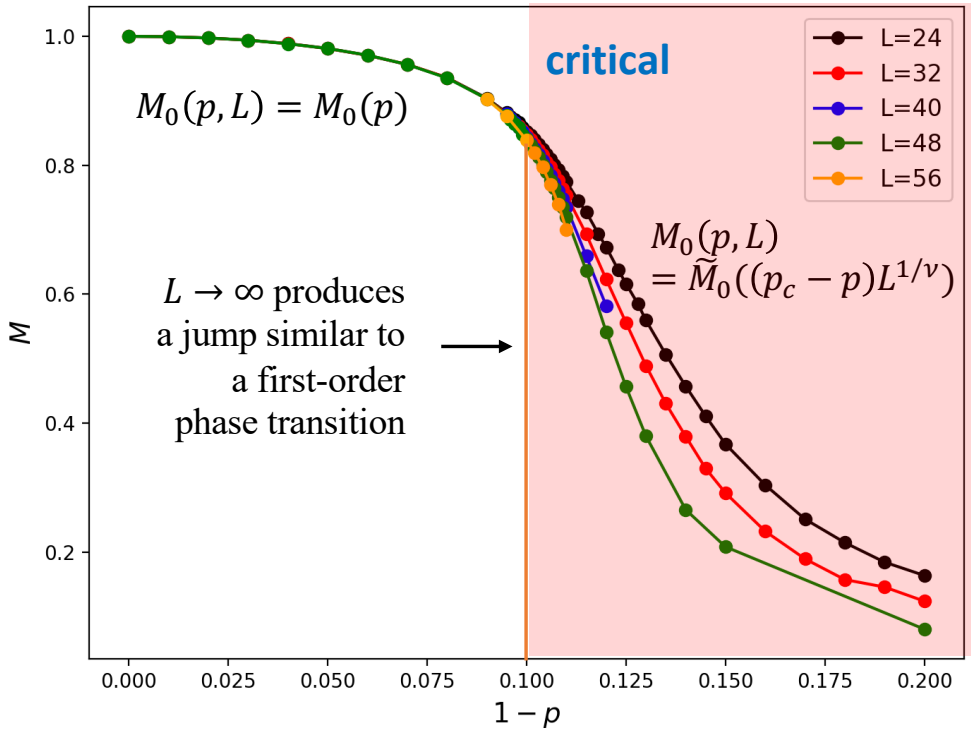
$$1 - p_c = 0.9 \quad \beta = 0 \quad \nu = 3/2$$

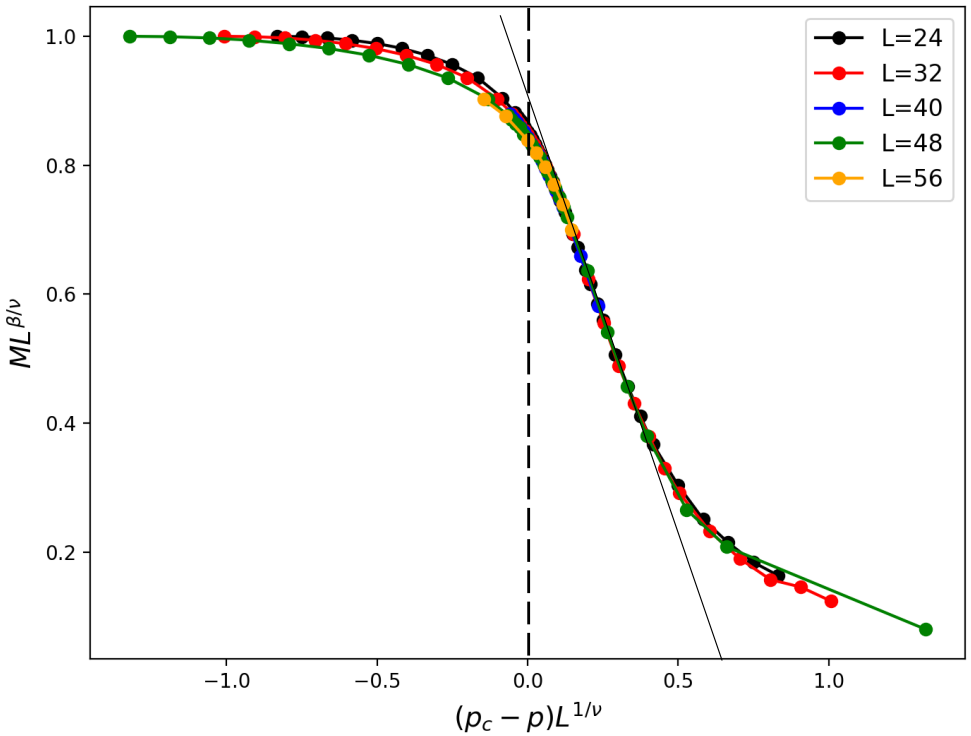




Improved data collapse when small sizes are excluded

no scaling $1 - p_c = 0.9$ $\beta = 0$ $\nu = 3/2$





$$M_0(p, L) = \begin{cases} M_0(p) & p > p_c \\ \tilde{M}_0\left((p_c - p)L^{1/\nu}\right) & p < p_c \end{cases}$$

$p > p_c$, only single spin or small cluster excitations. Weak finite-size effect.

