Finite-size scaling of the Kuramoto model at criticality

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Motivation

Kuramoto Model

- N coupled oscillators.
- Each oscillator is characterized by its phase $\boldsymbol{\theta_k(t)}$ $(k \in I)$
  - In this talk, N is assuemd even ($\boldsymbol{N = 2M}$)
  - Index set $\boldsymbol{I := \{\pm 1, \ldots, \pm M\}}$
- Each oscillator has (time-independent) natural frequency $\boldsymbol{\omega_k}$, satisfying

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k\in I} \Theta(\omega_k - a)\Theta(b - \omega_k) = \int_a^b g(\omega)\, d\omega,$$

  for any $-\infty \le a < b \le \infty$.
- Time-evolution equation (**all-to-all** interaction)

$$\frac{d\theta_k}{dt} = \omega_k + \frac{K}{N} \sum_{j\in I} \sin(\theta_j - \theta_k),$$

  where K is the interaction strength $(K > 0)$

Order Parameter and Phase Transitions

- Order Parameter

$$\Delta_N(t) := \frac{1}{N}\sum_{k\in I}\exp\left(i\theta_k(t)\right) =: r_N(t)\exp\left(i\psi_N(t)\right),$$

where $r_N(t) := |\Delta_N(t)|$ and $\exp(i\psi_N(t)) := \Delta_N(t)/r_N(t)$.

Order parameter R and dynamic fluctuation χ

$$R_N := \lim_{t\to\infty}\overline{r_N}(t), \quad R := \lim_{N\to\infty} R_N,$$

$$\chi_N := N\lim_{t\to\infty}\left[\overline{r_N^2}(t) - \overline{r_N}(t)^2\right], \quad \chi := \lim_{N\to\infty}\chi_N.$$

$$\overline{r_N}(t) = \frac{1}{t}\int_0^t |\Delta_N(t')|\,dt', \quad \overline{r_N^2}(t) = \frac{1}{t}\int_0^t |\Delta_N(t')|^2\,dt'$$

- Consider a symmetric and unimodal g with a parabolic shape

$$g(\omega) = \frac{3}{2}(1 - 4\omega^2)\Theta(1 - 2|\omega|)$$

- Phase transition and critical behavior (K_c : critical point)

$$R \sim \begin{cases} 0 & K \leq K_c \\ |K - K_c|^{\beta}, & K > K_c \end{cases}$$

$$\chi \sim \begin{cases} |K - K_c|^{-\gamma'} & K < K_c \\ |K - K_c|^{-\gamma}, & K > K_c \end{cases}$$

- Finite-size scaling at the critical point

$$R_N \sim N^{-\beta/\bar{\nu}_c}$$

Naive discussion for $\bar{\nu}_c$: Motivation

- Equation of motion revisited

$$\frac{d\theta_k}{dt} = \omega_k + \frac{K}{N}\sum_{j\in I}\sin(\theta_j - \theta_k) = \omega_k + \frac{K}{N}\sum_{j\in I}\Im\left(e^{i\theta_j}e^{-i\theta_k}\right)$$

$$= \omega_k + K\Im e^{-i\theta_k}\underbrace{\left(\frac{1}{N}\sum_{j\in I}e^{i\theta_j}\right)}_{=:\Delta_N(t)} = \omega_k - Kr_N(t)\sin(\theta_k - \psi_N),$$

  where $r_N(t) := |\Delta_N(t)|$, $e^{i\psi_N(t)} = \Delta_N(t)/r_N(t)$.

- If $\omega_1 = \omega_2$ and $\theta_1(t=0) = \theta_2(t=0)$, then $\boldsymbol{\theta_1(t) = \theta_2(t)}$ for all t.

$$\frac{1}{2}\left(e^{i\theta_1} + e^{i\theta_2}\right) = e^{i\theta_1}$$

- Consider L oscillators and assume $\bar{\nu}_c$ for this case is known.
- trival extension: If L^α oscillators with ω_1 and L^α oscillators with ω_2 and so on, then $N = L^{1+\alpha}$ and

$$R_N = L^{-\beta/\bar{\nu}_c} = N^{-\beta/(1+\alpha)\bar{\nu}_c}$$

- Still,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k\in I} \Theta(\omega_k - a)\Theta(b - \omega_k) = \int_a^b g(\omega)\, d\omega.$$

- Infinite number of possible $\bar{\nu}_c$.
- We wanted to find a nontrivial case.

Finite-size scaling exponent $\bar{\nu}_c$

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Motivation

Finite-size scaling exponent $\bar{\nu}_c$

Mean-field self-consistent equation

Summary and Discussion

Frequency sampling rules

- Two rules are typically considered in the literature
  - random sampling : ω_k's are i.i.d. r.v. drawn from $g(\omega)$
  - **regular sampling** to be discussed now.
- It is convenient to introduce x_k as

Definition of x_k

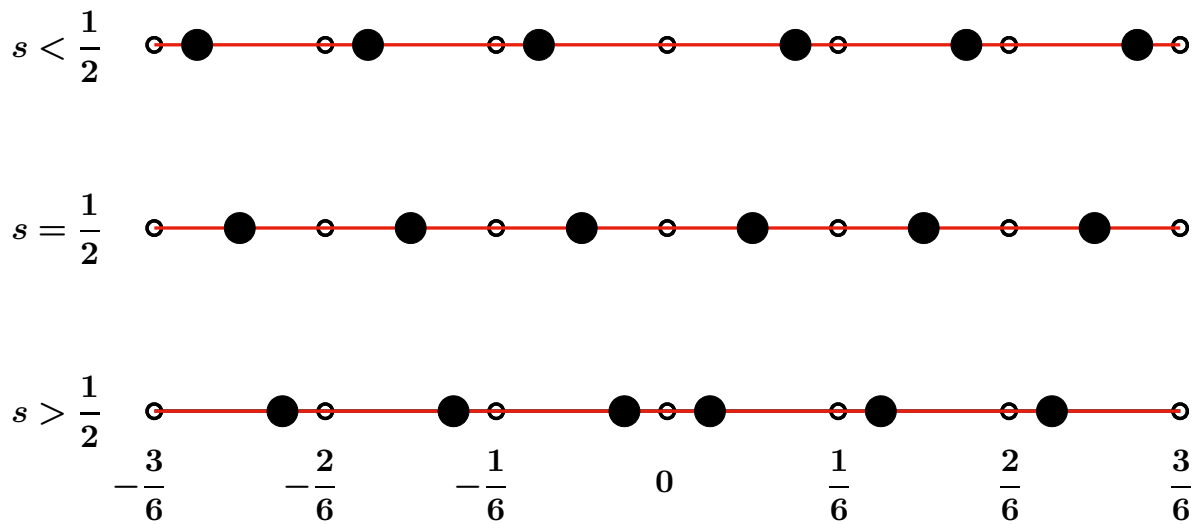
$$x_k := \int_0^{\omega_k} g(\omega)\, d\omega =: G(\omega_k), \quad \omega_k = G^{-1}(x_k).$$

