Coherence Enhanced by Dynamic Oscillators

EAJSSP 2025

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Jae Sung Lee

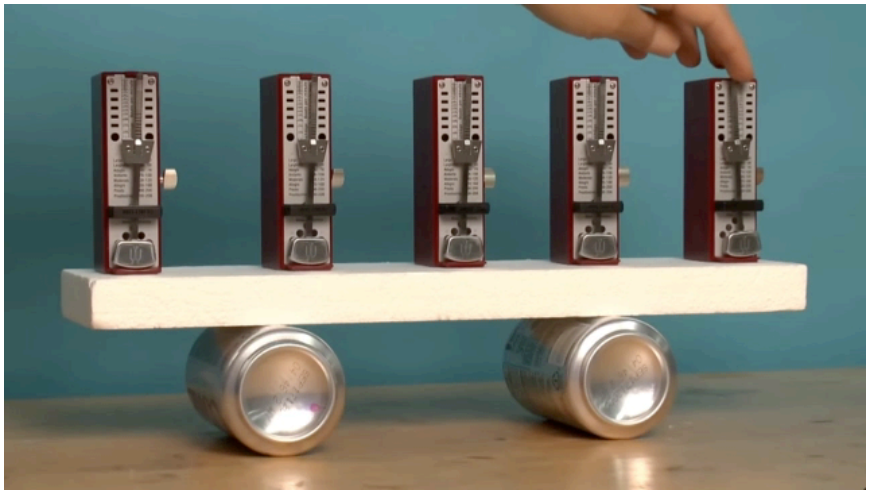
Korea Institute for Advanced Study (KIAS)

in collaboration with Hyunggyu Park (KIAS)

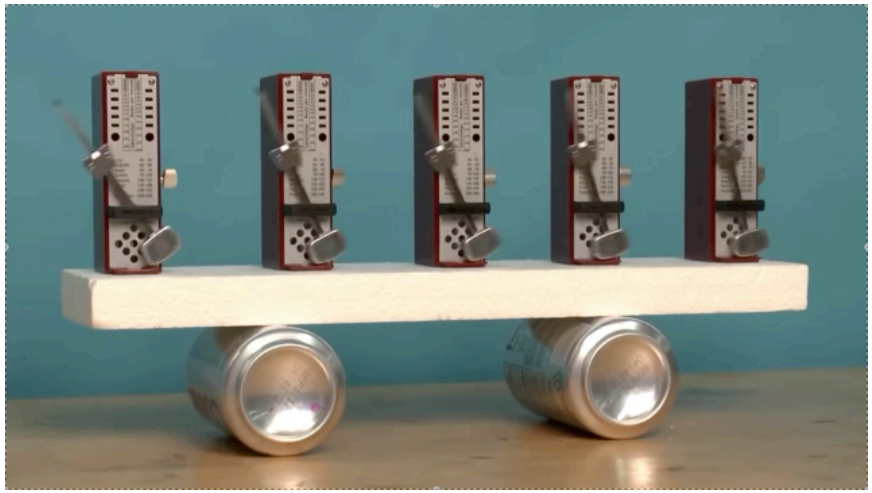
Hyunsuk Hong (Jeonbuk Univ.)

Synchronization Phenomena

metronome



UCLA, Department of Physics & Astronomy



after some time

Key ingredients of synchronization

1. *N* oscillators, each with its own phase & frequency
2. Interaction between them

Platform for Studying on Synchronization Phenomena

: Kuramoto model $\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}\sin(\theta_j - \theta_i)$



Boltzmann medalist 2025

- N oscillators : $i = 1, \cdots, N$
- θ_i : phase
- ω_i : natural frequency
- $K_{ij} = K$: interaction strength

Key ingredients of synchronization

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2. Interaction between them

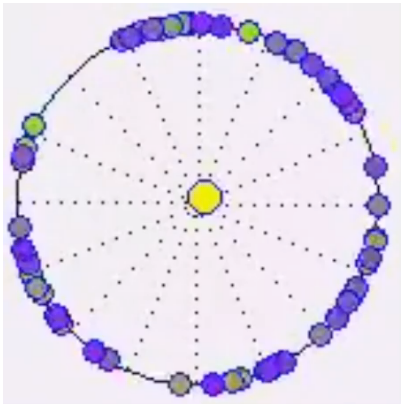
Platform for Studying on Synchronization Phenomena

: Kuramoto model $\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$

order parameter: $Z \equiv Se^{i\Phi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$

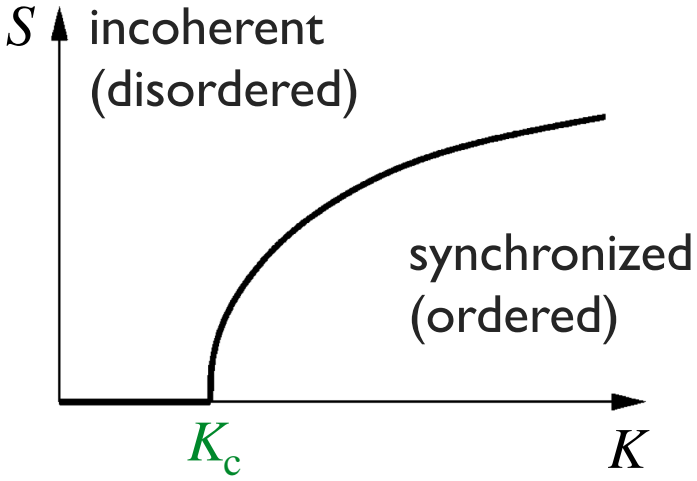
- Φ : mean phase
- S : (real) order parameter $(0 \le S \le 1)$

incoherent

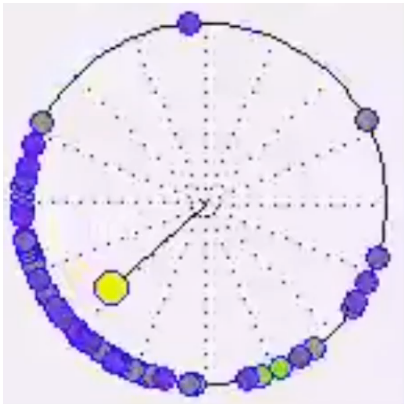


all moving with ω_i

→ all detrained (dynamic, $\dot{\theta} \neq 0$)



(partially) synchronized



partially phase-locked

(static, $\dot{\theta} = 0$)

partially detrained

(dynamic, $\dot{\theta} \neq 0$)

in moving Φ frame

static oscillators

dynamic oscillators do not

contribute to coherence

Main Message of This Study

static oscillators

dynamic oscillators do [redacted] contribute to coherence

→ Coherence can be enhanced by dynamic oscillators!!

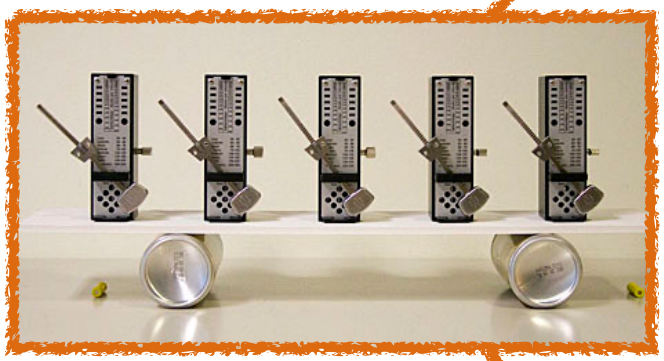
From a swarmalator model, we find novel dynamical modes:

- bounded oscillation
- full rotation with broken π-reflection symmetry

→ dynamic oscillators can contribute to coherence!