$p < p_c$, cascade of domain-inside-domains up to size $\xi \sim (p_c - p)^{2/3}$

Mapping of the domain-wall statistics to the DPRM

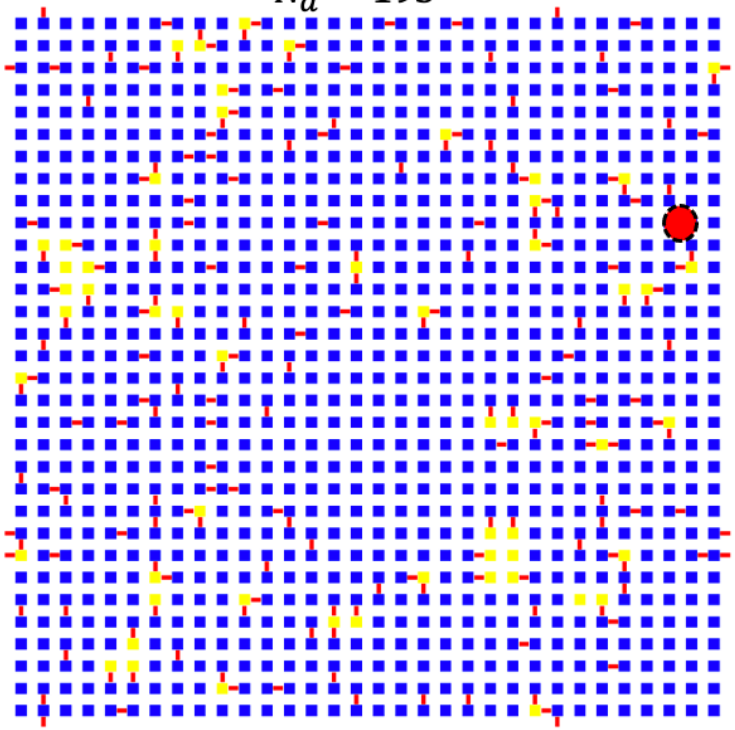
Energetics of cluster flip at domain boundary:

satisfied bonds $(E_b = -1) \leftrightarrow$ unsatisfied bonds $(E_b = 1)$

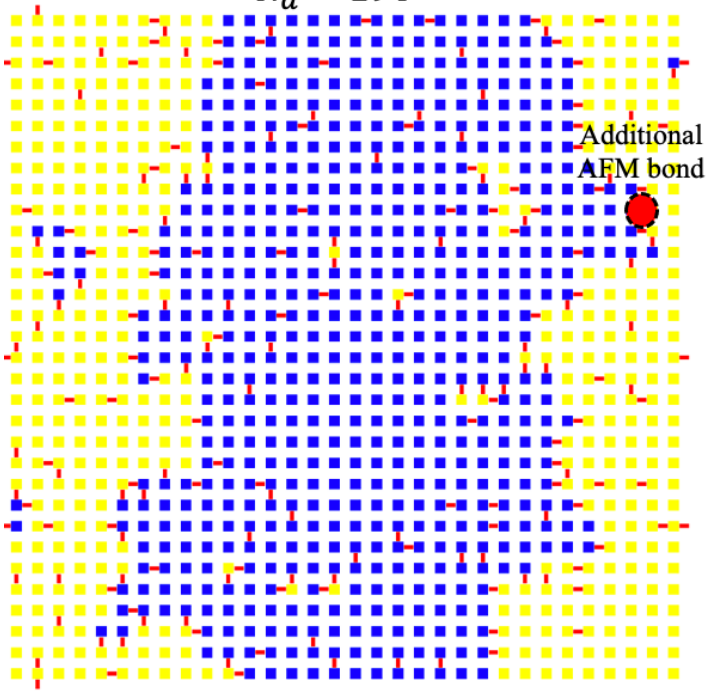
At p_{c0},

$E_{\mathrm{DW}} = 2(N_+ - N_-) = 0$

$N_a = 193$



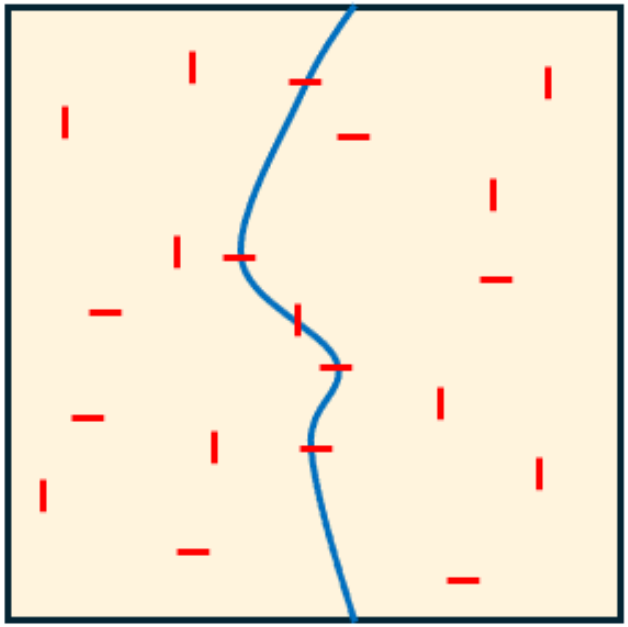
$N_a = 194$



Observations:

- **Compact domains**
- Satisfied bonds at domain boundary are **AFM bonds**
- Minimize $E_{\mathrm{DW}} \Longrightarrow$ **DW is essentially a directed path that**
  - passes through as many AFM bonds
  - passes through as few FM bonds

Why start from the largest length scale (quasi-first order transition)?



