


# Fluctuation-Response Inequalities for Nonequilibrium Steady States

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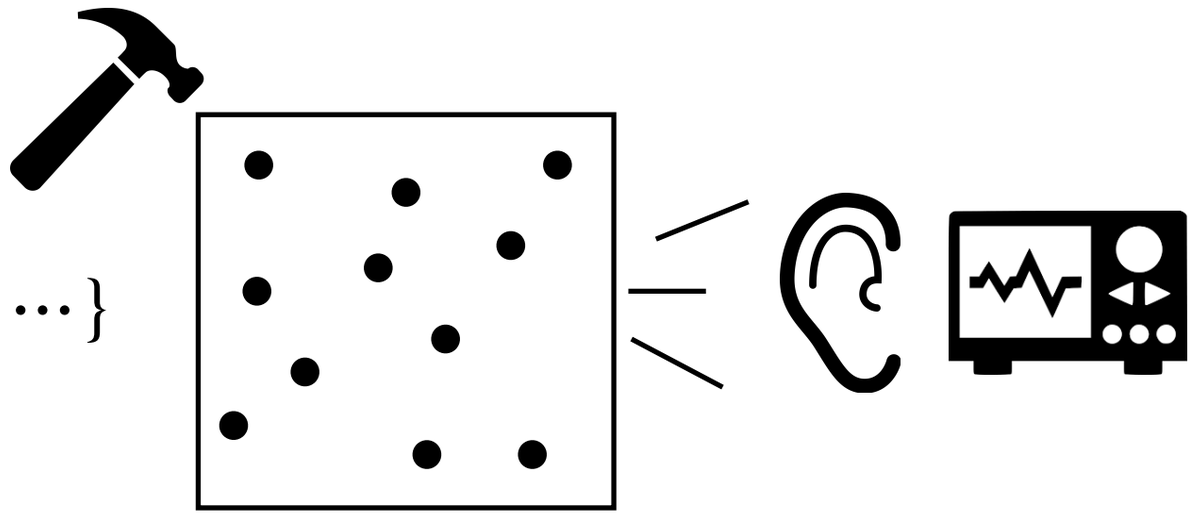
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Reference: Physical Review Letters **135**, 097101 (2025)

# Fluctuation-dissipation theorem

Near equilibrium,

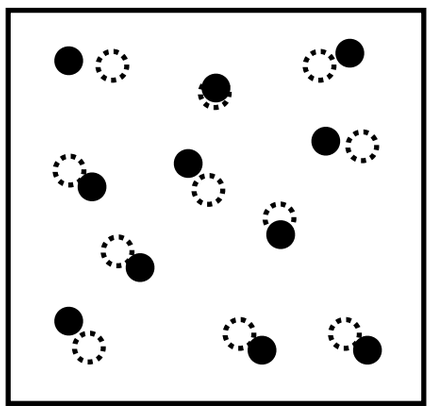


$\boldsymbol{F} \subset \{\Delta T,\ \Delta\mu, \cdots\}$

$\boldsymbol{J} \subset \{\dot{E},\ \dot{N}, \cdots\}$

$\boldsymbol{J} = \mathsf{L} \cdot \boldsymbol{F}$ (thermodynamic currents) $\propto$ (thermodynamic forces)

(dissipative) **response**



$\mathsf{C} \subset \left\{ \langle\langle \dot{N} \rangle\rangle_{\mathrm{eq}},\ \langle\langle \dot{E} \rangle\rangle_{\mathrm{eq}},\ \langle\langle \dot{N}, \dot{E} \rangle\rangle_{\mathrm{eq}}, \cdots \right\}$ **fluctuation**

$\mathsf{C} = 2\mathsf{L}$ : **Fluctuation-Dissipation Theorem (FDT)**

# FDT for nonequilibrium states

PRL **95,** 130602 (2005) PHYSICAL REVIEW LETTERS week ending 23 SEPTEMBER 2005

**Equality Connecting Energy Dissipation with a Violation of the Fluctuation-Response Relation**

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(Received 21 February 2005; published 21 September 2005)

(violation of FDT) $\propto$ (heat dissipation)

EUROPHYSICS LETTERS 1 May 2006

*Europhys. Lett.*, **74** (3), pp. 391–396 (2006)
DOI: 10.1209/epl/i2005-10549-4

**Restoring a fluctuation-dissipation theorem in a nonequilibrium steady state**

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*II. Institut für Theoretische Physik, Universität Stuttgart - 70550 Stuttgart, Germany*

received 25 November 2005; accepted in final form 1 March 2006
published online 24 March 2006

(FDT is still valid in a certain reference frame)