- $-\frac{1}{2} \leq x_k \leq \frac{1}{2}$.
- If $\omega_k < 0$, then $x_k < 0$.
- $G(-x) = G(x)$.

- For convenience, we assume $x_{-k} = -x_k$ ($\omega_{-k} = -\omega_k$)
- Our choice of a regular sampling

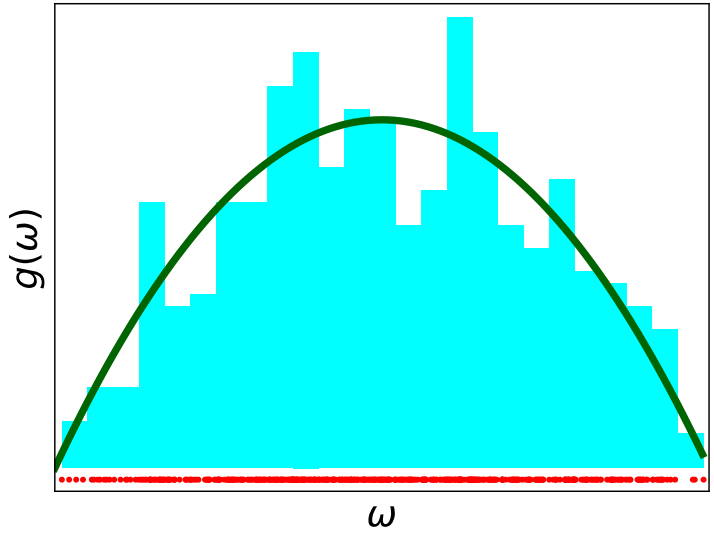
$$x_k = \begin{cases} \dfrac{k-s}{N}, & k > 0 \\ -x_{|k|}, & k < 0. \end{cases}$$

  where $0 \leq s \leq 1$ is a fixed paramter per frequency sampling.
- Illustration of x_k for $N = 6$

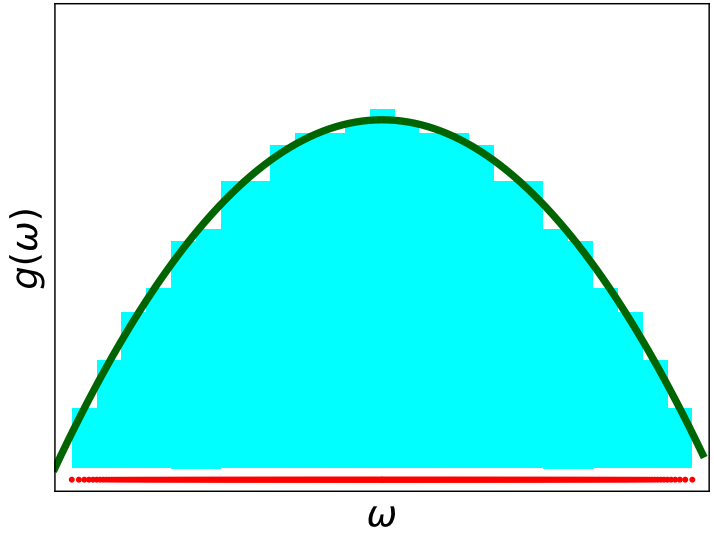


- Example : 512 oscillators with $g(\omega) = \frac{3}{2}(1 - 4\omega^2)\Theta(1 - 2|\omega|)$

random sampling



regular sampling



Mean-field self-consistent equation (MFSCE)

- Equation of motion revisited

$$\frac{d\theta_k}{dt} = \omega_k + \frac{K}{N}\sum_{j\in I}\sin(\theta_j - \theta_k) = \omega_k + \frac{K}{N}\sum_{j\in I}\Im\left(e^{i\theta_j}e^{-i\theta_k}\right)$$

$$= \omega_k + K\Im e^{-i\theta_k}\underbrace{\left(\frac{1}{N}\sum_{j\in I}e^{i\theta_j}\right)}_{=:\Delta_N(t)} = \omega_k - Kr_N(t)\sin(\theta_k - \psi_N),$$

where $r_N(t) := |\Delta_N(t)|$, $e^{i\psi_N(t)} = \Delta_N(t)/r_N(t)$.

- Mean-field assumptions
  - R_N is well defined.
  - $\psi_N(t)$ for large t is a constant ψ_0.
  - $r_N(t) = R_N$ for large t
- Mean-field equation of motion ($\theta_k \mapsto \theta_k + \psi_0$)

$$\frac{d\theta_k}{dt} = \omega_k - KR_N \sin\theta_k$$

  - **entrained** oscillators: $|\omega_k| < KR_N$ implies $\theta_k(t) \approx \sin^{-1} \frac{\omega_k}{KR_N}$
  - **running** oscillators: $|\omega_k| > KR_N$ implies $|\theta_k(t)|$ is a monotonously increasing function of t.