Single directed polymer in a 2D disordered medium

$$\Gamma(p,L) \approx \gamma(p)L + bL^{\omega}$$

1+1 DPRM (KPZ): $\omega = {}^{1}/_{3}$

$$\gamma(p) = c(p - p_c)$$

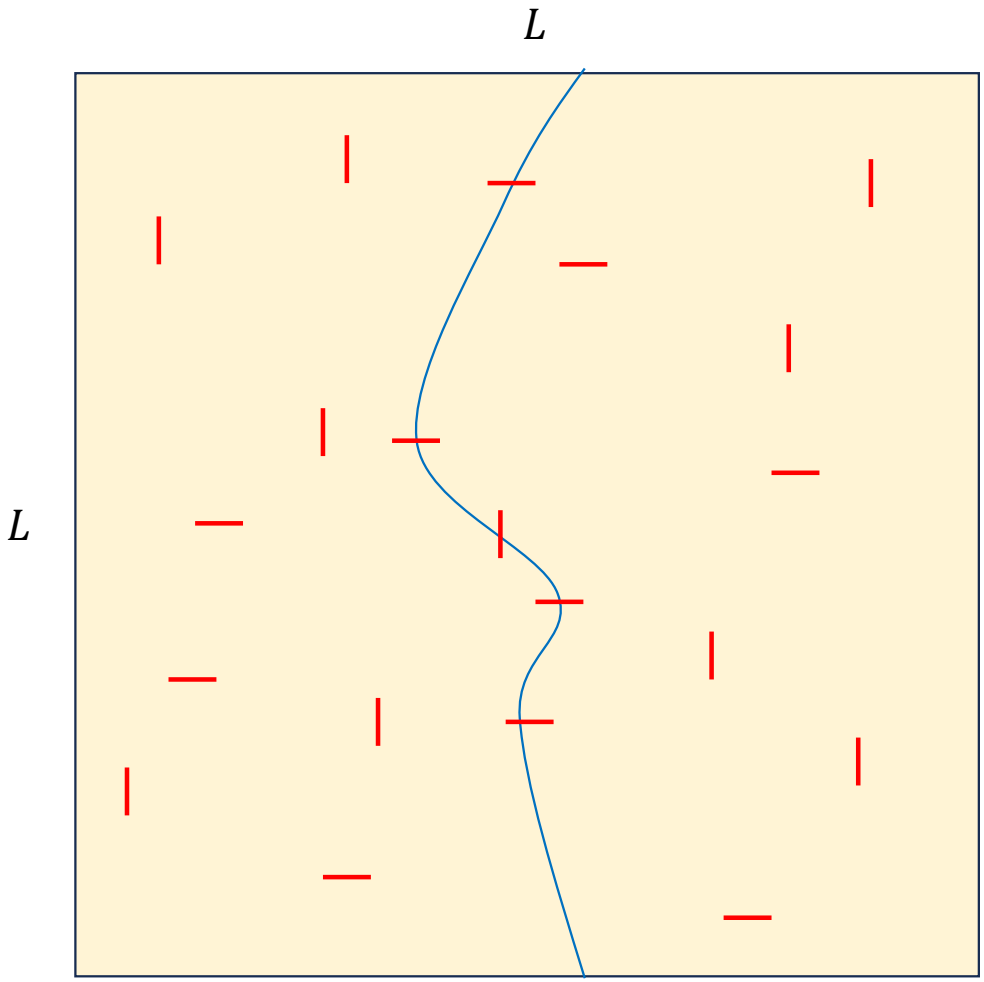
$$b, c > 0$$

$$\Gamma(p,L) = 0 \implies p_c(L) = p_c - \frac{b}{c}L^{-2/3}$$

Largest at $L = \infty$!

- FM phase first destabilizes on the largest scale
- **Inverse cascade** of large-to-small scales
- Width of the critical region scales as $\Delta p \sim L^{-1/\nu}$ with $\nu = \frac{1}{1-\omega} = \frac{3}{2}$
- Generalizes to directed surfaces in higher dimensions

Direct computation of the DW free energy/line tension



pbc: $s_{i+L\hat{x}} = s_i$, $s_{i+L\hat{y}} = s_i$

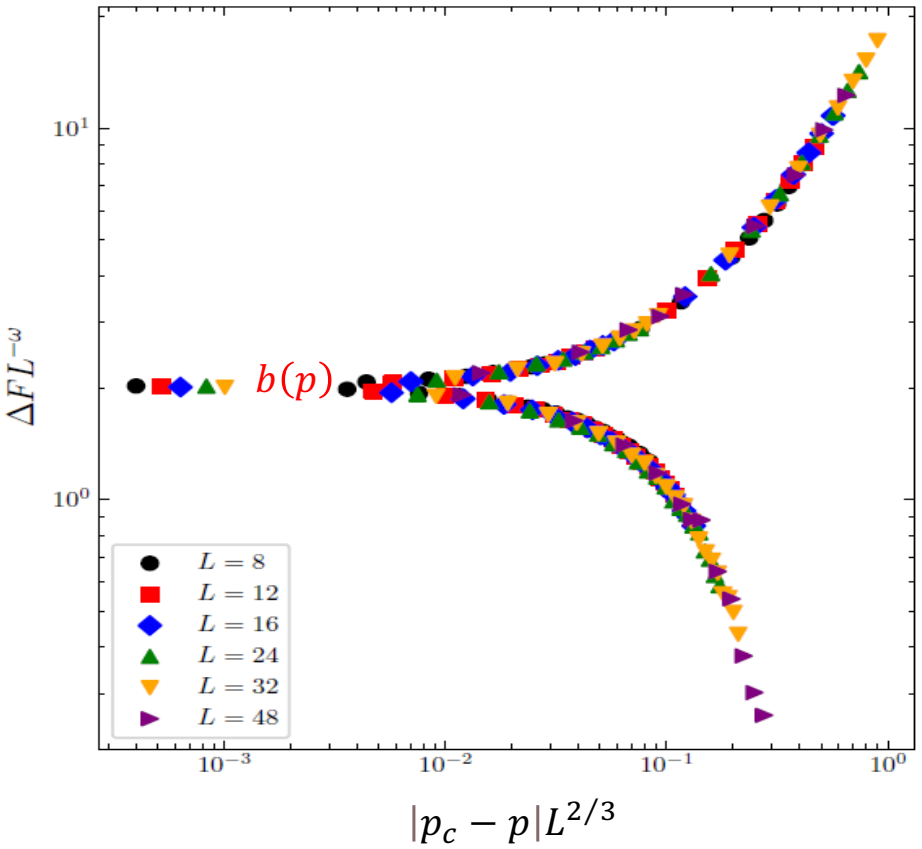
apbc: $s_{i+L\hat{x}} = -s_i$, $s_{i+L\hat{y}} = s_i$