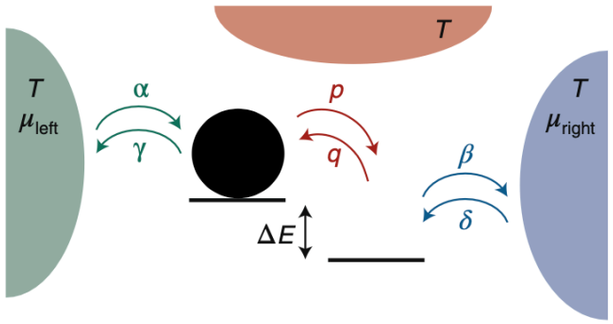
PRL **103,** 010602 (2009) PHYSICAL REVIEW LETTERS week ending 3 JULY 2009

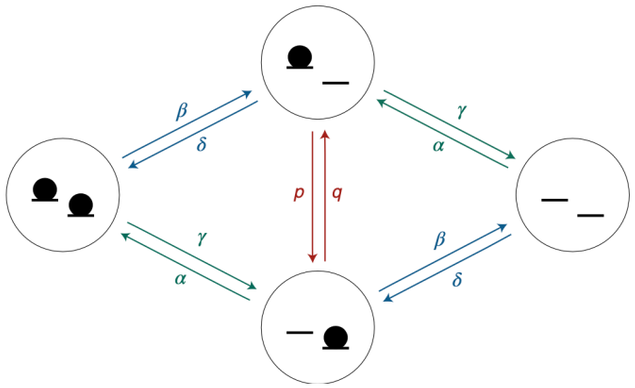
**Fluctuations and Response of Nonequilibrium States**

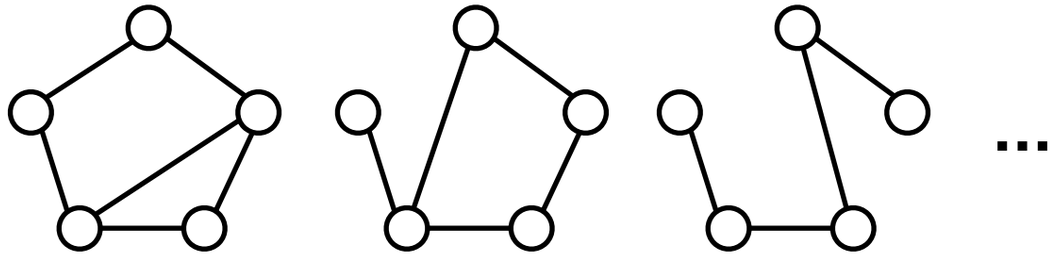
Marco Baiesi, Christian Maes, and Bram Wynants*
*Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven, 3001, Belgium*
(Received 23 February 2009; revised manuscript received 3 June 2009; published 2 July 2009)

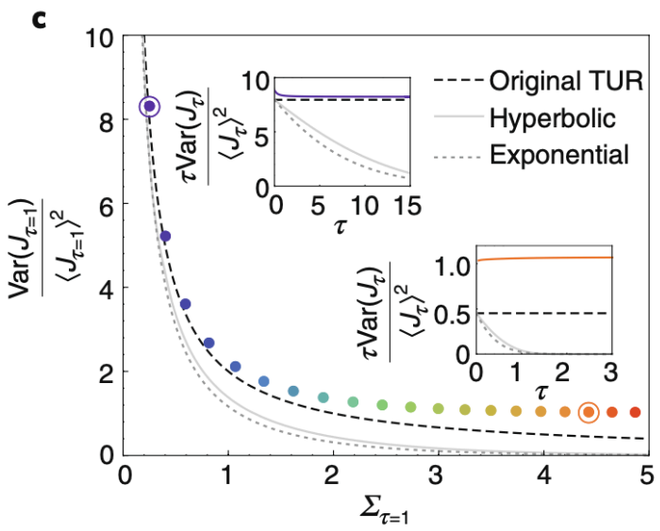
(FDT can be modified in a more general form)

# TUR and R-TUR



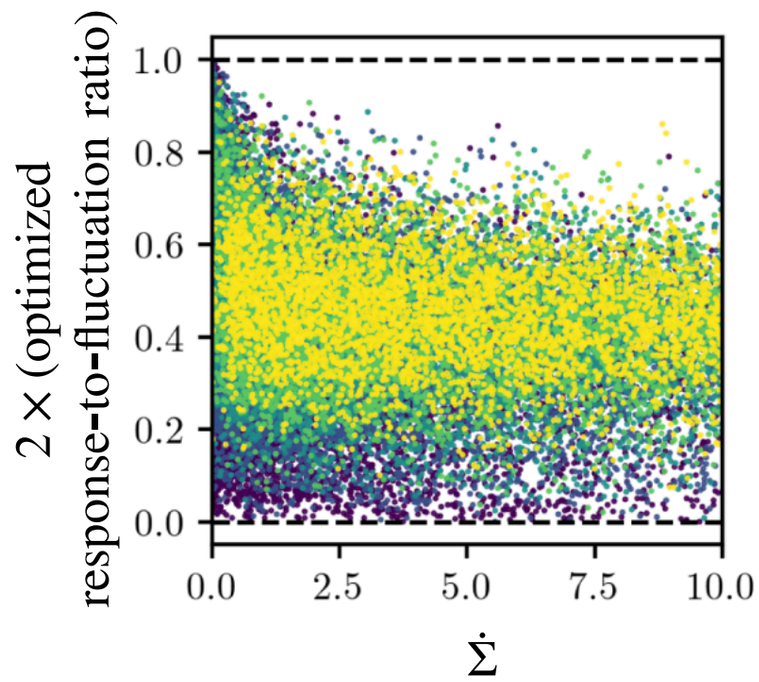






$$\frac{\langle\langle J\rangle\rangle}{\langle J\rangle^2} \geq \frac{2k_B}{\dot{\Sigma}} \overset{(=1)}{}$$