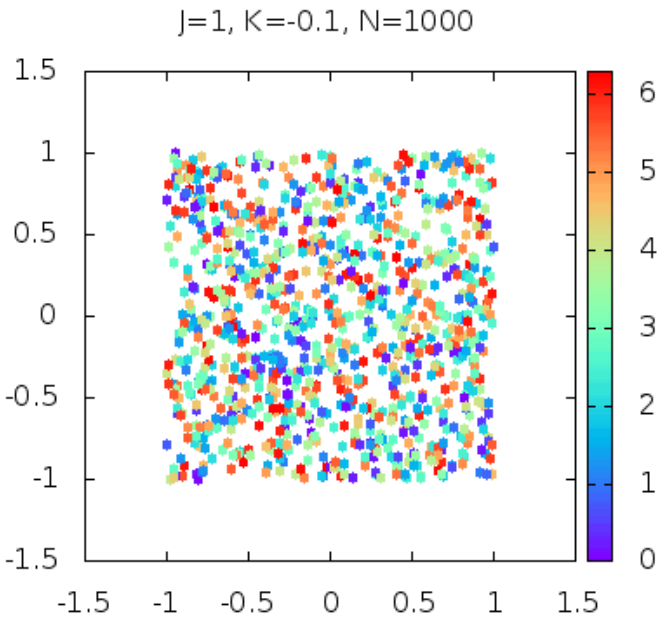
Swarmalator

: Oscillators that sync in phase and swarm in space





Swarmalator : phase and position dynamics are coupled.



color: phase

x, y coordinate: position

O'Keeffe, Hong, and Strogatz, Nature Commun. 8, 1504 (2017)

Swarmalator | analytically tractable model

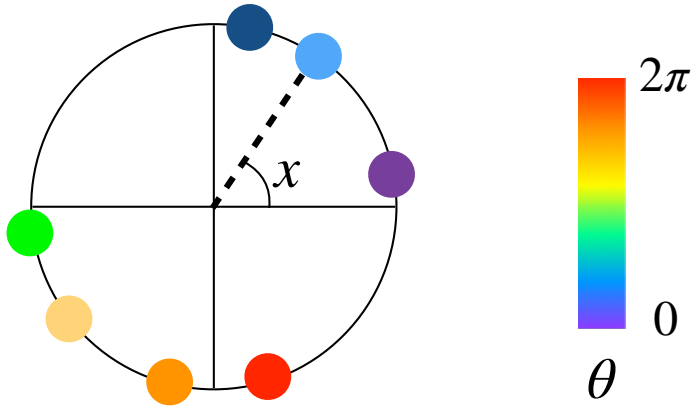
Yoon, O'Keeffe, Hong, Mendes, and Goltsev, PRL 129, 208002 (2022)

Dynamic equation for swarmalators

$$\dot{x}_i = v_i + \frac{1}{N}\sum_{j=1}^{N} J_{ij}(\theta_i, \theta_j)\sin(x_j - x_i)$$

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N} K_{ij}(x_i, x_j)\sin(\theta_j - \theta_i)$$

Schematic of the model



v : natural velocity

ω : natural frequency

drawn from $g_{\Delta_\omega}(\omega) = \frac{\Delta_\omega}{\pi}\frac{1}{\omega^2 + \Delta_\omega^2}$

x : position variable $(0 \le x < 2\pi)$

θ : phase variable $(0 \le \theta < 2\pi)$

$g_{\Delta_v}(v) = \frac{\Delta_v}{\pi}\frac{1}{v^2 + \Delta_v^2}$

$J_{ij}(\theta_i, \theta_j) = J\cos(\theta_j - \theta_i)$: phase-dependent swarming

$K_{ij}(x_i, x_j) = K\cos(x_j - x_i)$: position-dependent synchronization

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$$J_{ij}(\theta_i, \theta_j) = J\cos(\theta_j - \theta_i)$$

$$K_{ij}(x_i, x_j) = K\cos(x_j - x_i)$$

What are proper order parameters? First trial: standard Kuramoto type

$$Z_x \equiv S_x e^{i\Phi_x} = \frac{1}{N}\sum_{j=1}^{N} e^{ix_j} = 0 \qquad Z_\theta \equiv S_\theta e^{i\Phi_\theta} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} = 0$$

Order parameters measuring correlation between space and phase

change of variables : $X_i \equiv x_i + \theta_i \qquad Y_i \equiv x_i - \theta_i$

$$Z_+ \equiv S_+ e^{i\Phi_+} = \frac{1}{N}\sum_{j=1}^{N} e^{iX_j} \qquad Z_- \equiv S_- e^{i\Phi_-} = \frac{1}{N}\sum_{j=1}^{N} e^{iY_j} \qquad (0 \le S_\pm \le 1)$$

ex) $x_i = \theta_i$ (perfect correlation) $\rightarrow Y_i = 0 \rightarrow S_- = 1$

Swarmalator | analytically tractable model

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Dynamic equation for X and Y

$$\dot{X}_i = \omega_{i+} + J_+ S_+ \sin(\Phi_+ - X_i) + J_- S_- \sin(\Phi_- - Y_i)$$

$$\dot{Y}_i = \omega_{i-} + J_- S_+ \sin(\Phi_+ - X_i) + J_+ S_- \sin(\Phi_- - Y_i)$$

where $J_\pm \equiv (J \pm K)/2 \qquad \omega_{i\pm} \equiv \omega_i \pm v_i$

Swarmalator | analytically tractable model

Yoon, O'Keeffe, Hong, Mendes, and Goltsev, PRL 129, 208002 (2022)

