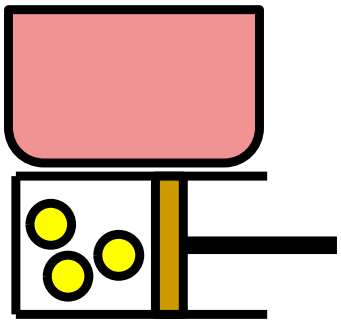
Recent theoretical advances on integrability and non-integrability in quantum many-body systems

Naoto Shiraishi (University of Tokyo)

N. Shiraishi, Europhys. Lett. 128 17002 (2019)
M. Yamaguchi, Y. Chiba, N. Shiraishi, arXiv:2411.02162 (accepted to Phys. Rev. Res.)
M. Yamaguchi, Y. Chiba, N. Shiraishi, arXiv:2411.02163 (accepted to Phys. Rev. B)
N. Shiraishi, J. Stat. Phys. 192, 170 (2025)
N. Shiraishi and M. Yamaguchi, Phys. Rev. B 113, L241111 (2026)
M. Sanatani, N. Shiraishi, and F. Ishii, in preparation

My interest (self-introduction)

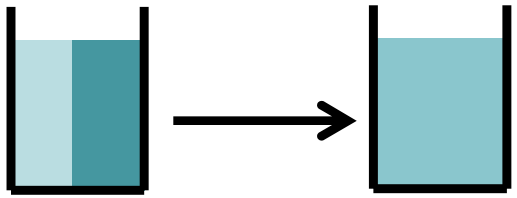
Stochastic thermodynamics



Trade-off between efficiency and power

NS, K. Saito, and H. Tasaki, PRL 117, 190601 (2016).
NS, K. Funo, and K. Saito, PRL 121, 070601 (2018)

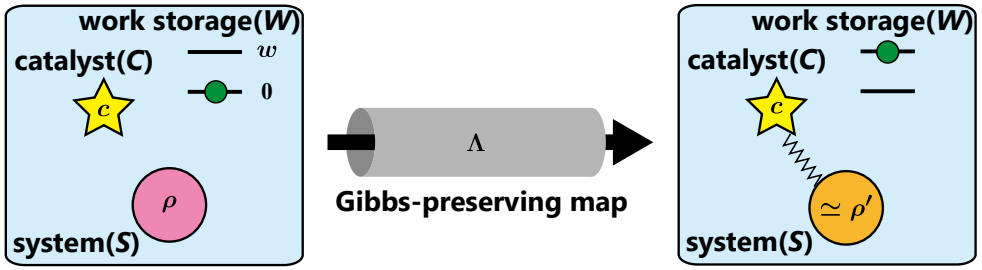
Quantum thermalization



Violation of ETH

NS and T. Mori, PRL 119, 030601 (2017)
NS and K. Matsumoto, Nat. Comm. 12, 5084 (2021)

Quantum thermodynamics



NS and T. Sagawa, PRL 126, 150502 (2021)
NS and R. Takagi, PRL 132, 180202 (2024)
NS and R. Takagi, arXiv:2501.05642

(Non-)integrability

$$\begin{array}{cccc} X & X & Z & \\ & & Y & Y \\ \hline -2 & X & X & X & Y \end{array}$$

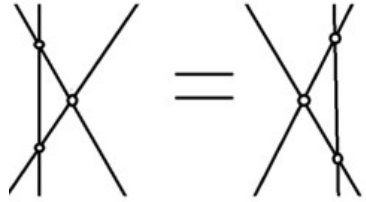
Proving non-integrability rigorously

This talk!

Quantum integrability

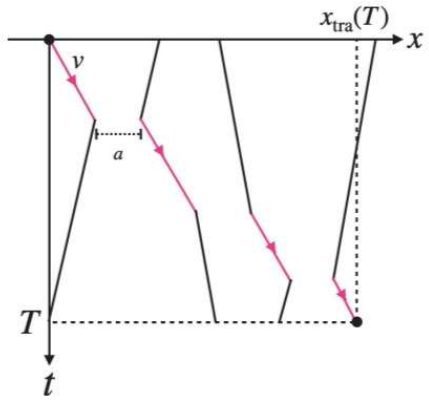
Quantum integrable systems have been studied for a long time:

- Various models are solved by the Yang-Baxter equation.



(Wikipedia “Yang-Baxter equation”)

- Dynamics of integrable systems are also investigated (relaxation to generalized Gibbs-ensemble, generalized hydrodynamics, etc)



(T. Yoshimura, JPS article, 79, 540 (2024))

Open problems on quantum integrability

On the other hand, many fundamental problems in quantum integrability have left unresolved.

- What is **quantum integrability**? (Yang-Baxter solvability vs local conserved quantities)
- How to **prove** non-integrability of a given model?
- How to **test** integrability of a given model? (Finding suitable R-matrix is an uneasy task)

Outline

Part I: Dichotomy theorem of local conserved quantities

Part II: Simple Hamiltonian criterion for Yang-Baxter integrability

Outline

Part I: Dichotomy theorem of local conserved quantities

Part II: Simple Hamiltonian criterion for Yang-Baxter integrability

Outline of Part I

Background

Result (S=1/2 spin chains)

Result (isotropic spin chains)

Outline of Part I

Background

Result (S=1/2 spin chains)

Result (isotropic spin chains)