J. M. Horowitz and T. R. Gingrich
[Nat. Phys. **16**, 15 (2020)]

**Thermodynamic Uncertainty Relation (TUR)**

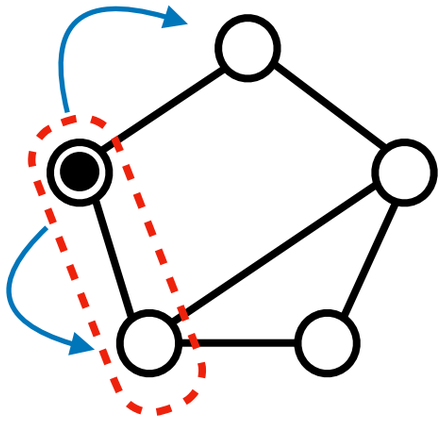



$$\underset{\text{fluctuation}}{\frac{(\partial_{B_{ij}}\langle J\rangle)^2}{\langle\langle J\rangle\rangle}} \leq \underset{\text{dissipation}}{\frac{\dot{\Sigma}}{2}}$$

**Response TUR (R-TUR)**

K. Ptaszyński *et al*. [PRL **133**, 227101 (2024)]

# Kinetic and entropic perturbations



$$\frac{dp_i}{dt} = \sum_{j(\neq i)} \left( W_{ij} p_j - W_{ji} p_i \right)$$

unique steady state

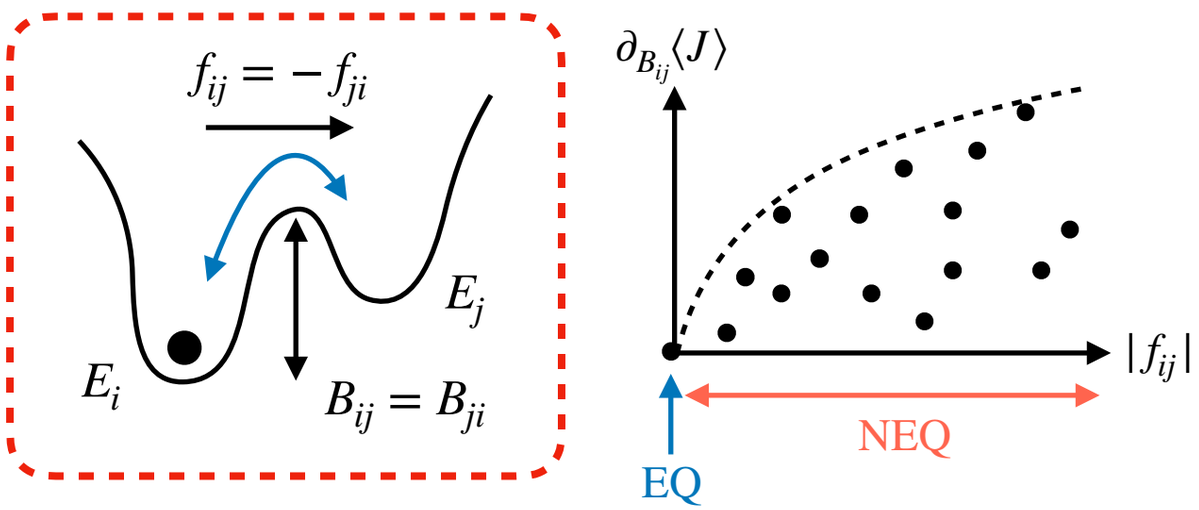
$$\lim_{t\to\infty} p_i = \pi_i$$

$$W_{ij} = e^{B_{ij}+F_{ij}/2}, \quad W_{ji} = e^{B_{ij}-F_{ij}/2}$$

($B_{ij}$ : symmetric, $F_{ij}$ : antisymmetric)

## [1] kinetic perturbation

$$\frac{\partial}{\partial B_{ij}} \equiv W_{ij}\frac{\partial}{\partial W_{ij}} + W_{ji}\frac{\partial}{\partial W_{ji}}$$