- self-consistent equation

$$
\begin{aligned}
R_N &= \frac{1}{N}\left|\sum_{k\in I} e^{i\theta_k}\right| = \frac{1}{N}\left|\sum_{|\omega_k|<KR_N} (\cos\theta_k + i\sin\theta_k)\right| \\
&= \frac{1}{N}\sum_{|\omega_k|<KR_N}\sqrt{1-\frac{\omega_k^2}{(KR_N)^2}} \\
&= \frac{1}{N}\sum_{|x_k|<G(KR_N)}\sqrt{1-\frac{G^{-1}(x_k)^2}{(KR_N)^2}} \\
&= \int_{-G(KR_N)}^{G(KR_N)}\sqrt{1-\frac{G^{-1}(x)^2}{(KR_N)^2}}\,dx + O(N^{-c})
\end{aligned}
$$

- random sampling : Monte Carlo approximation of the integral
- regular sampling : Riemann sum

Infinite N limit

- Using $g(\omega) = \frac{3}{2}\left(1 - 4\omega^2\right)\Theta(1 - 2|\omega|)$, we get

$$\begin{aligned} R &= \int_{-KR}^{KR} \sqrt{1 - \frac{x^2}{(KR)^2}} \left[g(0) - \frac{1}{2} g''(0) x^2 \right] dx \\ &= KR \int_{-1}^{1} \sqrt{1 - y^2} \left[g(0) - \frac{1}{2} g''(0) K^2 R^2 y^2 \right] dy \\ &= \frac{\pi}{2} g(0) KR - C(KR)^3 =: \frac{1}{K_c}(KR) - C(KR)^3, \end{aligned}$$

  where $K_c := 2/[\pi g(0)] = 4/(3\pi)$ and $C = \frac{\pi}{16} g''(0) = 3\pi/4 = 1/K_c$.
- For a general case, the above equation is an approximation for small R.

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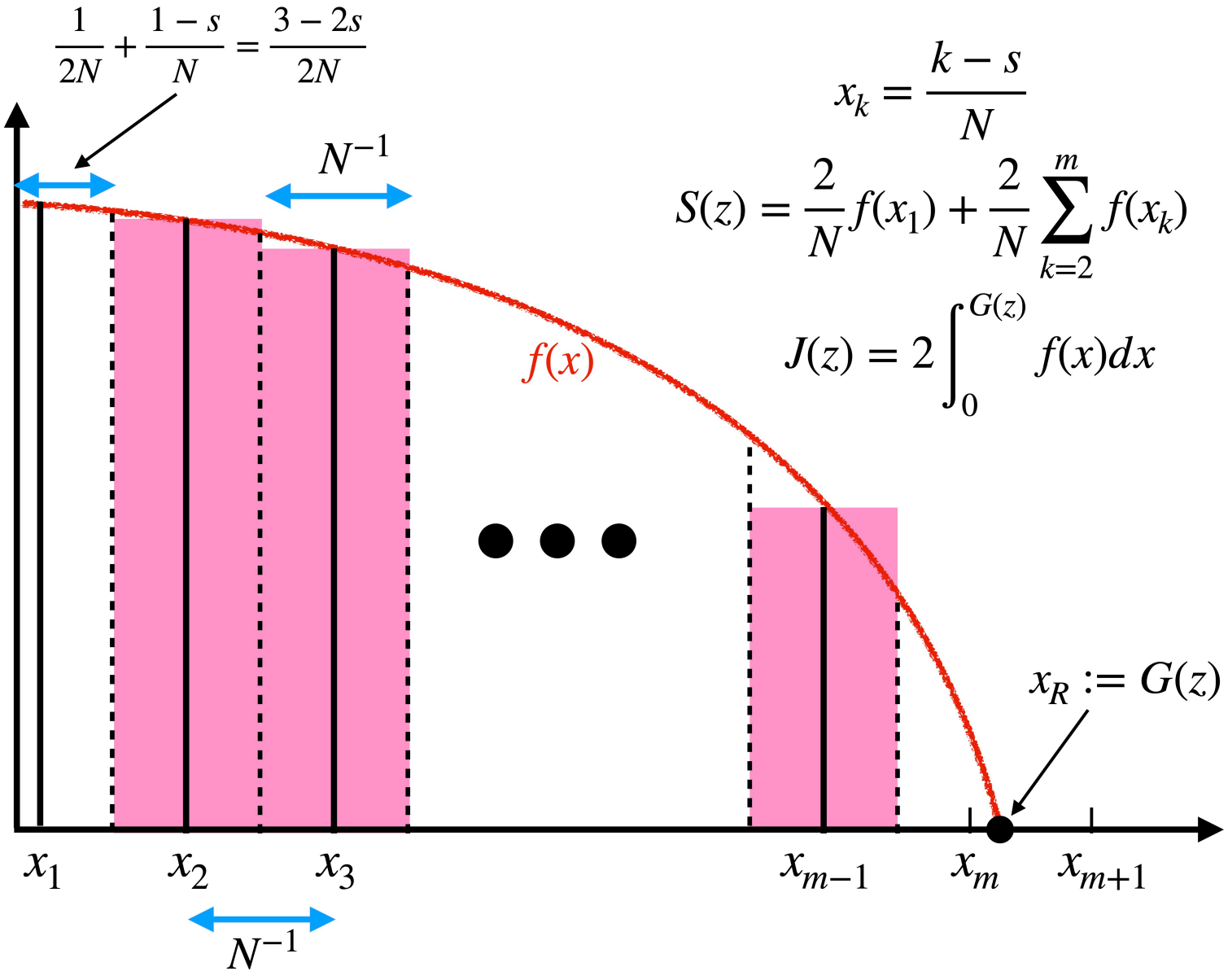
Motivation

Finite-size scaling exponent $\bar{\nu}_c$

Case with $s \neq \frac{1}{2}$

Summary and Discussion

MF prediction for the FSS when $s \neq \frac{1}{2}$



$$x_k = \frac{k-s}{N}$$

$$S(z) = \frac{2}{N} f(x_1) + \frac{2}{N} \sum_{k=2}^{m} f(x_k)$$

$$J(z) = 2 \int_0^{G(z)} f(x) dx$$

- For $s \neq \frac{1}{2}$, $\delta S = S(Z) - J(Z) \approx 2\left(\frac{1}{N} - \frac{3-2s}{2N}\right) = \frac{2s-1}{N}$.