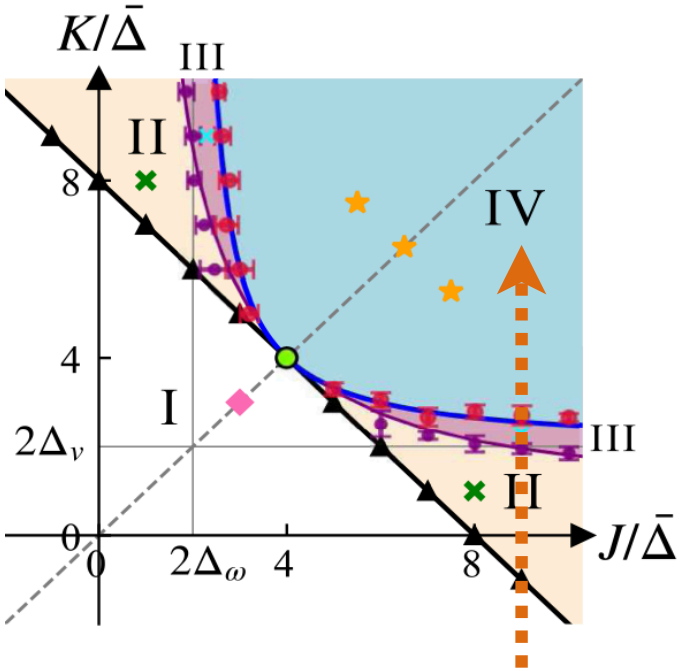
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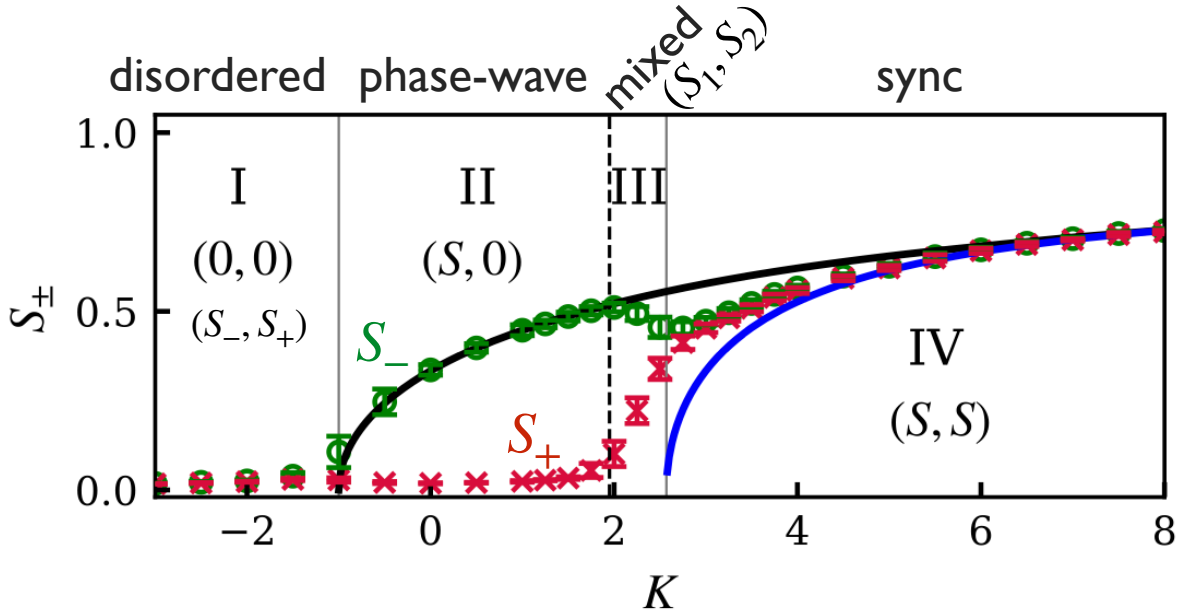
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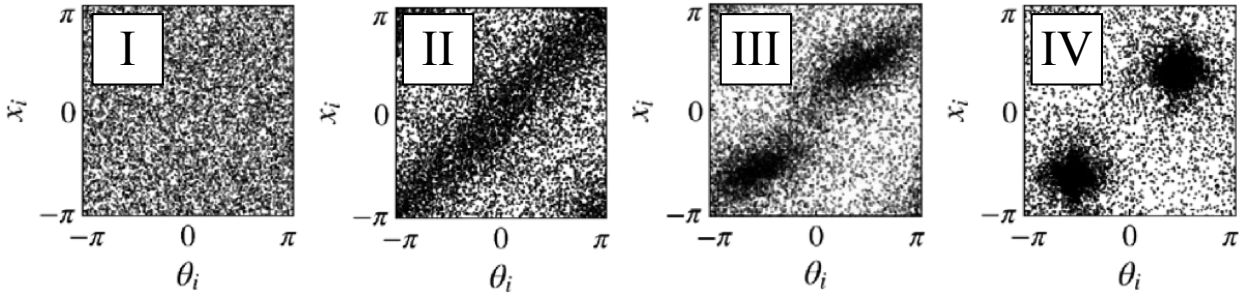
where $J_\pm \equiv (J \pm K)/2$ $\omega_{i\pm} \equiv \omega_i \pm \nu_i$

phase diagram



$\bar{\Delta} \equiv (\Delta_\omega + \Delta_\nu)/2$





Swarmalator | analytically tractable model

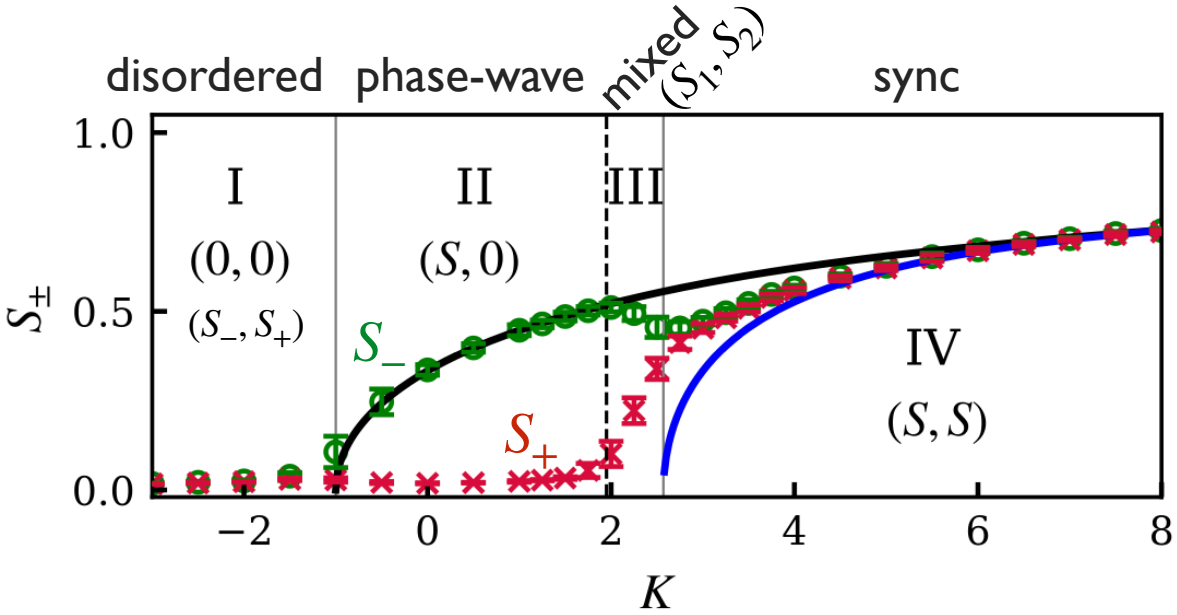
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These four phases appear to be commonly observed across swarmalator models.

Hong, O'Keeffe, **JSL**, and Park, PRR 5, 023105 (2023)

$$\dot{x}_i = \frac{J}{N}\sum_{j=1}^{N} \sin(x_j - x_i)\cos(\theta_j - \theta_i) + \xi_i^x \qquad \dot{\theta}_i = \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)\cos(x_j - x_i) + \xi_i^\theta$$