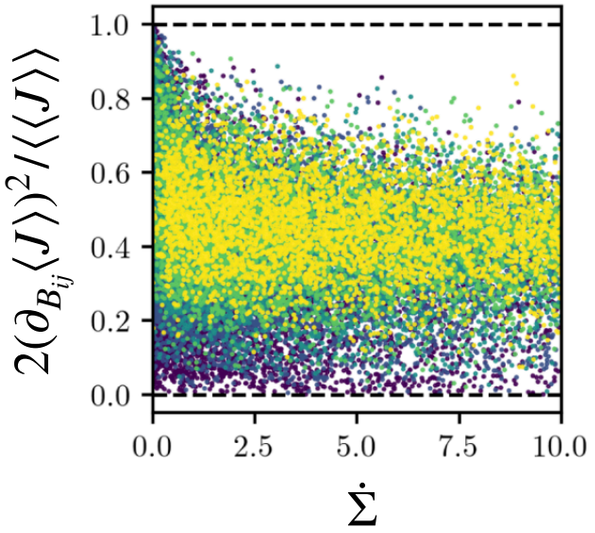

## [2] entropic perturbation

$$\frac{\partial}{\partial F_{ij}} \equiv \frac{1}{2}\left( W_{ij}\frac{\partial}{\partial W_{ij}} - W_{ji}\frac{\partial}{\partial W_{ji}} \right)$$

$$\frac{\partial\langle J\rangle}{\partial F_{ij}} \neq 0 \text{ in EQ}$$

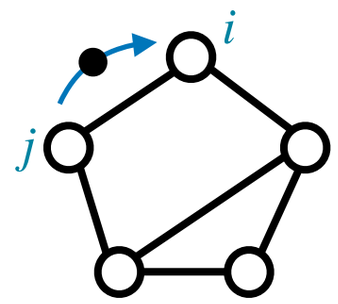
$$\text{cf.)}\ \frac{\pi_i^{\mathrm{eq}}}{\pi_j^{\mathrm{eq}}} = \frac{W_{ij}}{W_{ji}} = e^{-\beta(E_j - E_i)}$$

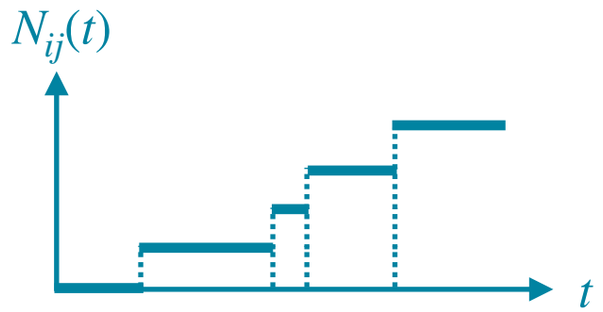
# A little more detail




$$\frac{(\partial_{B_{ij}}\langle J\rangle)^2}{\langle\langle J\rangle\rangle} \leq \frac{\dot{\Sigma}}{2}$$

$$J(\tau) \equiv \frac{1}{\tau}\int_0^\tau dt \sum_{i\neq j} \Lambda_{ij}\dot{N}_{ij}(t) \qquad (=-\Lambda_{ji})$$

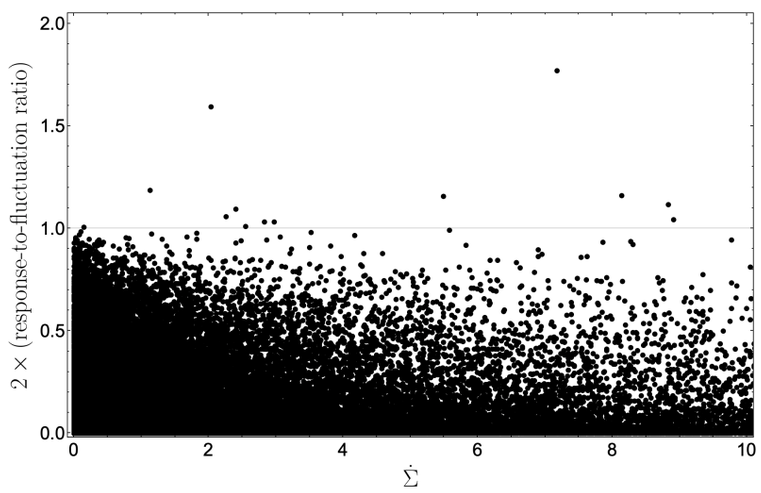



$$\langle J\rangle \equiv \lim_{\tau\to\infty}\langle J(\tau)\rangle = \sum_{i<j}\Lambda_{ij}\underbrace{(W_{ij}\pi_j - W_{ji}\pi_i)}_{\mathcal{J}_{ij}} = \sum_{i<j}\Lambda_{ij}\mathcal{J}_{ij}$$

$$\langle\langle J\rangle\rangle \equiv \lim_{\tau\to\infty}\tau\mathrm{Var}\{J(\tau)\}$$

## What about finite observation times?




$$\frac{(\partial_{B_{ij}}\langle J\rangle)^2}{\tau\mathrm{Var}\{J\}} \not\leq \frac{\dot{\Sigma}}{2}$$

# From static to dynamic response

$$\frac{(\partial_{B_{ij}}\langle J\rangle)^2}{\langle\langle J\rangle\rangle} \leq \frac{\dot{\Sigma}}{2}$$