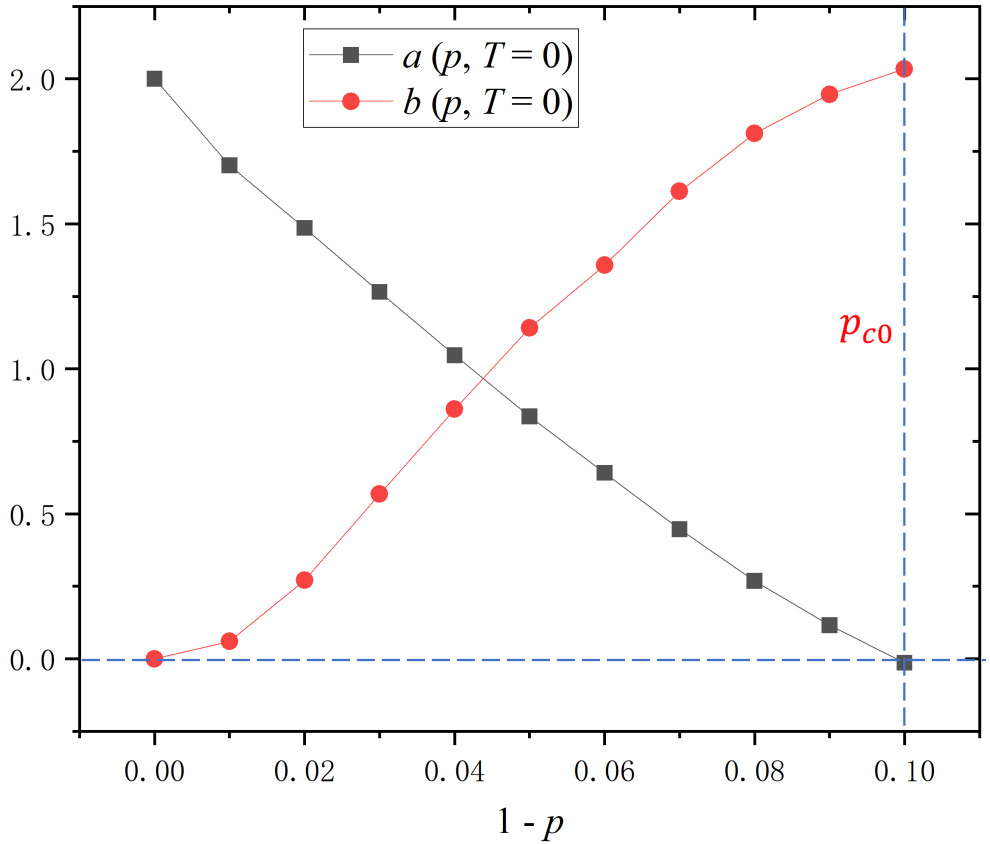
DW free energy/line tension:

$$\Gamma \equiv F_- - F_+ = \Gamma(p, T, L)$$

Finite-size scaling of line tension at $T = 0$

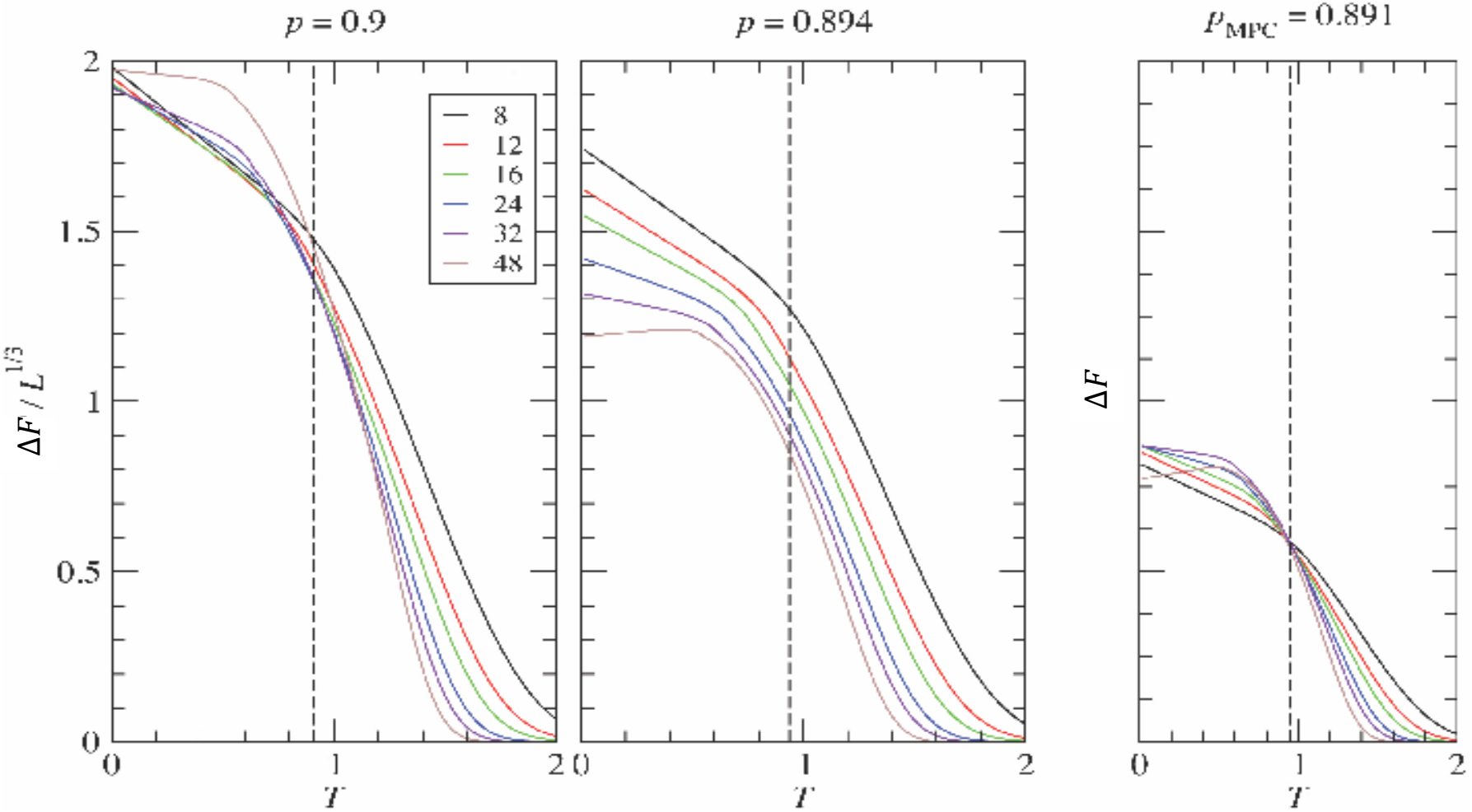
$$\Gamma(p, L) \approx a(p)L + b(p)L^{1/3}$$