MFSCE for $s \neq \frac{1}{2}$

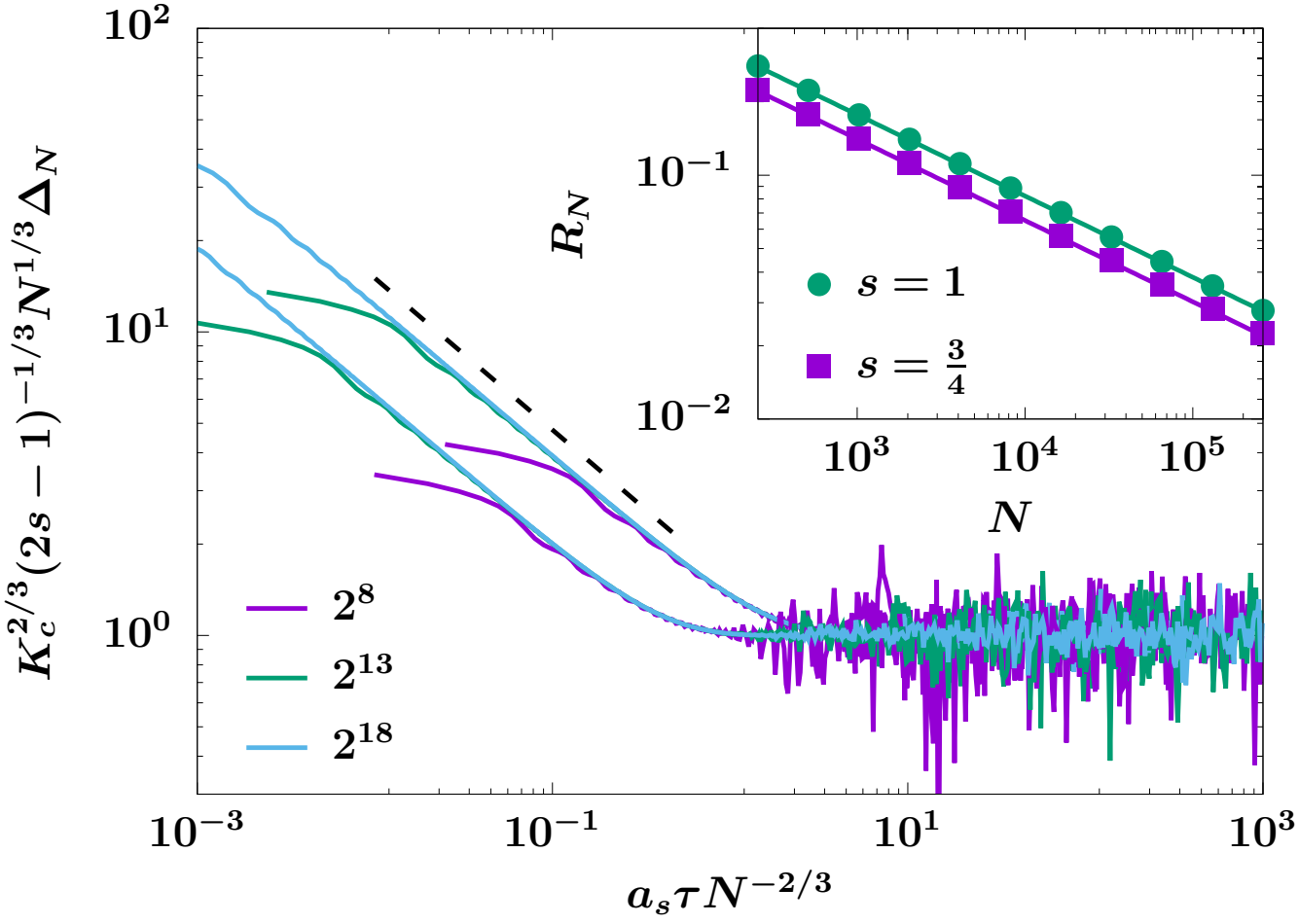
$$Z^3 \approx \varepsilon Z + K_c \frac{2s-1}{N}, \quad Z = KR_N, \quad \varepsilon \propto (K - K_c).$$

- At the critical point ($K = K_c$),

$$R_N \approx \left(\frac{2s-1}{K_c^2}\right)^{1/3} N^{-1/3} \sim N^{-\beta/\bar{\nu}_c}$$

- MF theory predicts $\bar{\nu}_c = \frac{3}{2}$ for $s > \frac{1}{2}$ (not reported before).
- If $s < \frac{1}{2}$, R_N is negative: failure of the MFSCE.