R-TUR

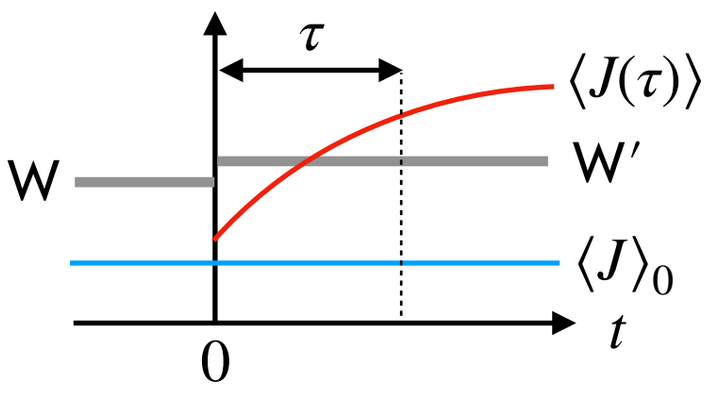




$$\frac{(\partial_\theta\langle X\rangle)^2}{\mathrm{Var}\{X\}} \leq I(\theta)$$

Cramér-Rao bound

$$W_{ij} \mapsto W'_{ij} = W_{ij} + (\partial_\theta W_{ij})\Delta\theta \qquad \text{where } \Delta\theta \in \{\Delta B_{ij}, \Delta F_{ij}\}$$




$$\langle J\rangle_0 \mapsto \langle J(\tau)\rangle = \langle J\rangle_0 + \Delta J(\tau)$$

$$R_\theta(\tau) = \lim_{\Delta\theta\to 0} \frac{\Delta J(\tau)}{\Delta\theta}$$

cf.) $\lim_{\tau\to\infty} R_\theta(\tau) = \lim_{\Delta\theta\to 0} \frac{\langle J(\infty)\rangle - \langle J\rangle_0}{\Delta\theta} = \frac{\partial\langle J\rangle}{\partial\theta}$

# Fluctuation-response inequalities



$$\sum_{\alpha,\beta} R_{\theta_\alpha}(\tau)[\mathrm{I}^{-1}(\tau)]_{\theta_\alpha\theta_\beta}R_{\theta_\beta}(\tau) \le \mathrm{Var}\{\Theta(\tau)\}$$

general observable

where $\mathrm{I}_{\theta_\alpha\theta_\beta}(\tau) = -\left\langle \partial_{\theta_\alpha}\partial_{\theta_\beta} \ln \mathscr{P}[\{s(t)\}_{t=0}^{\tau}] \right\rangle_0$

path probability of a trajectory $\{s(t)\}_{t=0}^{\tau}$

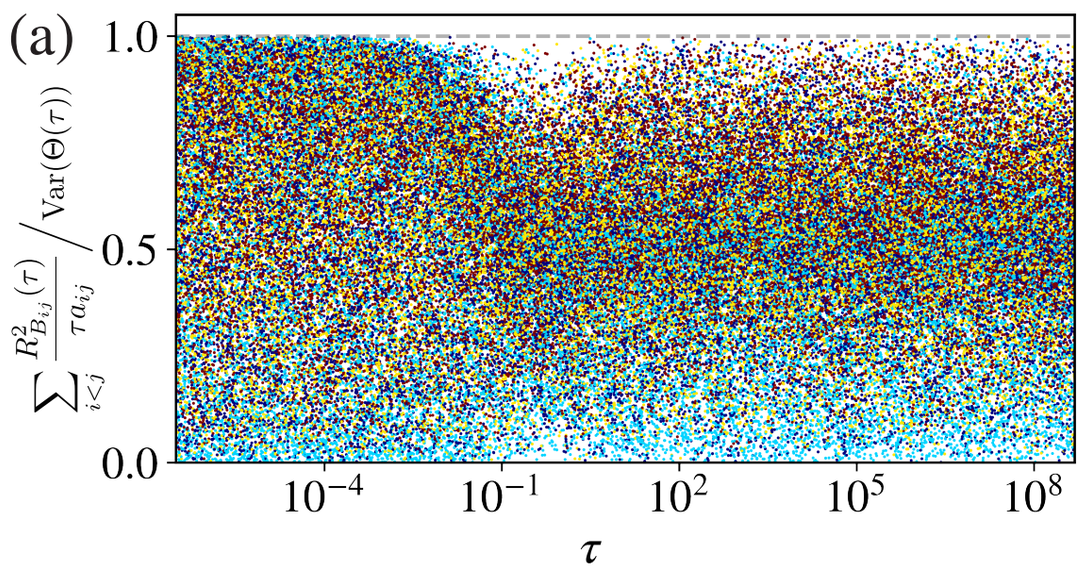
$\theta_\alpha \mapsto B_{ij}$ : $\mathrm{I}_{B_{ij}B_{i'j'}}(\tau) = \tau\delta_{ii'}\delta_{jj'}a_{ij}$