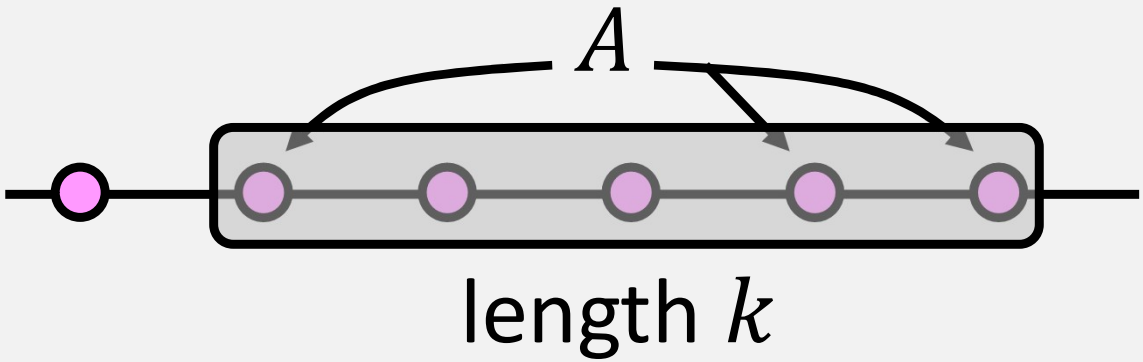
local conserved quantity approach

In this part, we focus on **nontrivial local conserved quantities** (CQ) as characterization of integrability.

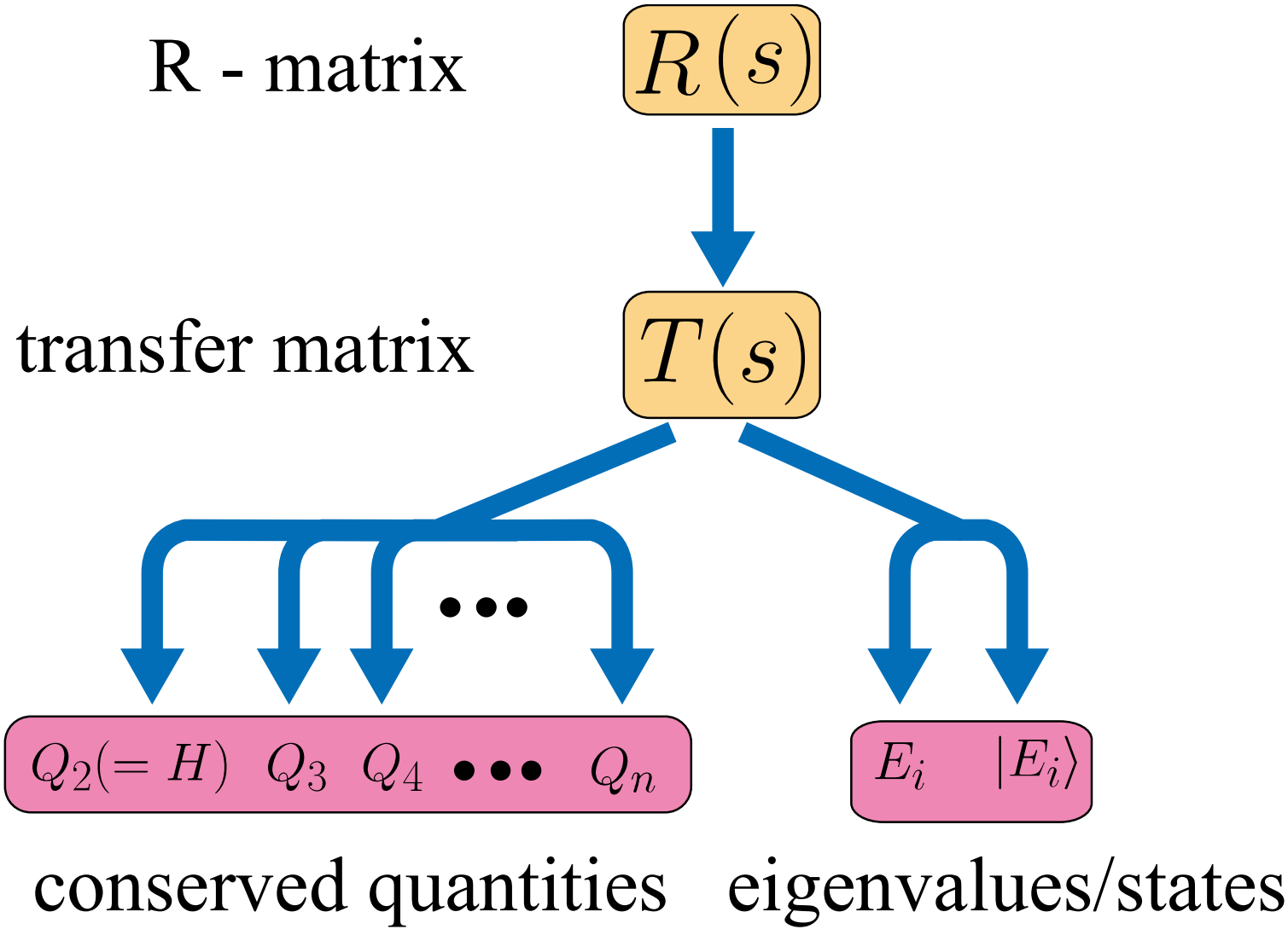
local = a sum of (spatially) local quantities.
nontrivial = Its support is larger than Hamiltonian's.

(J.-S. Caux, J. Mossel, J. Stat. Mech. P02023 (2011),
C. Gogolin and J. Eisert, Rep. Prog. Phys. 79, 056001 (2016).)

k-local operator: supported by k contiguous sites.