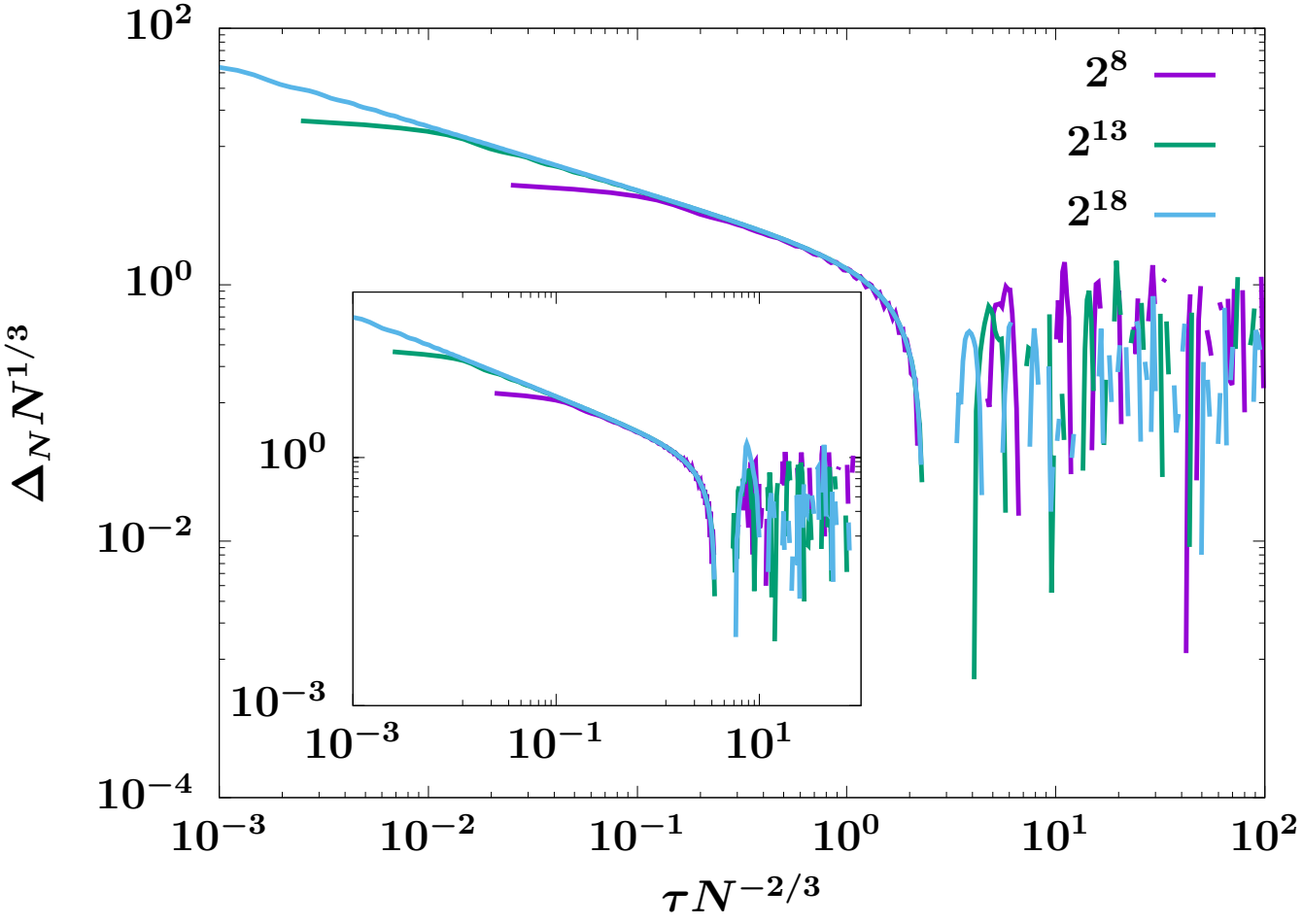
Data collapse at criticality for $s > \frac{1}{2}$



Crossover time scale : $\tau_N^{-1/2} \sim N^{-1/3} \Rightarrow \tau_N \sim N^{2/3}$

Triumph of the mean field theory!

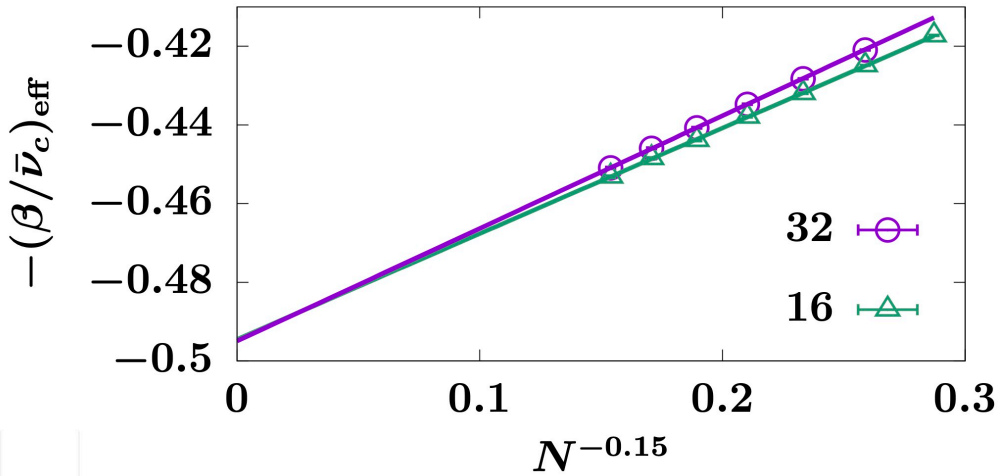
Data collapse at criticality for $s < \frac{1}{2}$



$\bar{\nu} = \frac{3}{2}$? Again triumph of the mean field theory?

R_N for $s < \frac{1}{2}$

$$R_N \sim N^{-\beta/\bar{\nu}_c} \Rightarrow -\left(\beta/\bar{\nu}_c\right)_{\text{eff}} = \frac{\ln R_N - \ln R_{N/b}}{\ln b}$$