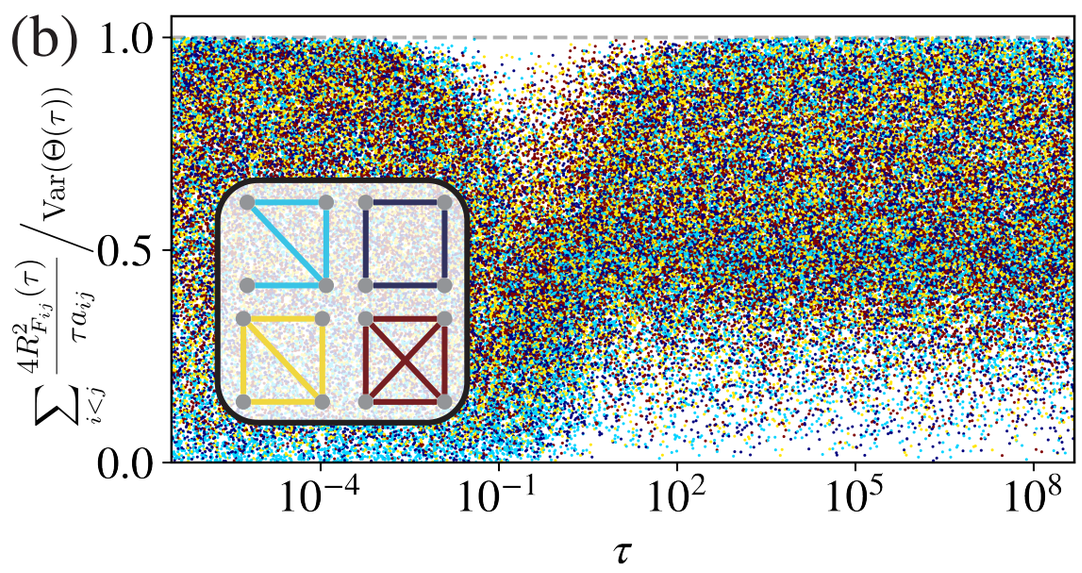
$(= W_{ij}\pi_j + W_{ji}\pi_i)$

$$\sum_{i<j} \frac{R_{B_{ij}}^2(\tau)}{\tau a_{ij}} \le \mathrm{Var}\{\Theta(\tau)\}$$

$\theta_\alpha \mapsto F_{ij}$ : $\mathrm{I}_{F_{ij}F_{i'j'}}(\tau) = \frac{1}{4}\tau\delta_{ii'}\delta_{jj'}a_{ij}$

$$\sum_{i<j} \frac{4R_{F_{ij}}^2(\tau)}{\tau a_{ij}} \le \mathrm{Var}\{\Theta(\tau)\}$$



# Response uncertainty relations

$$\tau \mathrm{Var}\{\Theta(\tau)\} \geq \sum_{i<j} \frac{R_{B_{ij}}^2(\tau)}{a_{ij}} \geq \frac{R_{B_{ij}}^2(\tau)}{a_{ij}} \geq \frac{R_{B_{ij}}^2(\tau)}{\sum_{i<j} a_{ij}}$$




$$\langle\langle J \rangle\rangle \geq \frac{2(\partial_{B_{ij}} \langle J \rangle)^2}{\dot{\Sigma}}$$

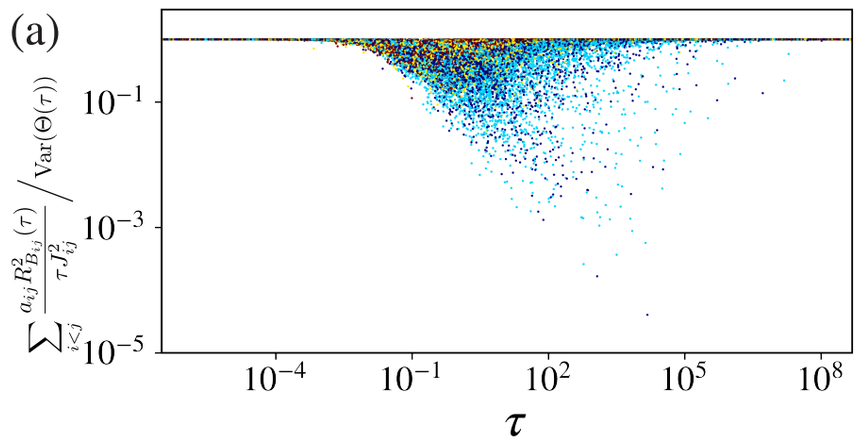
**R**esponse **K**inetic **U**ncertainty **R**elation (R-KUR)

R-TUR
for infinite observation times



$$\sum_{i<j} \frac{4R_{F_{ij}}^2(\tau)}{\tau a_{ij}} \leq \mathrm{Var}\{J(\tau)\}$$

$$\frac{R_{B_{ij}}(\tau)}{R_{F_{ij}}(\tau)} = \frac{2\mathcal{J}_{ij}}{a_{ij}} \quad \text{for currents}$$