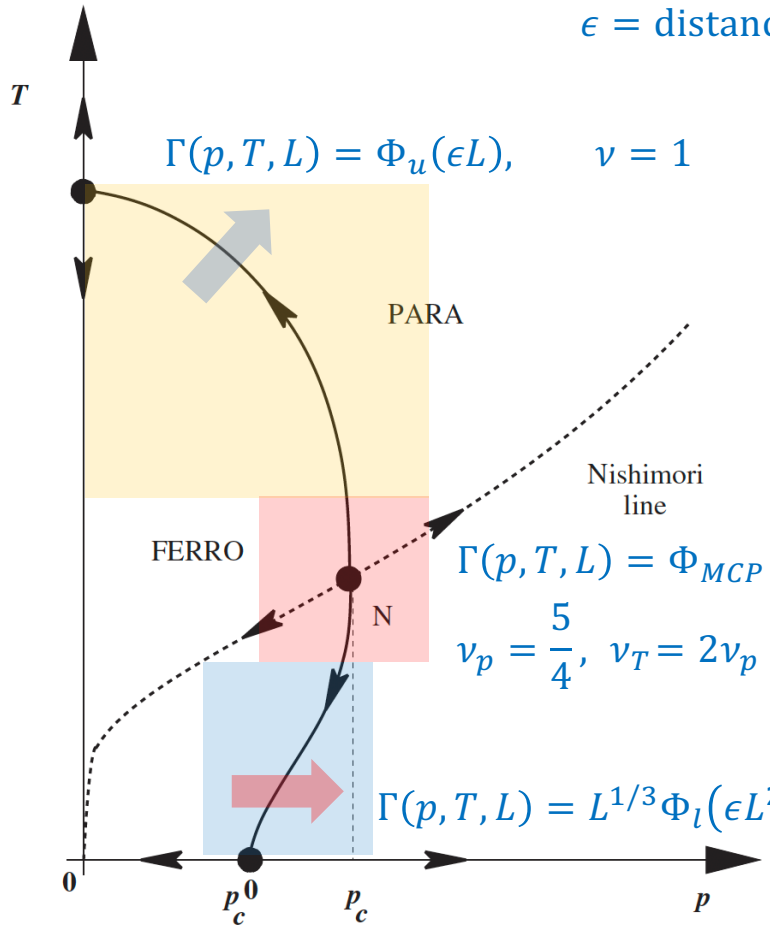
Finite-size scaling of line tension at various p, T

Multi-critical point



Finite-size scaling in the neighborhood of the transition line / RG flow

$\epsilon =$ distance to the phase boundary



$\Gamma(p,T,L) = \Phi_u(\epsilon L), \qquad \nu = 1$

These scaling forms and exponents fit out numerical data quite nicely.

$\Gamma(p,T,L) = \Phi_{MCP}\left((p-p_{MCP})L^{4/5}, (T-T_{MCP})L^{2/5}\right),$

$\nu_p = \frac{5}{4}, \ \nu_T = 2\nu_p = \frac{5}{2}$ (parabolic phase boundary)

$\Gamma(p,T,L) = L^{1/3}\Phi_l\left(\epsilon L^{2/3}\right), \qquad \nu = \frac{3}{2}$

Does the same mixed order transition happen in 3D EA model?