$$\bar{\nu}_c \approx 1.02 \neq \frac{3}{2}$$

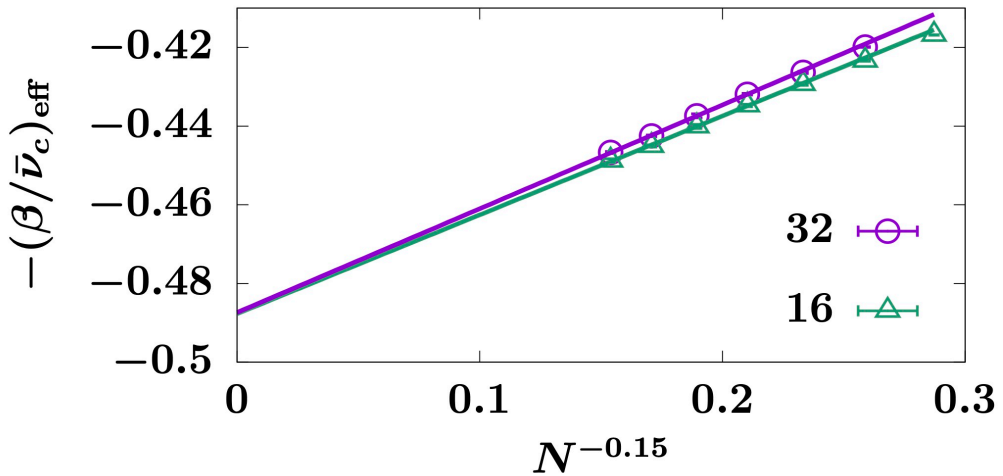


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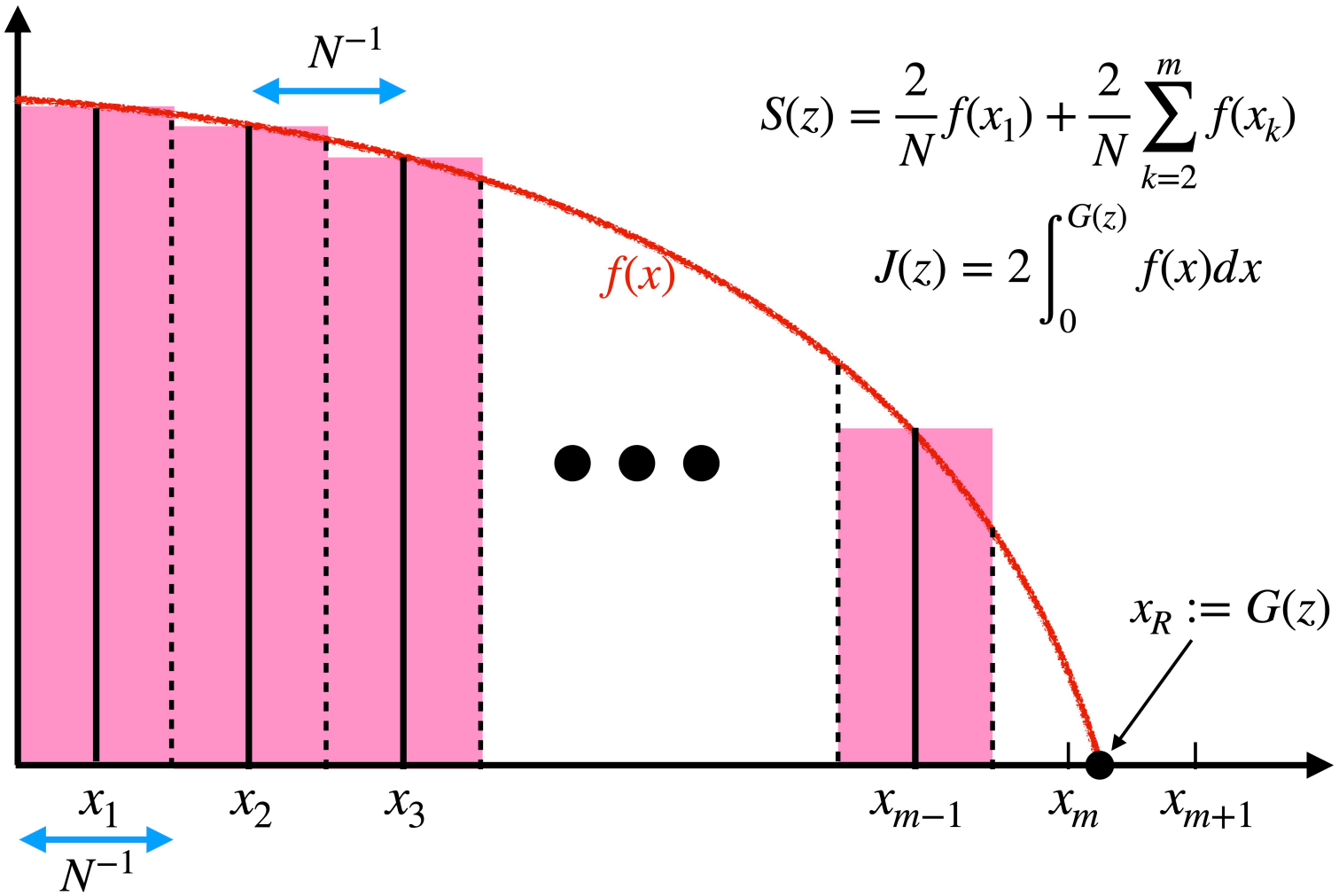
Motivation