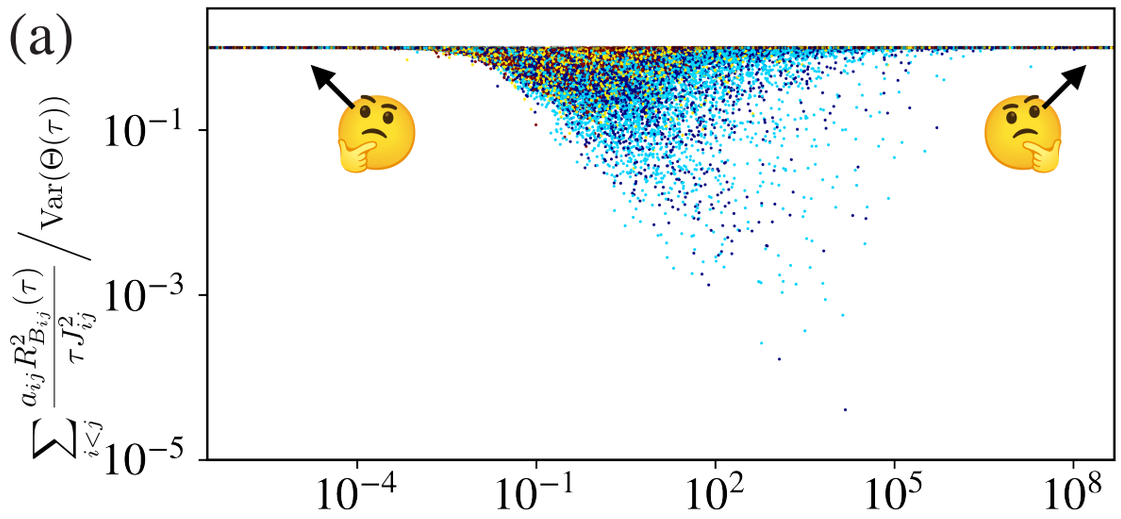



$$\sum_{i<j} \frac{a_{ij} R_{B_{ij}}^2(\tau)}{\tau \mathcal{J}_{ij}^2} \leq \mathrm{Var}\{J(\tau)\}$$

$$\tau \mathrm{Var}\{J(\tau)\} \geq \sum_{i<j} \frac{a_{ij} R_{B_{ij}}^2(\tau)}{\mathcal{J}_{ij}^2} \geq \frac{a_{ij} R_{B_{ij}}^2(\tau)}{\mathcal{J}_{ij}^2} \geq \frac{R_{B_{ij}}^2(\tau)}{\sum_{i<j} \mathcal{J}_{ij}^2 / a_{ij}} \geq \frac{2R_{B_{ij}}^2(\tau)}{\dot{\Sigma}}$$

R-TUR for finite observation times

# Fluctuation-response relations




$$\partial_{F_e}\langle J\rangle = \frac{1}{2}[\underbrace{\mathsf{T}\cdot\mathsf{P}}_{2\mathsf{L}}\cdot\mathbf{\Lambda}]_e \quad \text{where } [\mathsf{T}]_{ee'} = \delta_{ee'}a_e \quad (e:\text{edge})$$

$$\langle\langle J\rangle\rangle = \mathbf{\Lambda}^{\mathrm{T}}\cdot\underbrace{\mathsf{P}^{\mathrm{T}}\cdot\mathsf{T}\cdot\mathsf{P}}_{\mathsf{C}}\cdot\mathbf{\Lambda}$$

in equilibrium,

$$\mathsf{T}\cdot\mathsf{P} = \mathsf{P}^{\mathrm{T}}\cdot\mathsf{T} \;\;\&\;\; \mathsf{P}^2 = \mathsf{P} \longrightarrow \mathsf{C} = 2\mathsf{L}$$

time-reversal symmetry  projective property

$$\sum_{i<j}\frac{a_{ij}(\partial_{B_{ij}}\langle J\rangle)^2}{\mathscr{J}_{ij}^2} = \langle\langle J\rangle\rangle$$

$$\updownarrow \quad \frac{\partial_{B_{ij}}\langle J\rangle}{\partial_{F_{ij}}\langle J\rangle} = \frac{2\mathscr{J}_{ij}}{a_{ij}}$$

$$\sum_{i<j}\frac{4(\partial_{F_{ij}}\langle J\rangle)^2}{a_{ij}} = \langle\langle J\rangle\rangle$$