Preliminary results of the 3D RBIM

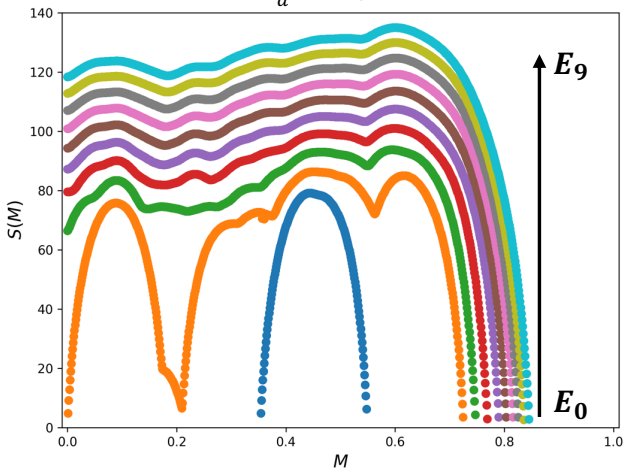
$N = 12\times12\times12$

$S(E,M) = \ln g(E,M)$

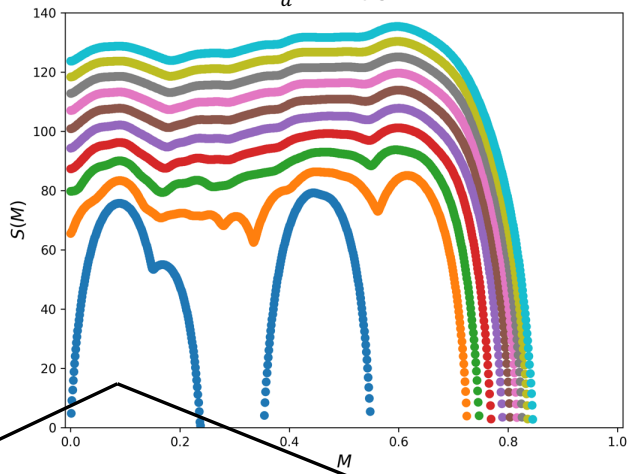
N_a = # of antiferromagnetic bonds

One specific series of disorder configurations

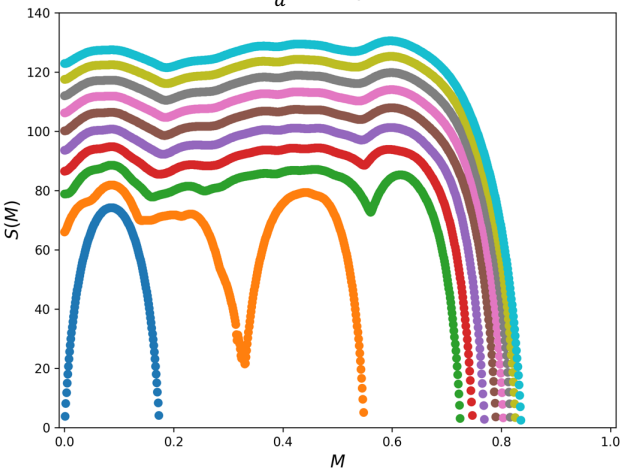
$N_a = 1172$



$N_a = 1173$

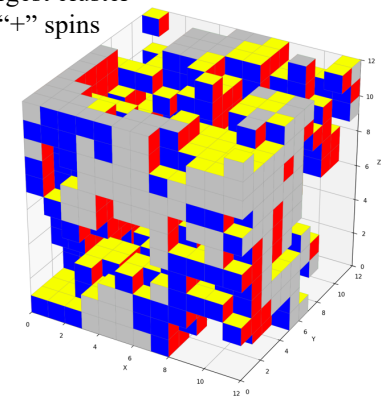


$N_a = 1174$

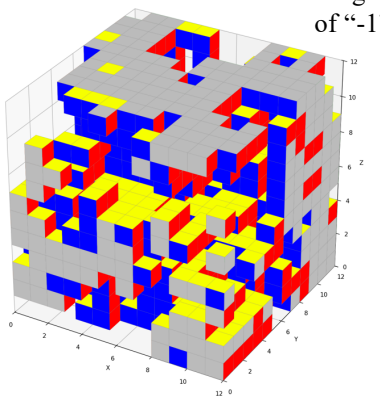


A typical ground state configuration

Largest cluster of "+" spins



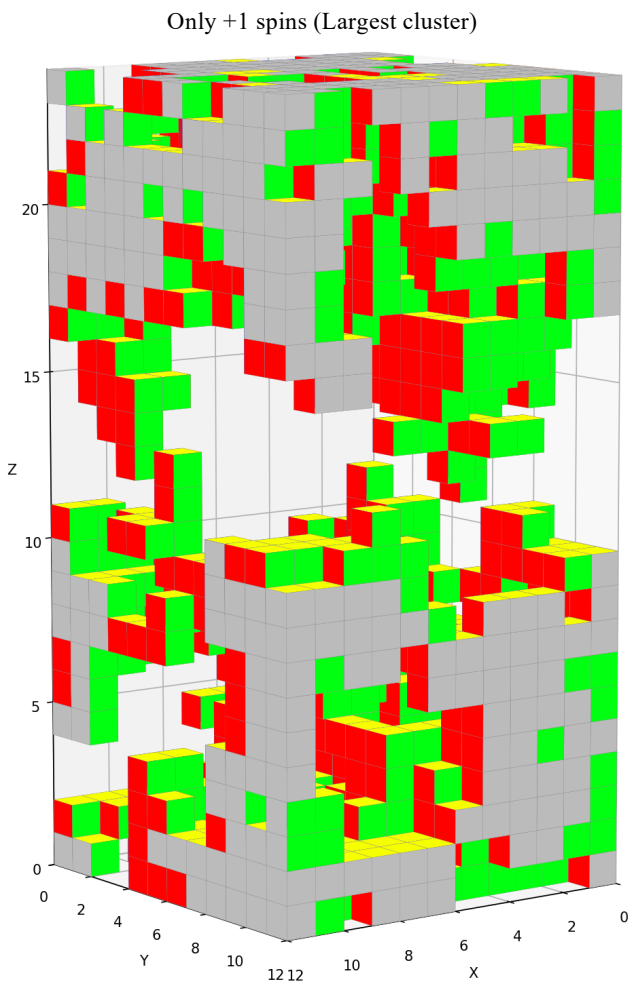
Largest cluster of "-1" spins



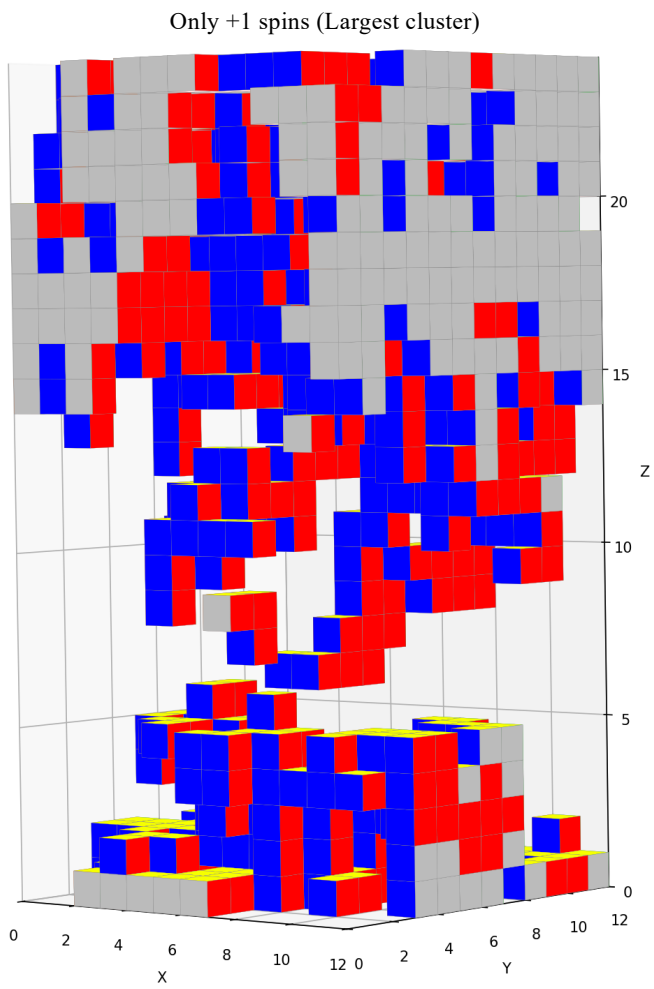
Preliminary results of the 3D RBIM

$$N = 12\times12\times24$$
$$p = 0.77$$

Sample 1



Sample 2



Energetics of domain walls in *d* = *D*+1 dimensions

$$E = A(p - p_c)L^D + bL^\omega$$

$$\nu^{-1} = D - \omega$$

DS Fisher, PRL **56**, 1964 (1996)
P Chauve, P Le Doussal, JK Wiese, PRL **86**, 1785 (2001)