Swarmalator | analytically tractable model

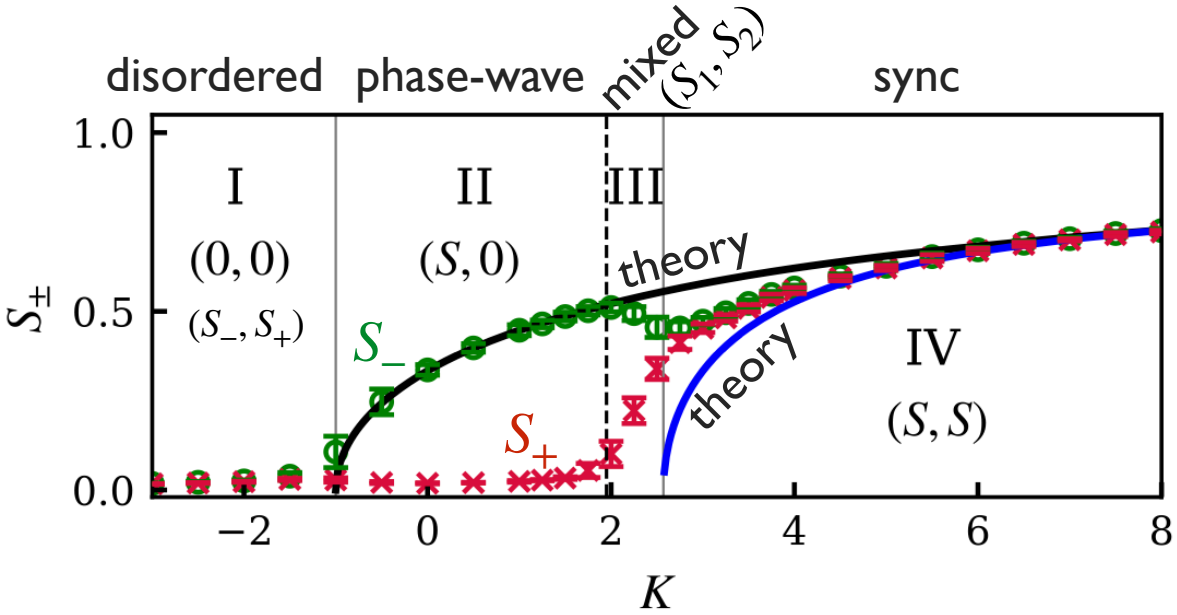
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Issues in Phase III : inconsistency between theory and numerics

- theory assumption: ordering comes only from static (phase-locked, $\dot{X}_i = 0$) ones
- phase transition occurs earlier than expected by theory → coherence is enhanced
- mechanism underlying this enhancement has not yet been uncovered.

Swarmalator | analytically tractable model

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where $J_\pm \equiv (J \pm K)/2$ $\omega_{i\pm} \equiv \omega_i \pm \nu_i$

Purpose of our study : unveil the underlying mechanism

Simplified model : $\omega_i = \nu_i$ (perfect correlation in natural freq)

$$\dot{X}_i = 2\omega_i + J_+ S_+ \sin(\Phi_+ - X_i) + J_- S_- \sin(\Phi_- - Y_i)$$

$$\dot{Y}_i = J_- S_+ \sin(\Phi_+ - X_i) + J_+ S_- \sin(\Phi_- - Y_i)$$

ω_i drawn from $g(\omega) = \frac{\gamma}{\pi} \frac{1}{\omega^2 + \gamma^2}$

→ analytically more tractable, but still retains essential features

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Swarmalator | analytically more tractable model

Hong, Lee, and Park, submitted

Case I : $J_- = 0 \;\; (J = K) \rightarrow X$ and Y are decoupled

$\dot{X}_i = 2\omega_i + J_+ S_+ \sin(\Phi_+ - X_i) \rightarrow$ standard Kuramoto (contiuous transition at $J_+ = 4\gamma$)

$\dot{Y}_i = J_+ S_- \sin(\Phi_- - Y_i) \rightarrow$ Watanabe-Strogatz (discontinuous transition at $J_+ = 0$)

Watanabe and Strogatz, Physica D 74, 197 (1994)

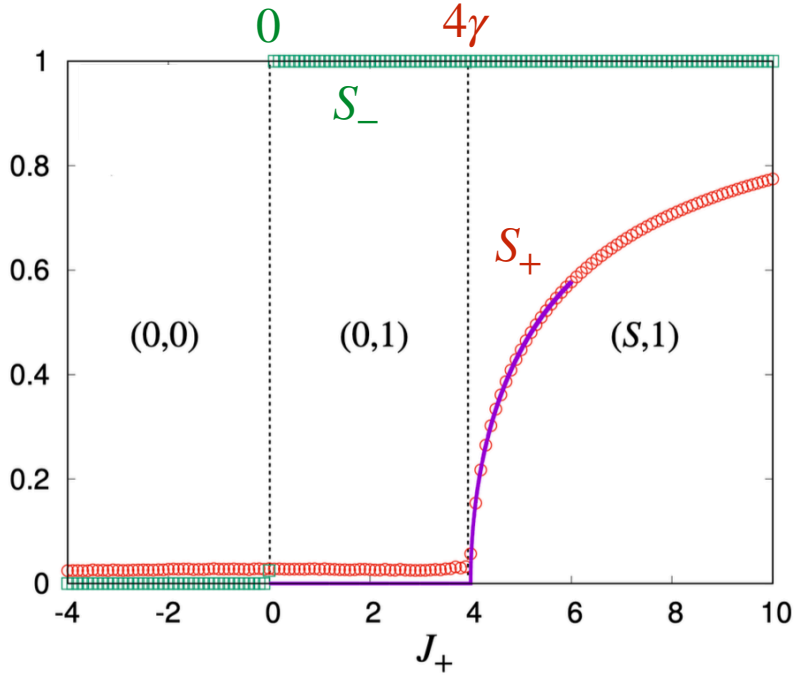
Decomposition of order parameter S_-

$$Z_- \equiv S_- e^{i\Phi_-} = \frac{1}{N}\sum_{j=1}^{N} e^{iY_j} \rightarrow \text{set } \Phi_- = 0$$

$$S_- = S_-^{\mathrm{s}} + S_-^{\mathrm{d}} = \frac{1}{N}\sum_{j\in\Lambda_{\mathrm{s}}} e^{iY_j} + \frac{1}{N}\sum_{j\in\Lambda_{\mathrm{d}}} e^{iY_j}$$

Λ_{s} : set of static oscillators $(\dot{Y}_i = 0)$

Λ_{d} : set of dynamic oscillators $(\dot{Y} \neq 0)$



In the ordered state $(S_- = 1)$

all oscillators are static and locked at a single phase $\quad S_-^{\mathrm{s}} = 1$

no dynamic oscillator $\quad S_-^{\mathrm{d}} = 0$

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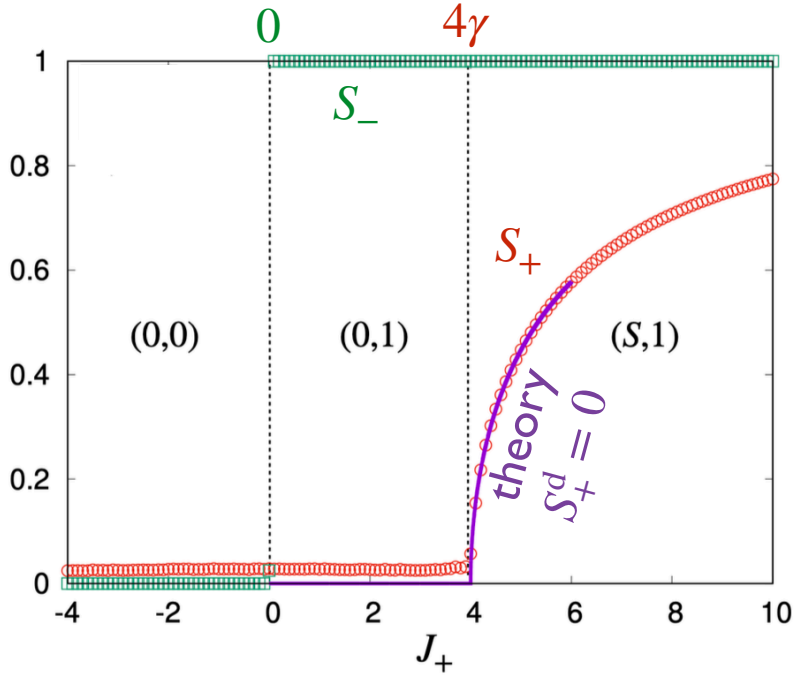
Decomposition of order parameter S_+