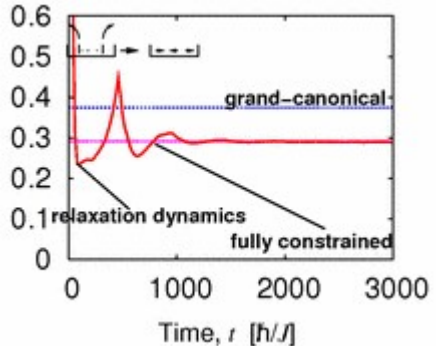
Why local conserved quantities matter in Yang-Baxter solvable systems



Why local conserved quantities matter in physics

Integrable systems show various **anomalous behaviors**

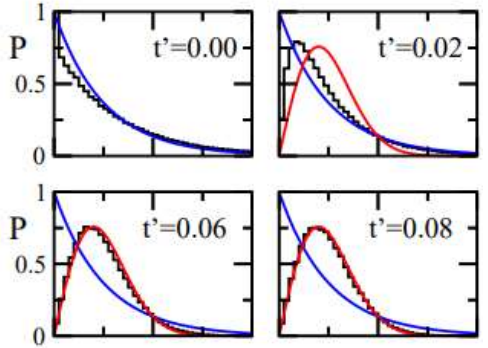
- Absence of thermalization



- Anomalous transport

(M. Rigol, et al., PRL 98, 050405 (2007))

- Level statistics
  (Poisson vs Wigner-Dyson)



(L. F. Santos and M. Rigol, PRE 81, 036206 (2010))

The existence of local conserved quantities causes these anomalous properties.

Dichotomy conjecture

All existing models satisfy this dichotomy:

Dichotomy conjecture

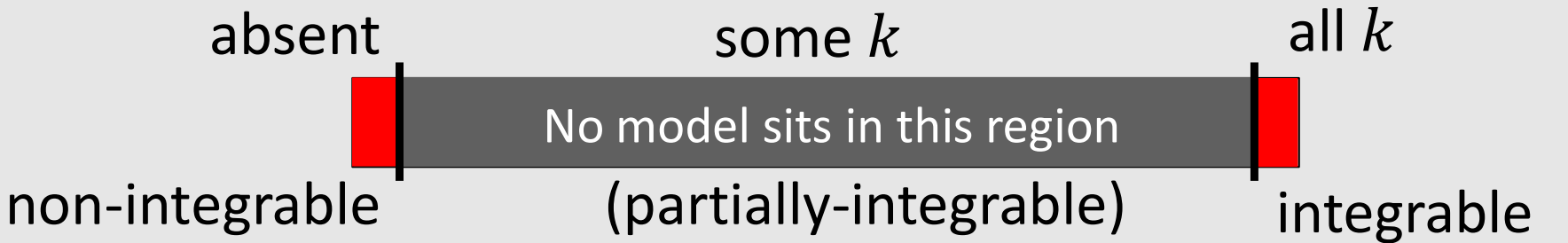
A locally-interacting system possesses

(i) k-local CQ for all k (completely integrable), or

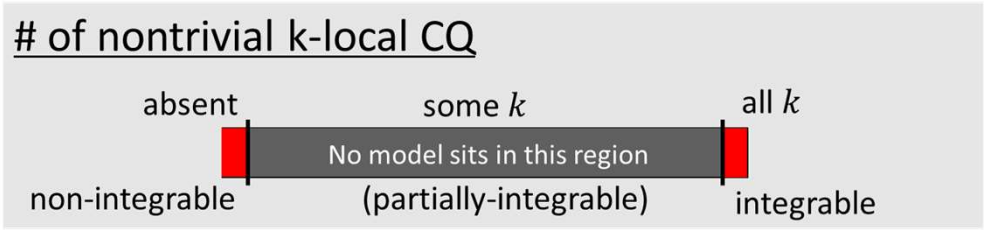
(ii) no nontrivial local CQ (non-integrable).

(Related conjectures: M. P. Grabowski and P. Mathieu, J. Phys. A: Math. Gen. 28, 4777 (1995), T. Gombor and B. Pozsgay, Phys. Rev. E 104, 054123(2021))

# of nontrivial k-local CQ



Our goal of this part



Dichotomy conjecture is highly plausible.

But, can we show this conjecture rigorously?

→**Yes!** We prove the dichotomy in

- S=1/2 chains with NN & NNN interaction
- isotropic spin chains.

Outline of Part I

Background

Result (S=1/2 spin chains)

Result (isotropic spin chains)

Classification of NN and NNN symmetric S=1/2 chain

Theorem: An $S = 1/2$ chain with Hamiltonian

$$H = \sum_{i=1}^{L} \sum_{\alpha,\beta\in\{X,Y,Z\}} J^2_{\alpha\beta}\sigma^\alpha_i\sigma^\beta_{i+2} + \sum_{i=1}^{L} \sum_{\alpha,\beta\in\{X,Y,Z\}} J^1_{\alpha\beta}\sigma^\alpha_i\sigma^\beta_{i+1} + \sum_{i=1}^{L} \sum_{\alpha\in\{X,Y,Z\}} h_\alpha\sigma^\alpha_i$$

with $J^2_{\alpha\beta} = J^2_{\beta\alpha}, \quad J^1_{\alpha\beta} = J^1_{\beta\alpha}$ is **nonintegrable** (no local CQ) **except for known integrable models** (with all k-local conserved quantities).

(NN case: M. Yamaguchi, Y. Chiba, N. Shiraishi, arXiv:2411.02162 / arXiv:2411.02163)

(NNN case: N. Shiraishi, J. Stat. Phys. 192, 170 (2025))

$S = 1/2$ chains with inversion-symmetric, nearest neighbor (NN) and next-nearest neighbor (NNN) interactions are completely classified!

Implications

This is the **first complete classification of integrable/non-integrable** for a wide class of systems

- **No missing integrable systems** in this class.
- **No intermediate (partially-integrable) systems.**
  →**Dichotomy is proven** in these classes.