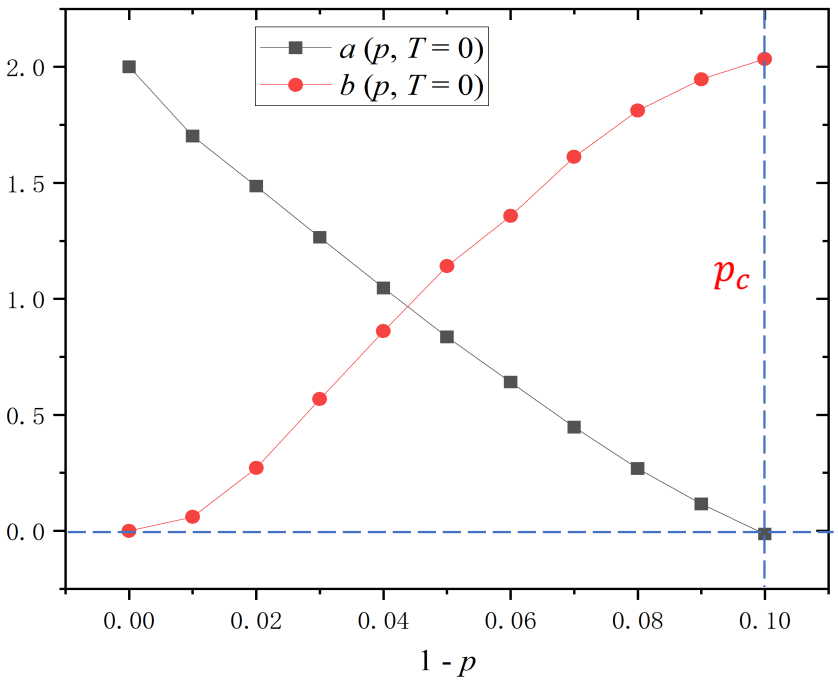
$$\omega = D - 2 + 2\zeta,$$

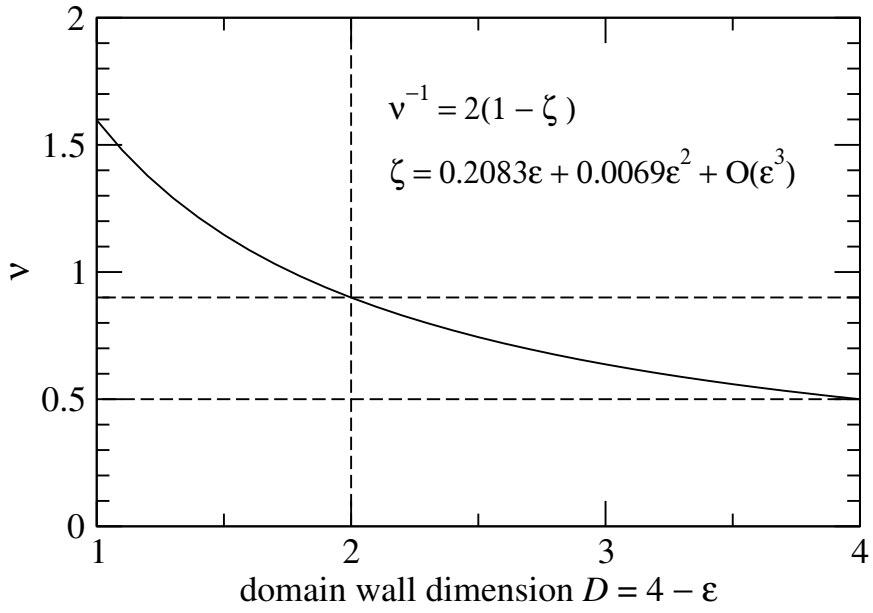
$$\zeta = 0.2083\epsilon + 0.0069\epsilon^2, \epsilon = 4 - D$$