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$$S_+ = S_+^{\mathrm{s}} + S_+^{\mathrm{d}} = \frac{1}{N}\sum_{j\in\Lambda_{\mathrm{s}}} e^{iX_j} + \frac{1}{N}\sum_{j\in\Lambda_{\mathrm{d}}} e^{iX_j}$$

Λ_{s} : set of static oscillators $(\dot{X}_i = 0)$

Λ_{d} : set of dynamic oscillators $(\dot{X} \neq 0)$



In the ordered state

static $|\omega| \le \omega_{\mathrm{c}} \equiv J_+ S_+/2 \qquad S_+^{\mathrm{s}} = S_+$

dynamic $|\omega| > \omega_{\mathrm{c}} \qquad S_+^{\mathrm{d}} = 0$ no contribution from dynamic oscillators

Swarmalator | analytically more tractable model

Hong, Lee, and Park, arXiv …

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Why $S_+^{\mathrm{d}} = 0$?

$$S_+^{\mathrm{d}} = \frac{1}{N}\sum_{j\in\Lambda_{\mathrm{d}}} e^{iX_j} = \int_{|\omega|>\omega_{\mathrm{c}}} d\omega g(\omega) \int_{-\pi}^{\pi} dX e^{iX} P_\omega^{\mathrm{d}}(X) = \frac{\mathrm{sgn}(\omega)}{2\pi} \frac{\sqrt{(2\omega)^2 - (J_+S_+)^2}}{2\omega - J_+S_+ \sin X}$$

$P_\omega^{\mathrm{d}}(X)$: pdf of dynamic X oscillators for given ω

$$= \int_{|\omega|>\omega_{\mathrm{c}}} d\omega g(\omega) \int_{-\pi}^{\pi} dX (\cos X + i \sin X) P_\omega^{\mathrm{d}}(X)$$

imaginary part should vanish

$$= \int_{|\omega|>\omega_{\mathrm{c}}} d\omega g(\omega) \int_{-\pi}^{\pi} dX \cos X \; P_\omega^{\mathrm{d}}(X)$$

odd function with respect to $\pi/2$ — even function with respect to $\pi/2$

$$= 0$$

$$P_\omega^{\mathrm{d}}(X) = P_\omega^{\mathrm{d}}(\pi - X)$$

No contribution from dynamic oscillators due to π-reflection symmetry of pdf

→ If the symmetry is broken, dynamic oscillators can contribute to the ordering.

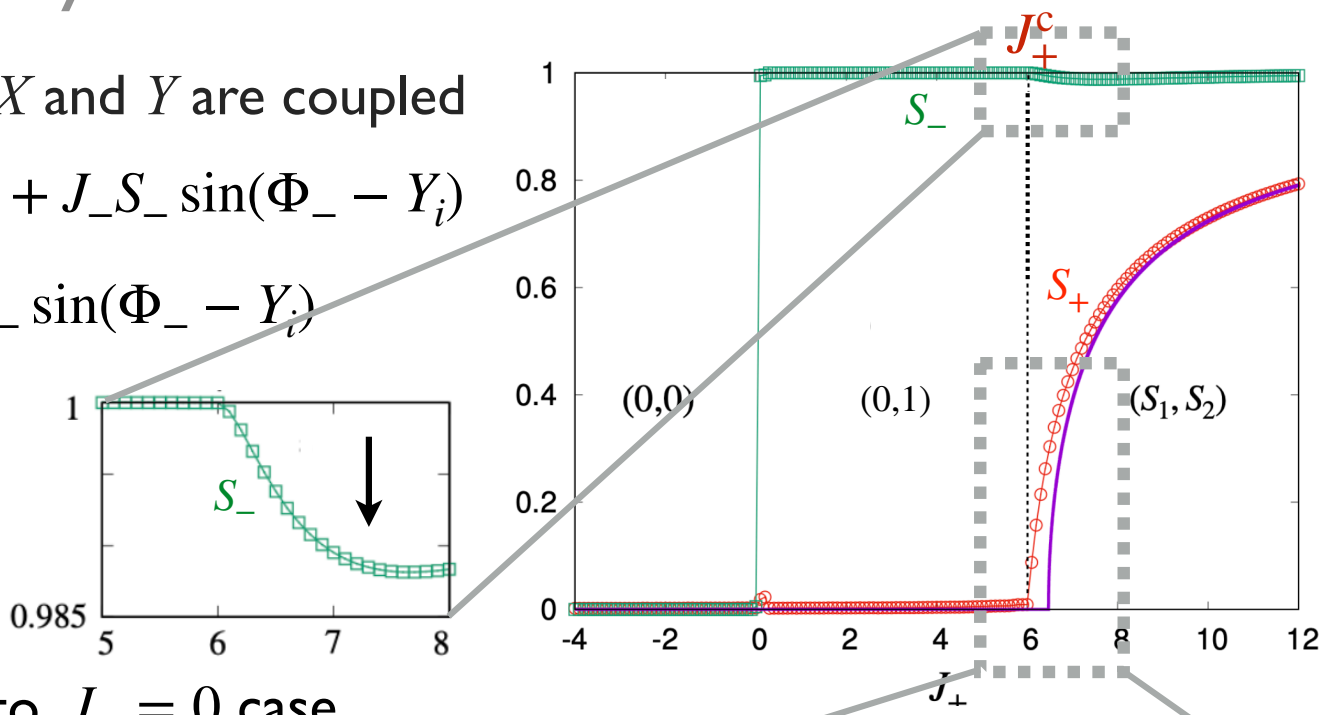
Swarmalator | analytically more tractable model

Hong, Lee, and Park, submitted

Case II : $J_- \neq 0 \quad (J \neq K) \rightarrow X$ and Y are coupled

$$\dot{X}_i = 2\omega_i + J_+ S_+ \sin(\Phi_+ - X_i) + J_- S_- \sin(\Phi_- - Y_i)$$

$$\dot{Y}_i = J_- S_+ \sin(\Phi_+ - X_i) + J_+ S_- \sin(\Phi_- - Y_i)$$



Overall feature looks similar to $J_- = 0$ case

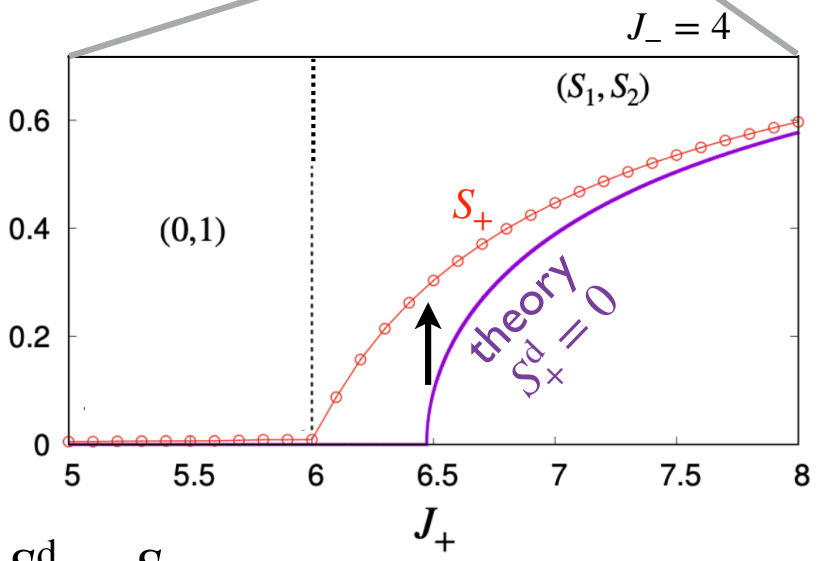
Essential difference

Both $S_\pm$ experience transition at J_+^c.