List of known integrable models

Classical: $H = \sum_i \left[J^2_Z Z_i Z_{i+2} + J^1_{ZZ} Z_i Z_{i+1} + h_Z Z_i \right] .$

XY model: (+z field) $H = \sum_i [J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + h_Z Z_i]$

XXZ model: (+z field) $H = \sum_i [J_X X_i X_{i+1} + J_X Y_i Y_{i+1} + J_Z Z_i Z_{i+1} + h_Z Z_i]$

XYZ model: $H = \sum_i [J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1}]$

Bethe solvable: $H = \sum_i \left[J^2_Z Z_i Z_{i+2} + J^1_{XZ} (X_i Z_{i+1} + Z_i X_{i+1}) + h_Y Y_i \right] .$

Preliminary process for NN case: global spin rotation

Since the interaction matrix J is symmetric, we can diagonalize it by an orthogonal matrix.

$$\begin{pmatrix} J_{XX} & J_{XY} & J_{XZ} \\ J_{YX} & J_{YY} & J_{YZ} \\ J_{ZX} & J_{ZY} & J_{ZZ} \end{pmatrix} \longrightarrow \begin{pmatrix} J'_{XX} & & \\ & J'_{YY} & \\ & & J'_{ZZ} \end{pmatrix}$$

$$\begin{pmatrix} h_X \\ h_Y \\ h_Z \end{pmatrix} \longrightarrow \begin{pmatrix} h'_X \\ h'_Y \\ h'_Z \end{pmatrix}$$

We re-define J' and h' as J and h for simplicity.

Standard form

Our Hamiltonian is reduced to

$$H = \sum_i J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1}$$
$$+ h_X X_i + h_Y Y_i + h_Z Z_i$$

The treatment is qualitatively different depending on the **rank of *J*** (i.e., how many J_X, J_Y, J_Z take nonzero values)

(Here we denote Pauli matrices $\sigma^X, \sigma^Y, \sigma^Z$ by X, Y, Z)

But…

How can we prove non-integrability at all?

Proof of non-integrability: basic idea

Take **XYZ chain with z magnetic field (rank 3)** as an example. $H = \sum_i (J_X X_i X_{i+1} + J_Y Y_i Y_{i+1} + J_Z Z_i Z_{i+1} + h Z_i)$

A candidate of k-local CQ is expanded as

$$Q = \sum_i q_{XX\cdots X} X_i X_{i+1} \cdots X_{i+k-1} + q_{XX\cdots Y} X_i X_{i+1} \cdots Y_{i+k-1} + \cdots$$

$$\cdots + q_{ZI\cdots IZ} Z_i I_{i+1} \cdots Z_{i+k-1}$$

at most 4^k terms!

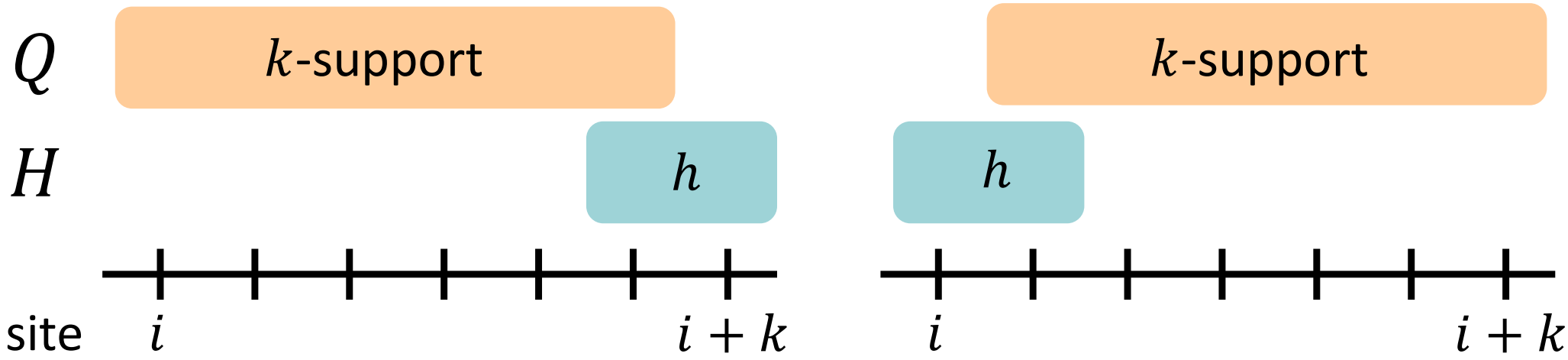
$[\boldsymbol{Q}, \boldsymbol{H}] = \boldsymbol{0}$ yields many linear relations on $q_{\boldsymbol{A}}$.

→ Using them, we show $q_{\boldsymbol{A}} = 0$ for all $\boldsymbol{A}$.

Proof of non-integrability: outline 1

Step 1: Consider $k+1$-support operator in $[Q, H]$.