Finite-size scaling exponent $\bar{\nu}_c$

Case with $s = \frac{1}{2}$

Summary and Discussion

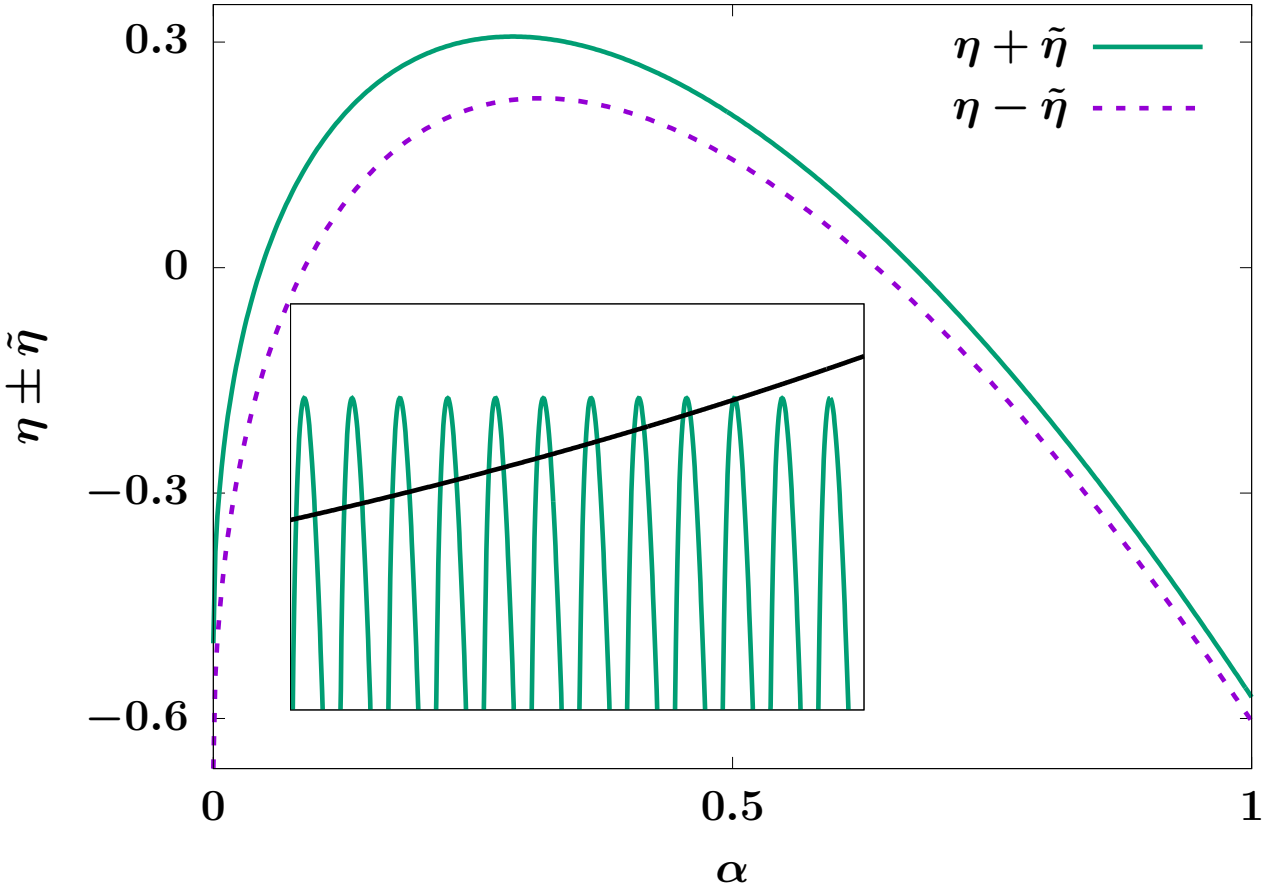
MFSCE



$$S(z) = \frac{2}{N} f(x_1) + \frac{2}{N} \sum_{k=2}^{m} f(x_k)$$

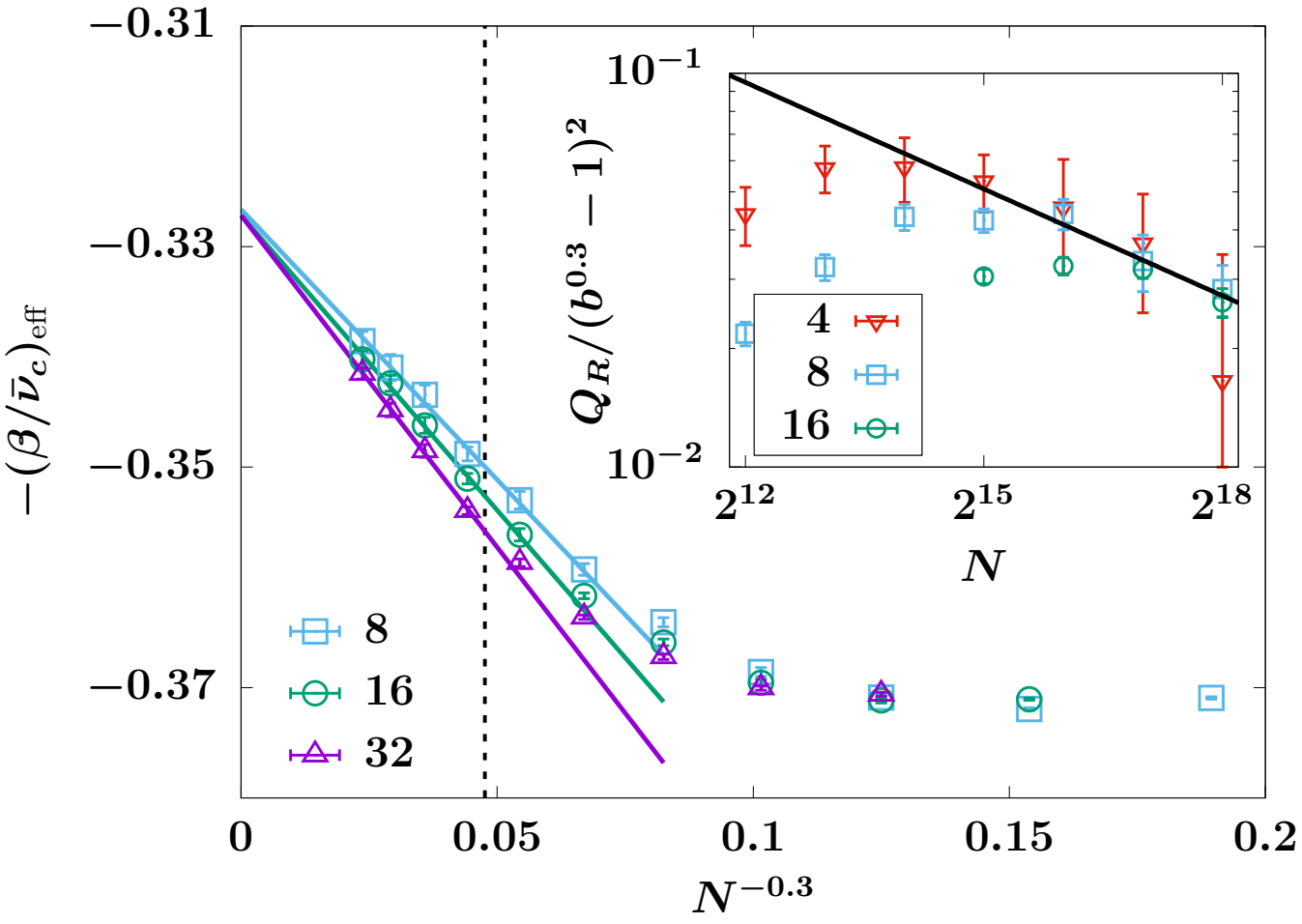
$$J(z) = 2 \int_0^{G(z)} f(x) dx$$

- MFSCE $Z^3 \approx \varepsilon Z + \dfrac{K_c}{\sqrt{g(0)}} \eta_0(Z)(ZN^3)^{-1/2}$.
- If $\eta_0(Z)$ is a constant, $Z^{7/2} \sim N^{-3/2}$ or $\bar{\nu}_c = \frac{7}{6}$. But,



- too many solutions!

Numerical results for $\beta/\bar{\nu}_c$



previous report in the literature : $\beta/\bar{\nu}_c \approx 0.4$