S_- : slightly reduced

S_+ : enhanced than expected

$\rightarrow$ implies that $S_+^d > 0$



$\therefore$ need to calculate $S_\pm^s$ and $S_\pm^d$ separately $\Rightarrow S_\pm^s + S_\pm^d = S_\pm$

Swarmalator | analytically more tractable model

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$\therefore$ need to calculate $S_\pm^{\mathrm{s}}$ and $S_\pm^{\mathrm{d}}$ separately

In (S_1,S_2) state one can show that

static $\quad |\omega| \leq \omega_{\mathrm{c}} \equiv aS_+/2$

dynamic $\quad |\omega| > \omega_{\mathrm{c}}$

$$a = \frac{J_+^2 - J_-^2}{2J_+}$$

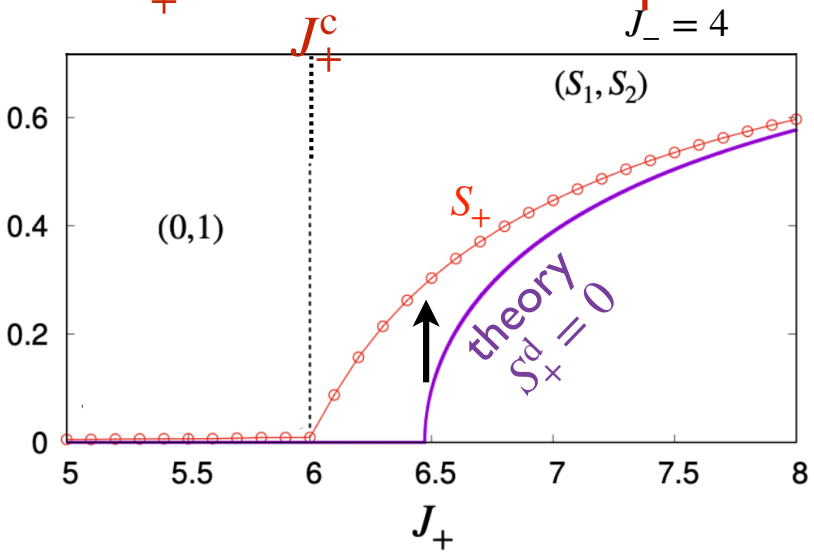
$$S_+^{\mathrm{d}} = \int_{|\omega|>\omega_{\mathrm{c}}} d\omega g(\omega) \int_{-\pi}^{\pi} dX e^{iX} P_\omega^{\mathrm{d}}(X)$$

$$S_-^{\mathrm{d}} = \int_{|\omega|>\omega_{\mathrm{c}}} d\omega g(\omega) \int_{-\pi}^{\pi} dY e^{iY} P_\omega^{\mathrm{d}}(Y)$$

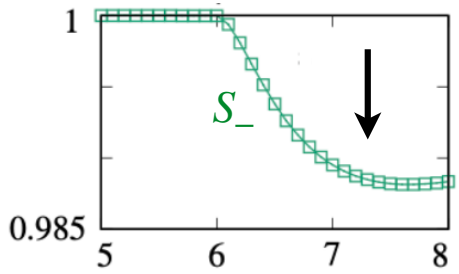
Analytic calculation is intractable in general.

Perturbation when $S_+ \ll 1$ near $J_+ = J_+^{\mathrm{c}}$

S_+ : enhanced than expected



S_- : slightly reduced



Swarmalator | analytically more tractable model

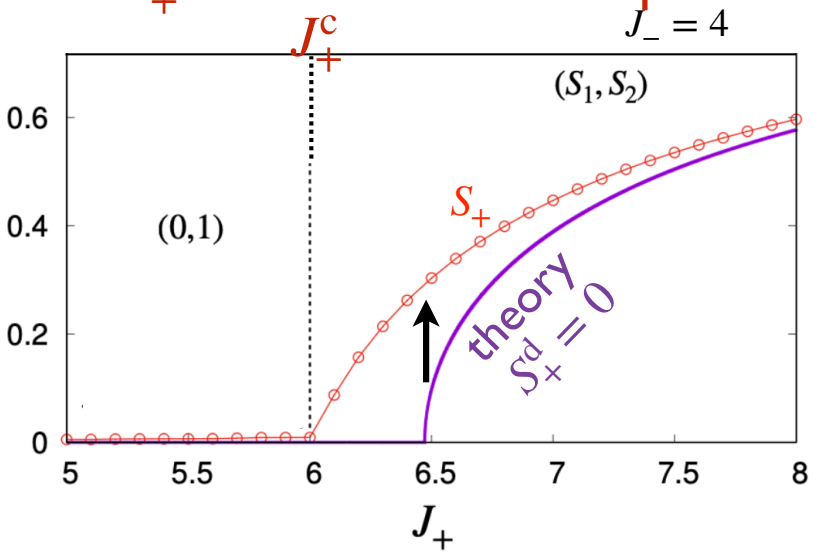
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S_+ : enhanced than expected



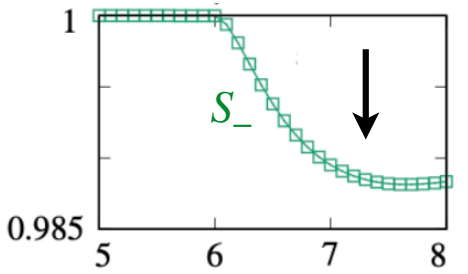
Perturbation when $S_+ \ll 1$ near $J_+ = J_+^c$

$$X_\omega \equiv X - \Phi_+ = X_\omega^{(0)} + X_\omega^{(1)} S_+ + X_\omega^{(2)} S_+^2 + O(S_+^3)$$

$$Y_\omega \equiv Y - \Phi_- = Y_\omega^{(0)} + Y_\omega^{(1)} S_+ + Y_\omega^{(2)} S_+^2 + O(S_+^3)$$

for $|\omega| > \omega_c$: dynamic

S_- : slightly reduced



Solutions:

$$X_\omega^{(0)} = 2\omega t \qquad X_\omega^{(1)} = \frac{J_+}{2\omega}\cos 2\omega t - \frac{J_-^2 \sin(2\omega t + \beta_\omega)}{\sqrt{J_+^2 + 4\omega^2}} \qquad X_\omega^{(2)} = \cdots$$

: full rotation

$$Y_\omega^{(0)} = 0 \qquad Y_\omega^{(1)} = \frac{J_- \cos(2\omega t + \beta_\omega)}{\sqrt{J_+^2 + 4\omega^2}} \qquad Y_\omega^{(2)} = \cdots$$

: oscillation in a bounded region

Swarmalator | analytically more tractable model

Hong, Lee, and Park, submitted

Case II : $J_- \neq 0 \quad (J \neq K) \rightarrow X$ and Y are coupled

$$\dot{X}_i = 2\omega_i + J_+ S_+ \sin(\Phi_+ - X_i) + J_- S_- \sin(\Phi_- - Y_i)$$