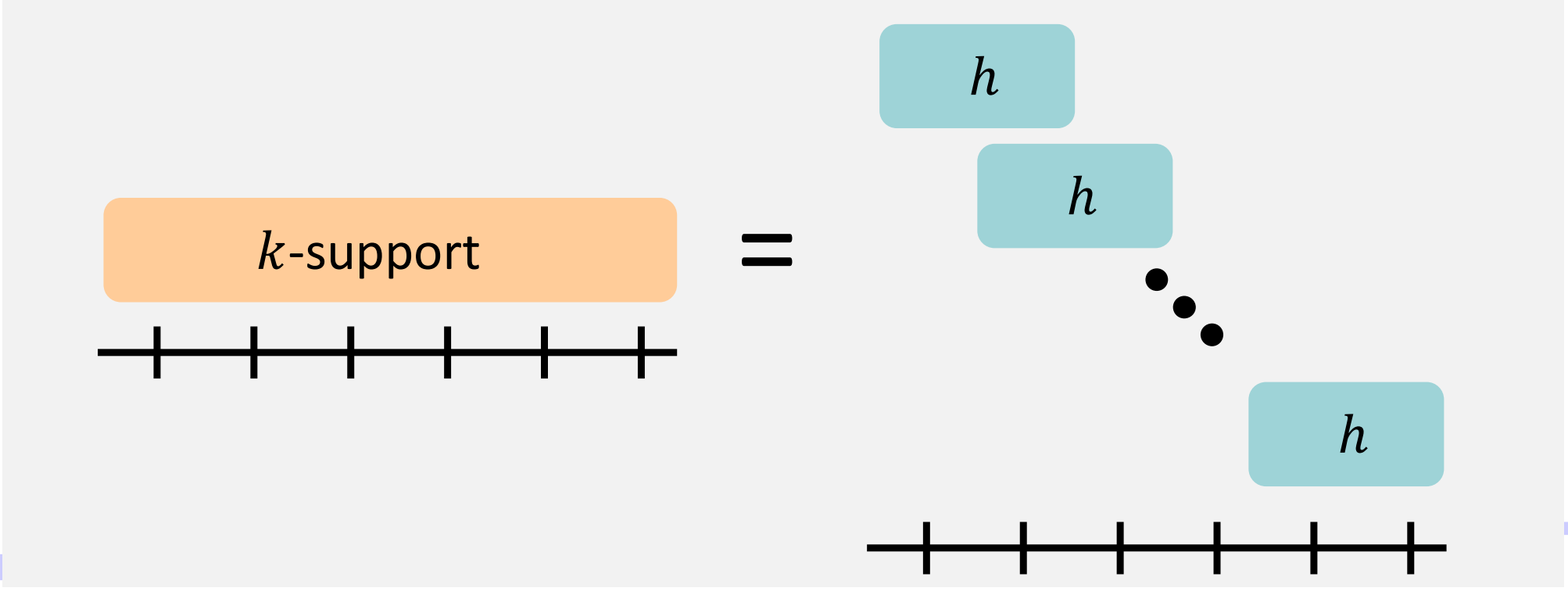
Possible commutators generating a given $k+1$-support operator are



Proof of non-integrability: outline 1

From this, we find that Q should be

$$Q = \sum_i [[\cdots [[h_{i,i+1}, h_{i+1,i+2}], h_{i+2,i+3}] \cdots], h_{i+k-2,i+k-1}]$$
$$+(\mathrm{k}-1 \text{ support terms}) + (\mathrm{k}-2 \text{ support terms}) \cdots$$



Proof of non-integrability: outline 2

Step 2: Consider k-support operator in $[Q, H]$.
We find a nontrivial relation on q_A.

Ex) $k = 3$.
We have

$$\begin{cases} h(q_{XYZ} + q_{YXZ}) + & J_Z q_{XY} = 0 \\ h(q_{YZX} + q_{XZY}) - J_Y(q_{YX} + q_{XY}) & = 0 \\ h(q_{XYZ} + q_{YXZ}) + J_Z q_{YX} & = 0 \end{cases}$$

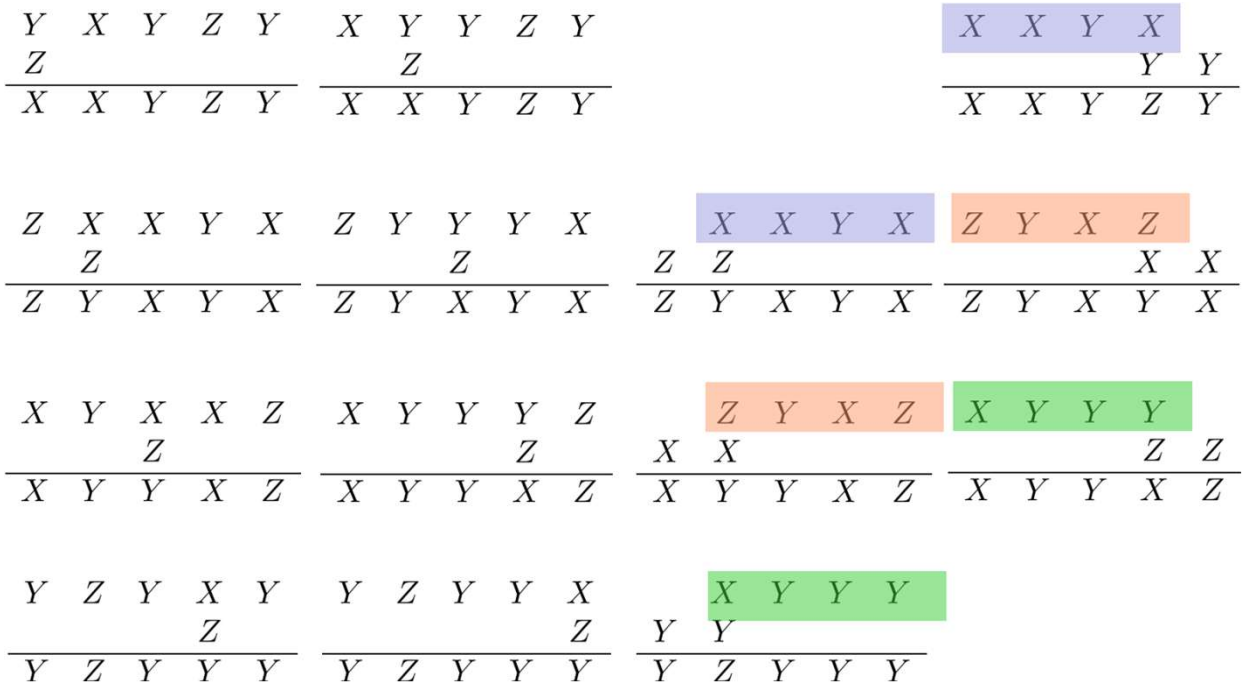
$$\Rightarrow 3h\left(1 - \frac{J_Y}{J_X}\right) q_{XYZ} = 0$$

Unless $h = 0$ (XYZ model) or $J_X = J_Y$ (XXZ model with a z-magnetic field), we have $\boldsymbol{q_{XYZ} = 0}$.