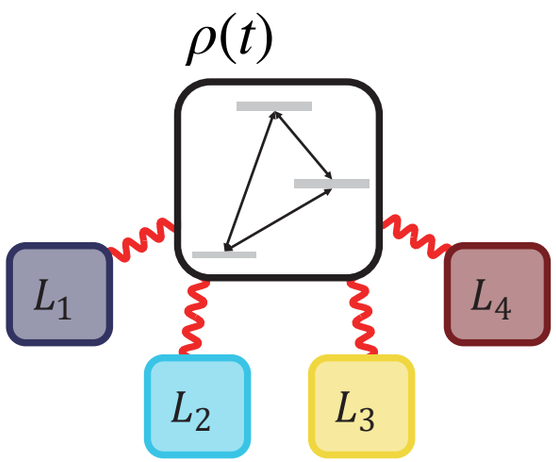
**fluctuation** **response**

$\mathsf{C} = 2\mathsf{L}$

**FDT**

T. Aslyamov *et al*. [PRL **134**, 157101 (2025)]

# Quantum extension



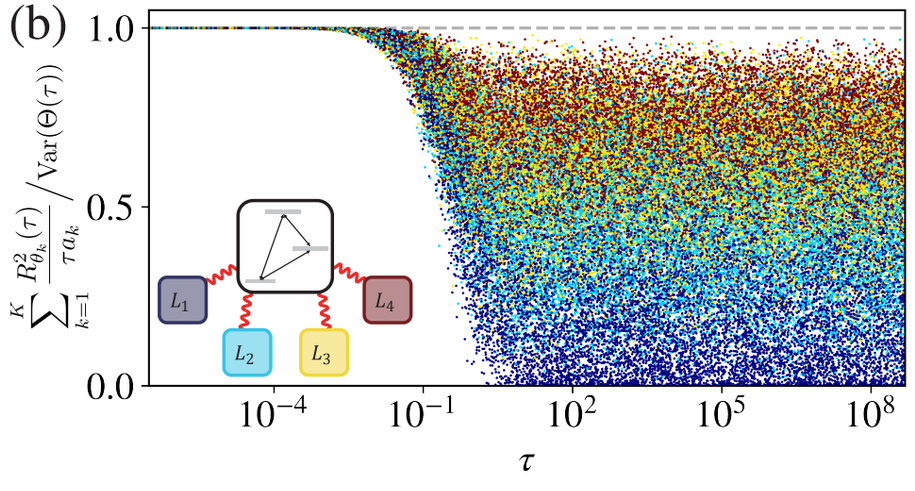

$$\dot{\rho}(t) = -i[H, \rho(t)] + \sum_{k=1}^{K} \mathscr{D}[L_k^{\theta_k}]\rho(t)$$

$$\mathscr{D}[L]\bullet \equiv L \bullet L^\dagger - \{L^\dagger L, \bullet\}/2$$

$$L_k^{\theta_k} = e^{\theta_k/2} L_k$$

$$\sum_{\alpha,\beta} R_{\theta_\alpha}(\tau)[\mathrm{I}_Q^{-1}(\tau)]_{\theta_\alpha\theta_\beta} R_{\theta_\beta}(\tau) \leq \mathrm{Var}\{\Theta(\tau)\} \quad \Rightarrow \quad \sum_k \frac{R_{\theta_k}^2(\tau)}{\tau a_k} \leq \mathrm{Var}\{\Theta(\tau)\} \quad \text{where } a_k = \mathrm{tr}[L_k^{\theta_k}\rho(L_k^{\theta_k})^\dagger]$$

$$\frac{R_{\theta_k}^2(\tau)}{\tau \sum_k a_k} \leq \mathrm{Var}\{\Theta(\tau)\} : \text{R-KUR}$$




**[1] no saturation at infinite observation times**

**[2] quantum R-TUR ?**

# Summary

**Fluctuation-Response Inequalities (FRIs)**

$$\sum_{i<j} \frac{a_{ij} R^2_{B_{ij}}(\tau)}{\tau \mathscr{J}^2_{ij}} \leq \mathrm{Var}\{J(\tau)\} \qquad \sum_{i<j} \frac{4 R^2_{F_{ij}}(\tau)}{\tau a_{ij}} \leq \mathrm{Var}\{\Theta(\tau)\}$$

long-time limit

**Fluctuation-Response Relations (FRRs)**

$$\sum_{i<j} \frac{a_{ij} (\partial_{B_{ij}} \langle J \rangle)^2}{\mathscr{J}^2_{ij}} = \langle\langle J \rangle\rangle \qquad \sum_{i<j} \frac{4 (\partial_{F_{ij}} \langle J \rangle)^2}{a_{ij}} = \langle\langle J \rangle\rangle$$

equilibrium limit

**Fluctuation-Dissipation Theorem (FDT)**

$C = 2L$

**R-TUR & R-KUR**

$$\frac{(\partial_{B_{ij}} \langle J \rangle)^2}{\langle\langle J \rangle\rangle} \leq \frac{\dot{\Sigma}}{2} \qquad \frac{(\partial_{F_{ij}} \langle J \rangle)^2}{\langle\langle J \rangle\rangle} \leq \dot{A}$$

**TUR & KUR**

$$\frac{\langle J \rangle^2}{\langle\langle J \rangle\rangle} \leq \frac{\dot{\Sigma}}{2} \qquad \frac{\langle J \rangle^2}{\langle\langle J \rangle\rangle} \leq \dot{A}$$

ありがとうございます




谢谢

謝謝

**Thank you for your attention!**

Reference: Physical Review Letters **135**, 097101 (2025)