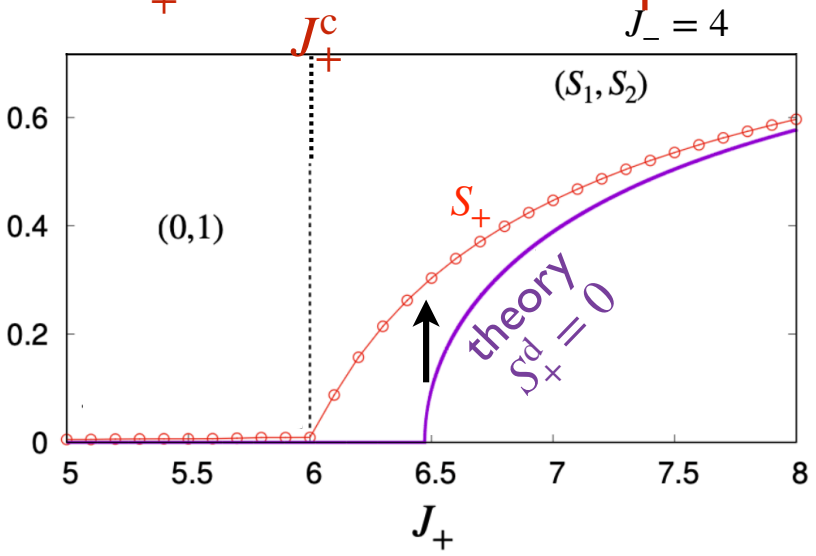
$$\dot{Y}_i = J_- S_+ \sin(\Phi_+ - X_i) + J_+ S_- \sin(\Phi_- - Y_i)$$

Perturbation when $S_+ \ll 1$ near $J_+ = J_+^c$

$$X_\omega \equiv X - \Phi_+ = X_\omega^{(0)} + X_\omega^{(1)} S_+ + X_\omega^{(2)} S_+^2 + O(S_+^3)$$

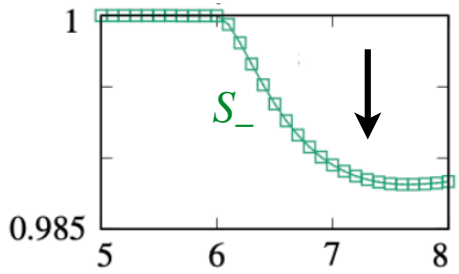
$$Y_\omega \equiv Y - \Phi_- = Y_\omega^{(0)} + Y_\omega^{(1)} S_+ + Y_\omega^{(2)} S_+^2 + O(S_+^3)$$

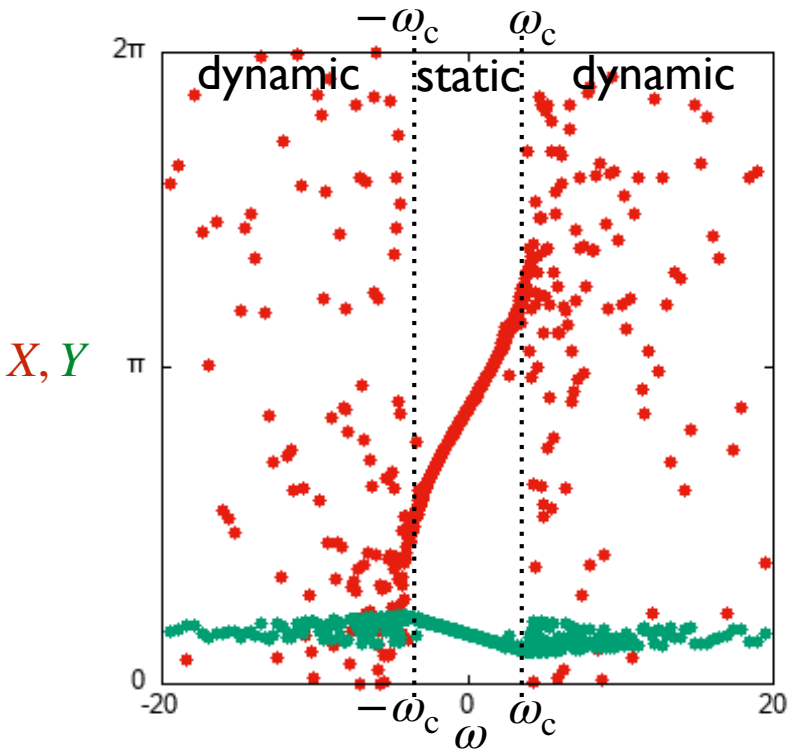
S_+ : enhanced than expected



for $|\omega| > \omega_c$: dynamic

S_- : slightly reduced





$$X_\omega^{(0)} = 2\omega t$$

: full rotation

$$Y_\omega^{(1)} = \frac{J_- \cos(2\omega t + \beta_\omega)}{\sqrt{J_+^2 + 4\omega^2}}$$

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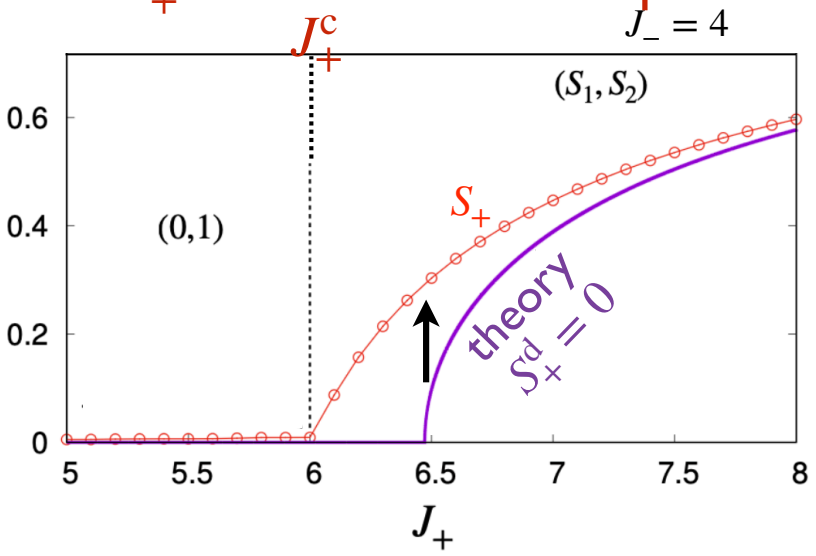
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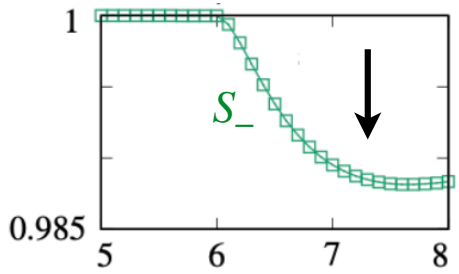
$$Y_\omega \equiv Y - \Phi_- = Y_\omega^{(0)} + Y_\omega^{(1)} S_+ + Y_\omega^{(2)} S_+^2 + O(S_+^3)$$

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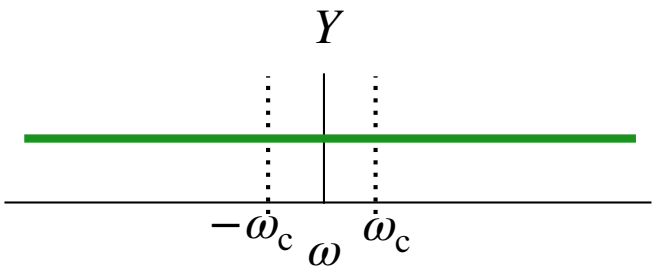
S_- : slightly reduced



Why S_- reduced?