Proof of non-integrability: outline 2

Step 2: Consider k-support operator in $[Q, H]$.
We find a nontrivial relation on $q_{\boldsymbol{A}}$.

We have a good sequence for general k.



How to find this sequence?

In fact, this sequence is a **simple elongation** of the case of $k = 3$.

One of k-support op. in this sequence satisfies

$$ZXYZZ \ldots ZZZZZZZY$$
$$YZXYZZ \ldots ZZZZZZX$$
$$\vdots$$

inserted for elongation

$$YZZZ \ldots ZXYZZ \ldots ZX$$
$$\vdots$$
$$YZZZ \ldots ZZZZZZXYZ$$

S=1/2 spin chains: other cases

The case of rank 2 is treated in a similar manner to rank 3.

(M. Yamaguchi, Y. Chiba, N. Shiraishi, arXiv:2411.02162 / arXiv:2411.02163)

The case of rank 1 is treated in a slightly different manner (The role of each step is not the same).

(Y. Chiba, Phys. Rev. B 109, 035123 (2024))

The case of next-nearest-neighbor interaction is treated in a similar manner, though we need extremely many case divisions and careful computations.

(N. Shiraishi, J. Stat. Phys. 192, 170 (2025))

Outline of Part I

Background

Result (S=1/2 spin chains)

Result (isotropic spin chains)

Motivation: dichotomy as a guidepost

All-or-zero dichotomy has been proven in some classes of models by completely classify integrability/nonintegrability of models.

However, dichotomy originally aims to serve as a guidepost for studying **unexplored models**.



Can we show the dichotomy for a broad class of models including unexplored ones?

Dichotomy of isotropic spin chain

Consider NN isotropic spin chain with general S:

$$H = \sum_i h_{i,i+1}, \quad h_{i,i+1} = \sum_{n=1}^{2S} J_n (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^n$$

We introduce $D := [[h_{12}, h_{23}], h_{12} + h_{23}]$

and define D^3 as its 3-support part.

Theorem

An isotropic (i.e., SU(2) sym.) spin chain has

(i) no nontrivial local CQ (non-integrable) if $\boldsymbol{D^3 \neq 0}$,

(ii) k-local CQ for all k (integrable) if $\boldsymbol{D^3 = 0}$.

(N. Shiraishi and M. Yamaguchi, Phys. Rev. B 113, L241111 (2026))

Remarks and comments

$$D = [[h_{12}, h_{23}], h_{12} + h_{23}]$$

- This dichotomy applies **unexplored** isotropic spin chains (e.g., $S = 100$ spin chains).
- The criterion of dichotomy is extremely simple and **easy to compute** for a given model.

Our criterion is useful for investigating new integrable models and studying **quantum many-body scars** (since scar is meaningful only in non-integrable models).

Proof outline: non-integrability 1

Step 1: Considering k-support op. in $[Q, H]$, we find

$$Q = \sum_i [[\cdots [[h_{i,i+1}, h_{i+1,i+2}], h_{i+2,i+3}] \cdots], h_{i+k-2,i+k-1}]$$
$$+(\mathrm{k}-1 \text{ support terms}) + (\mathrm{k}-2 \text{ support terms}) \cdots$$

(To show this, we use the fuzzy (non-commutative) spherical harmonics as basis matrices.)

In particular, a candidate of 3-local CQ is

$$Q^3 = \sum_i [h_{i,i+1}, h_{i+1,i+2}] + (2 \text{ support terms})$$

Proof of non-integrability: Step 2

Step 2: It suffices to consider the case of $k = 3$
(General k is reduced to $k = 3$ (A. Hokkyo, arXiv:2501.18400))

Since H is SU(2) sym. and $[Q^3, H] = 0$, Q^3 is SU(2) sym. and

$$Q^3 = \sum_i [h_{i,i+1}, h_{i+1,i+2}] + (2 \text{ support terms})$$

$[h_{i,i+1}, h_{i+1,i+2}]$: time anti-sym.

$(2 \text{ support terms})$ is also SU(2) sym.

$$\sum_{n=1}^{2S} c_n (\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^n$$

$(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^n$: time sym.

These two live in different sectors.
$\rightarrow D^3 \neq 0$ means $[Q^3, H] \neq 0$.

Proof of integrability

Let $B \coloneqq \sum_i i h_{i,i+1}$ be a boost operator.
We recursively define $Q^{k+1} := [B, Q^k]$. $(Q^2 = H)$
(Q^k is k-support operator)

We show that $D^3 = 0$ means $[Q^3, H] = [Q^4, H] = 0$.

It is recently shown that $[Q^3, H] = 0 \rightarrow [Q^k, H] = 0$.

(A. Hokkyo, arXiv:2508.20713: Explain in the next part)

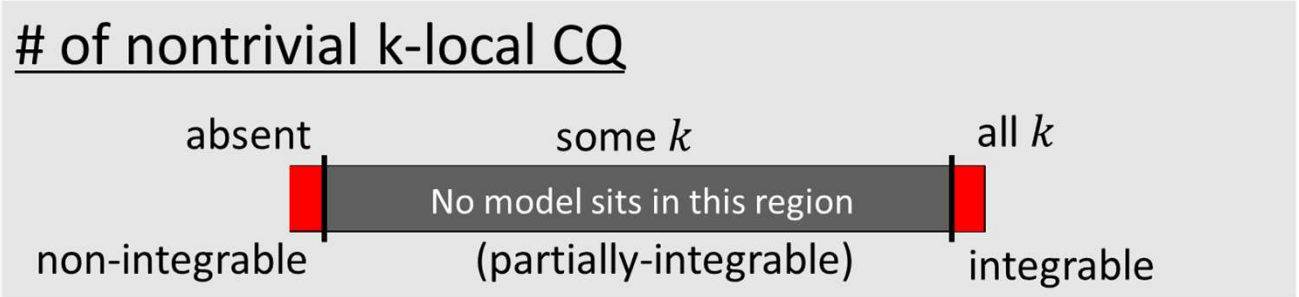
Hence, $D^3 = 0$ means $[Q^k, H] = 0$ for all k.