$D = 1$

$\omega = \frac{1}{3}$

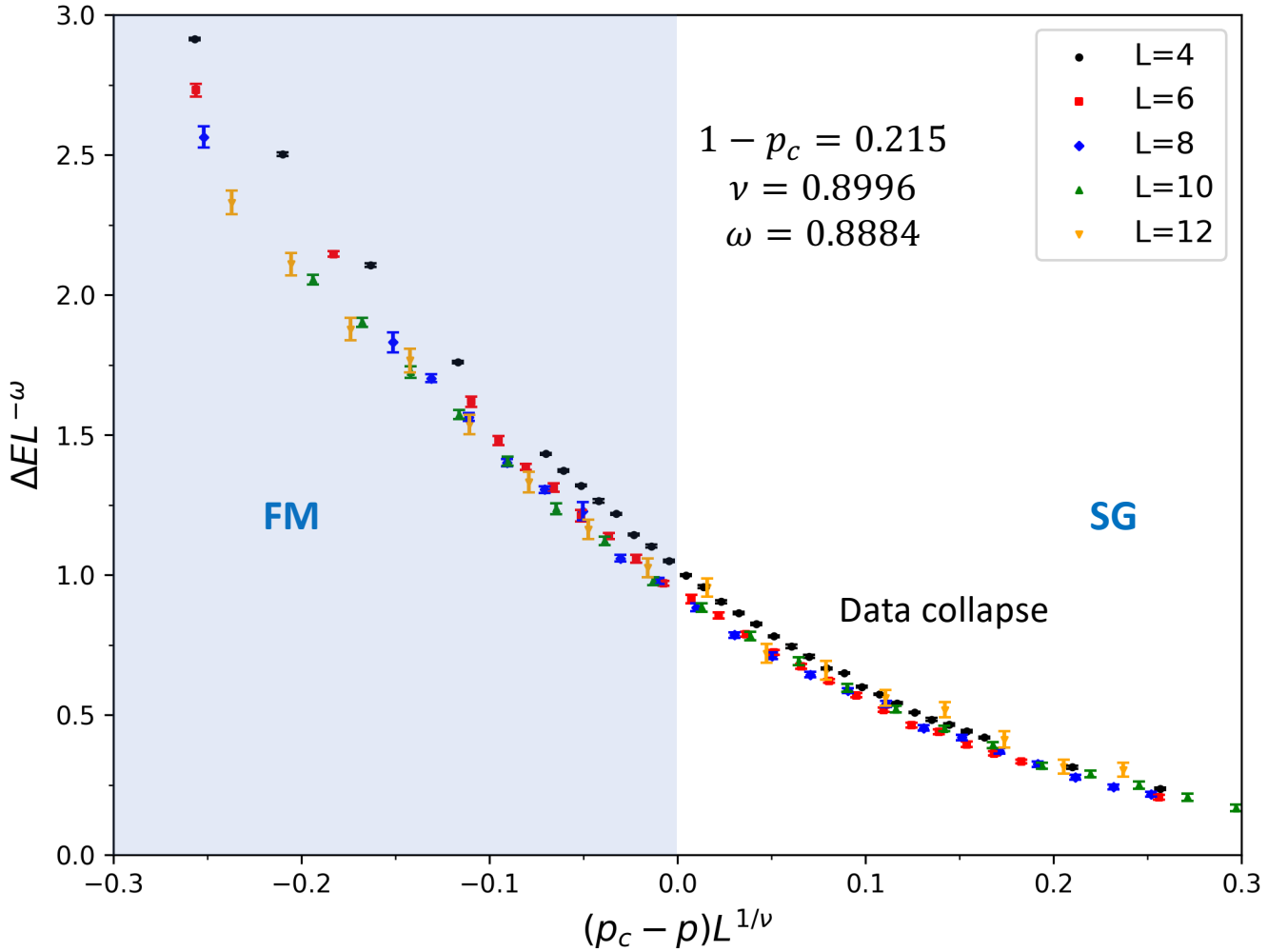
$\nu = 1.5$





Preliminary results of the 3D RBIM

Finite size scaling of ground state domain wall energy ΔE in 3D RBIM ($N = L^3$)



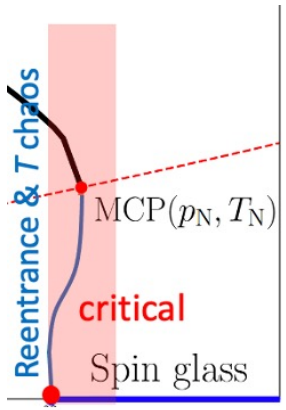
Summary:

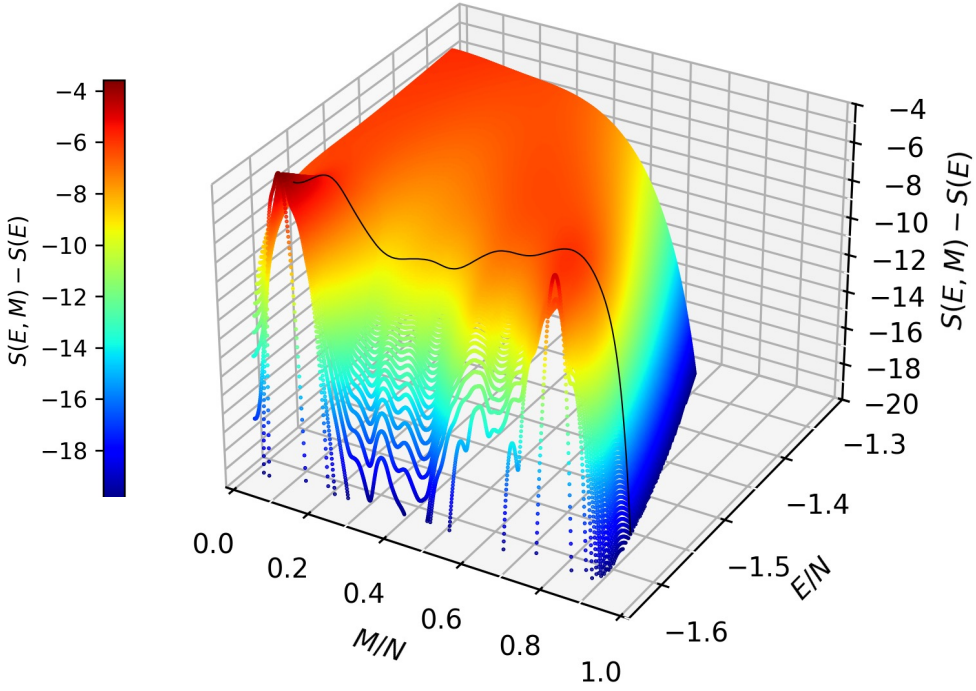
Zero temperature FM -> SG transition governed by DPRM physics

- ground state undergoes sudden change from FM to system-size coexisting domains
- inverse-cascade of domain-inside-domains, leading eventually to the spin-glass phase
- mixed-order transition with critical properties governed by DPRM exponents

Extensive degeneracy of the ground state

- glassiness: domain-wall localization freezes uncorrelated spin flips but the network may adopt different structures with extensive entropy
- re-entrance transition and relation to temperature chaos? (Nishimori et al., 2507.00276)






Thank you very much
for your attention!
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