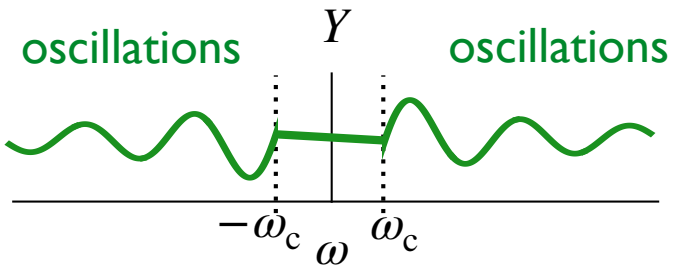
for $J_+ < J_+^c$

all oscillators are static and locked at a single phase



for $J_+ > J_+^c$

bounded oscillations
reduced coherenece



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Why S_+ enhanced? $\rightarrow S_+^{\mathrm{d}} \neq 0$

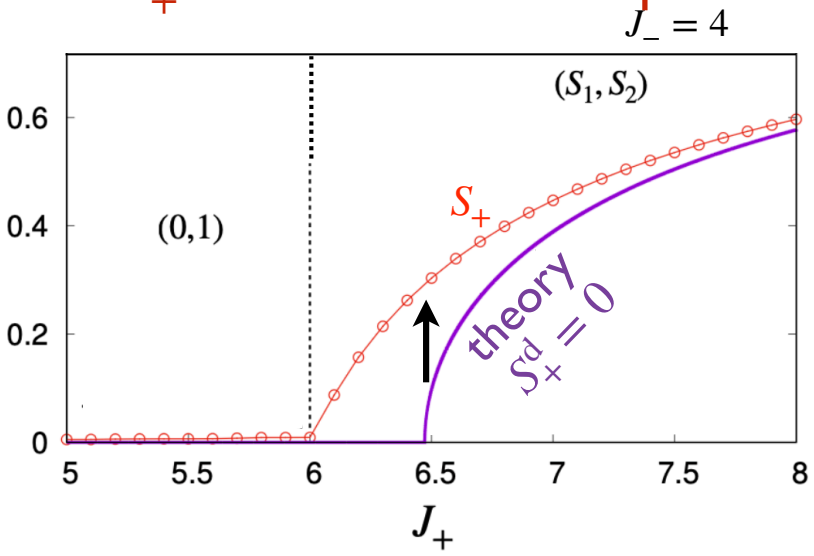
$$S_+^{\mathrm{d}} = \int_{|\omega|>\omega_c} d\omega g(\omega) \int_{-\pi}^{\pi} dX \cos X \; P_\omega^{\mathrm{d}}(X)$$

$$\therefore P_\omega^{\mathrm{d}}(X) \neq P_\omega^{\mathrm{d}}(\pi - X)$$

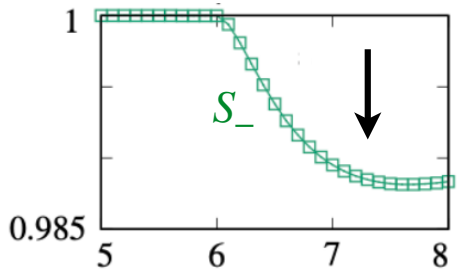
$$= c_1 S_+ - c_2 S_+^2 + O(S_+^3) > 0 \qquad \text{for } S_+ \ll 1$$

$$c_1 = \frac{J_-^2}{2J_+(J_+ + 2\gamma)} \qquad c_2 = \frac{2J_-^2(J_+^2 - J_-^2)}{3\pi\gamma J_+^3}$$

S_+ : enhanced than expected



S_- : slightly reduced



When π-reflection symmetry of pdf is broken,

→ dynamic oscillators can enhance synchronization

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Scaling behaviors

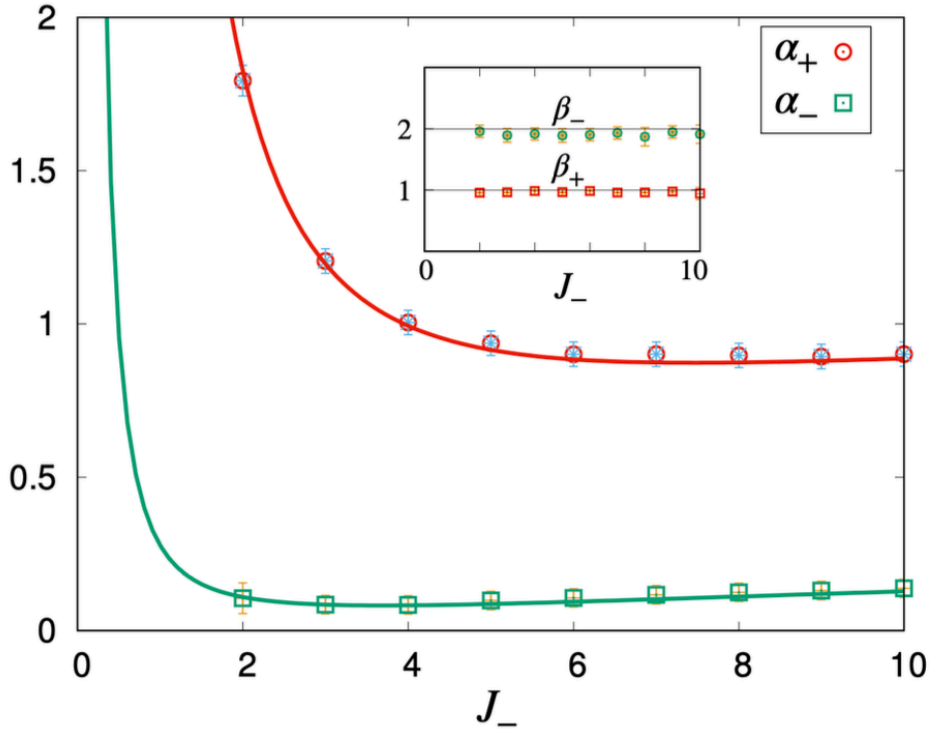
$$S_+ = \alpha_+ (J_+ - J_+^c)^{\beta_+}$$

$$\beta_+ = 1 \qquad \alpha_+ = \frac{3\pi {J_+^c}^3 (J_+^c - \gamma)}{8\gamma (J_+^c + 2\gamma)^2 ({J_+^c}^2 - 16\gamma^2)}$$

$$J_+^c = \gamma + \sqrt{J_-^2 + 9\gamma^2}$$

$$S_- = 1 - \alpha_- (J_+ - J_+^c)^{\beta_-}$$

$$\beta_- = 2 \qquad \alpha_- = \frac{J_-^2}{4J_+(J_+ + 2\gamma)} \alpha_+^2$$



Conclusions

1. We find two novel dynamical modes in the swarmalator model:
   - bounded oscillation
   - full rotation with broken π-reflection symmetry

2. These two dynamic modes contribute to order parameters.

3. We explicitly estimate the critical point and scaling behaviors.

Hong, Lee, and Park, appear soon in arXiv

Collaborators



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