Summary of Part I

- We establish dichotomy of no nontrivial local CQ or all k-local conserved CQ in
  ① S=1/2 chains with NN and NNN interaction
  ② NN isotropic spin chain.

- We completely classify int./non-int. for ①.

- The criterion for ② is very simple.



Outline

Part I: Dichotomy theorem of local conserved quantities

Part II: Simple Hamiltonian criterion for Yang-Baxter integrability

Outline of Part II

Background

Main result

Proof outline

Outline of Part II

Background

Main result

Proof outline

What is quantum integrability? solvability vs charges

Solvability

charges

Hamilton-Jacobi equation

classical



Liouville-Arnold theorem

first integrals

Yang-Baxter equation(YBE)

quantum



?

local conserved quantities

How Yang-Baxter equation solves a given model?

Find ***R*-matrix** satisfying this Yang-Baxter equation

$$R_{12}(s)R_{13}(s+t)R_{23}(t) = R_{23}(t)R_{13}(s+t)R_{12}(s)$$

Construct transfer matrix

$$T(s) := \mathrm{Tr}_a[R_{L,a}(s)R_{L-1,a}(s)\cdots R_{2,a}(s)R_{1,a}(s)]$$

Compute logarithmic derivatives

$$Q^{T;n} = \frac{d^{n-1}}{ds^{n-1}} \ln T(s)\Big|_{s=0} .$$

We set $Q^{T;2} = \boldsymbol{H}$.

From Hamiltonian to R-matrix

Find ***R*-matrix** satisfying this Yang-Baxter equation

$$R_{12}(s)R_{13}(s+t)R_{23}(t) = R_{23}(t)R_{13}(s+t)R_{12}(s)$$

Construct transfer matrix

$$T(s) := \mathrm{Tr}_a[R_{L,a}(s)R_{L-1,a}(s)\cdots R_{2,a}(s)R_{1,a}(s)]$$

Compute logarithmic derivatives

$$Q^{T;n} = \frac{d^{n-1}}{ds^{n-1}} \ln T(s)\Big|_{s=0}. \quad \text{We set } Q^{T;2} = \boldsymbol{H}.$$

As clearly seen, it is uneasy to find a proper R-matrix from a given Hamiltonian H.

A simple criterion to **determine whether a given H is Yang-Baxter integrable or not** is desirable.

Reshetikhin condition: a simple necessary condition for YBE

$$R_{12}(s)R_{13}(s+t)R_{23}(t) = R_{23}(t)R_{13}(s+t)R_{12}(s)$$

Expand R-matrix as (Π: swap operator)

$$\Pi R(s) = I + sh + s^2\check{R}^{(2)} + s^3\check{R}^{(3)} + \cdots$$

Seek $\check{R}^{(n)}$ by inserting this to YBE and comparing each order of s, t.

coefficients of st: $2(\check{R}^{(2)}_{23} - \check{R}^{(2)}_{12}) = h^2_{23} - h^2_{12},$

coefficients of st^2:

$$6(\check{R}^{(3)}_{12} - \check{R}^{(3)}_{23}) + [h_{12} + h_{23}, [h_{12}, h_{23}]] - h^3_{12} + h^3_{23} = 0.$$

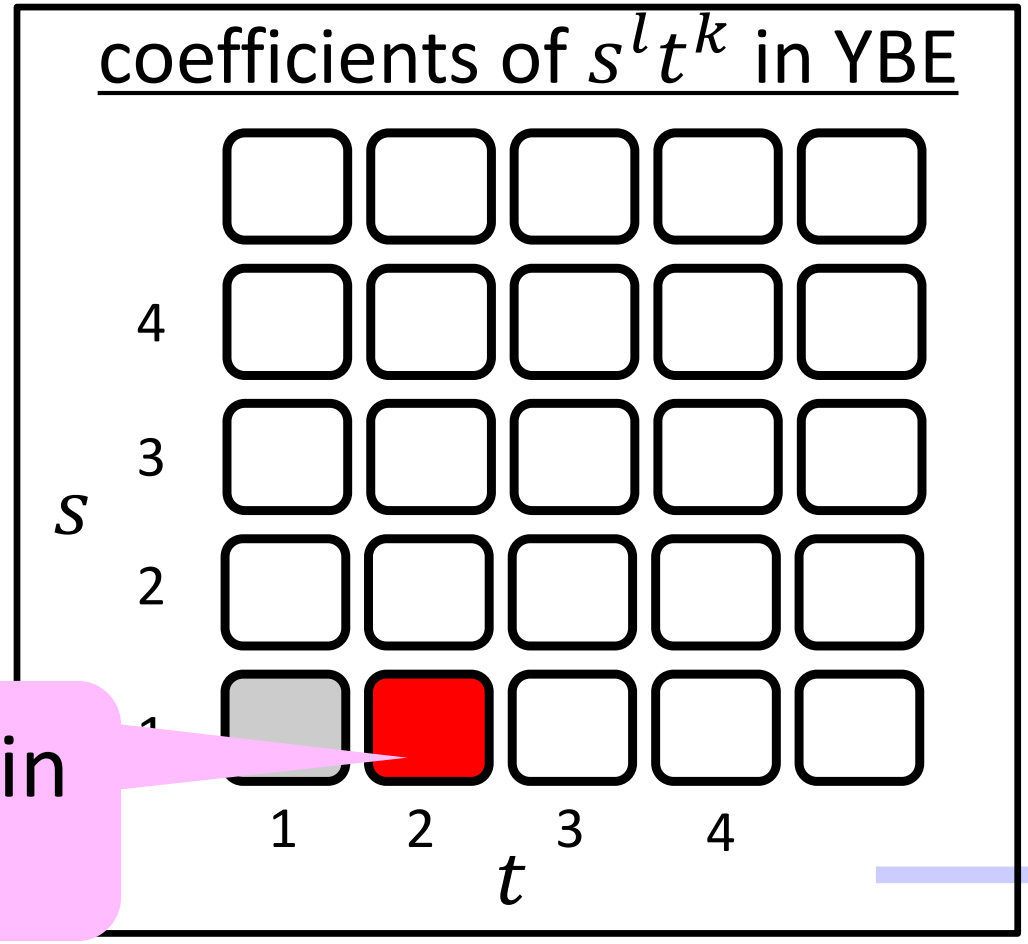
This is the first nontrivial constraint on h!