Summary and Discussion

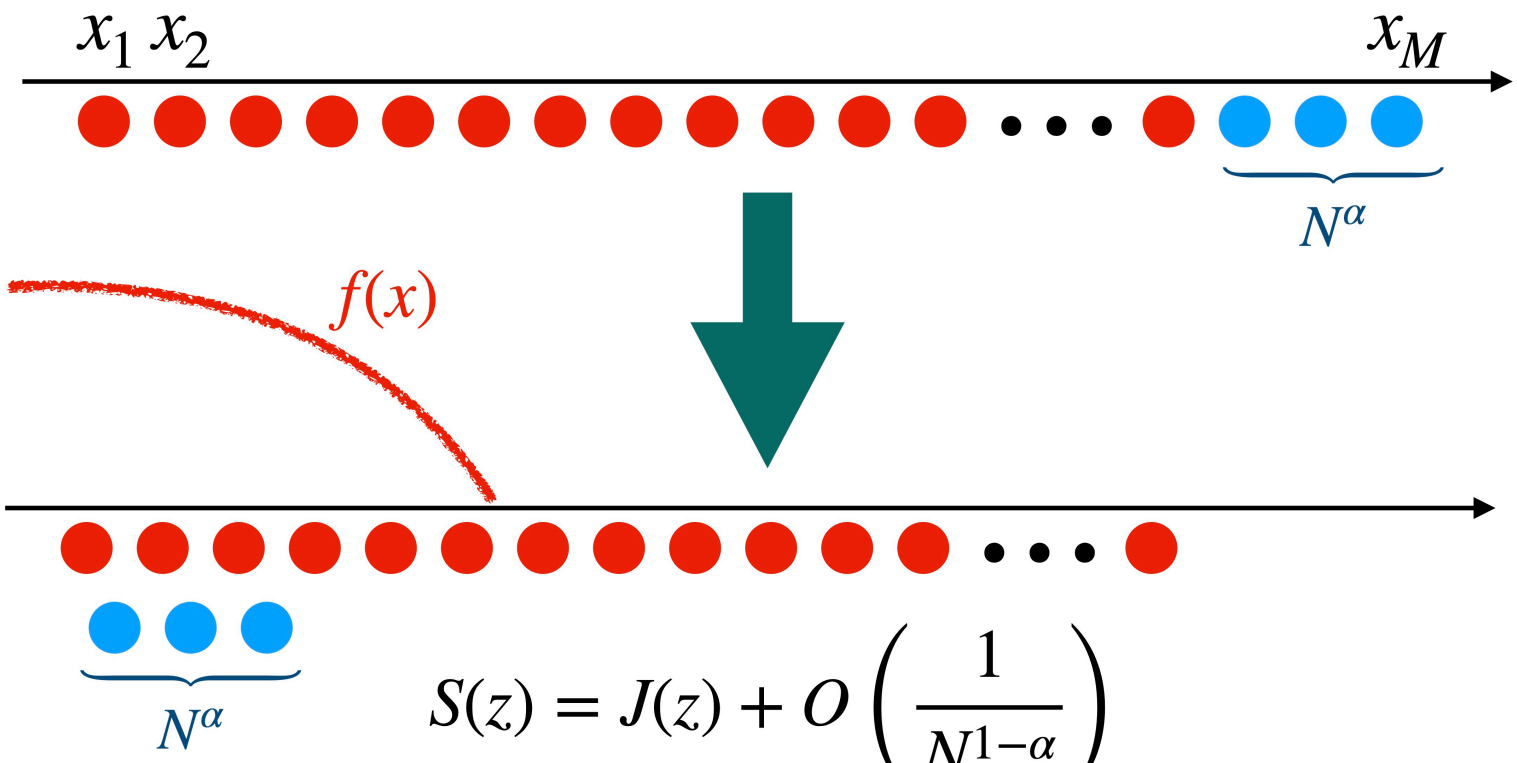
Summary of the critical volume exponents

- Finite-size scaling at criticality

$$R_N \sim N^{-\beta/\bar{\nu}_c}$$

- $\bar{\nu}_c$ does depend on the sampling rule
- $s > 1/2 : \bar{\nu}_c = 3/2$
- $s = 1/2 : \bar{\nu}_c \approx 1.5$
- $s < 1/2 : \bar{\nu}_c \approx 1.02$

Discussion: continuously varying $\bar{\nu}_c$



$$S(z) = J(z) + O\left(\frac{1}{N^{1-\alpha}}\right)$$

Assume $0 < \alpha < 1$

$$R_N^3 \sim N^{\alpha-1} \Rightarrow R_N \sim N^{-(1-\alpha)/3}, \quad \bar{\nu}_c = \frac{3}{2(1-\alpha)}$$