Reshetikhin condition: a simple necessary condition for YBE

Reshetikhin condition

$$[[h_{12}, h_{23}], h_{12} + h_{23}] = X_{23} - X_{12}$$

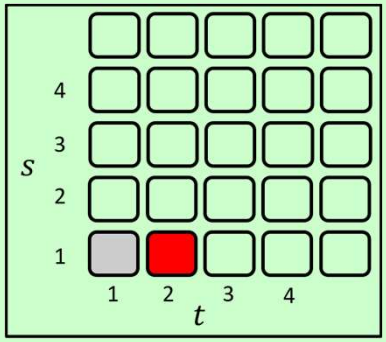
Of course, this is only a **necessary condition** for YBE.



Note: many faces of Reshetikhin condition

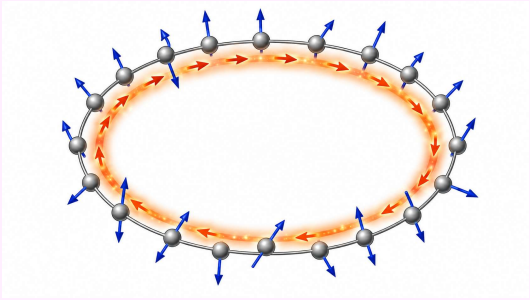
Lowest-order relation of YBE

$$[[h_{12}, h_{23}], h_{12} + h_{23}] = X_{23} - X_{12}$$



Conservation of global heat current

$$[J_E, H] = 0$$



Conservation of Q^3 generated by boost operator

$$[Q^3, H] = 0$$

Reshetikhin condition is also **sufficient** for YBE?

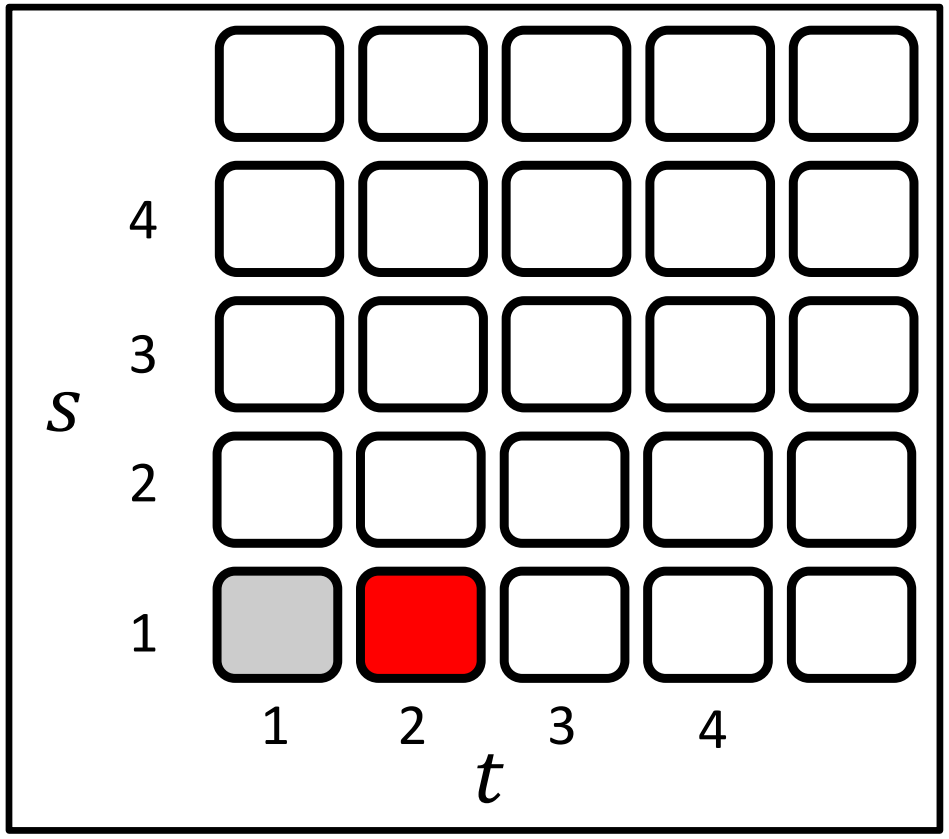
Of course, this is only a **necessary condition** for YBE. However, curiously, all known models satisfying the Reshetikhin condition are finally shown to be Yang-Baxter solvable.

Conjecture (tangential star-triangle hypothesis)
If a Hamiltonian satisfies the Reshetikhin condition, then it accompanies R-matrix with YBE.

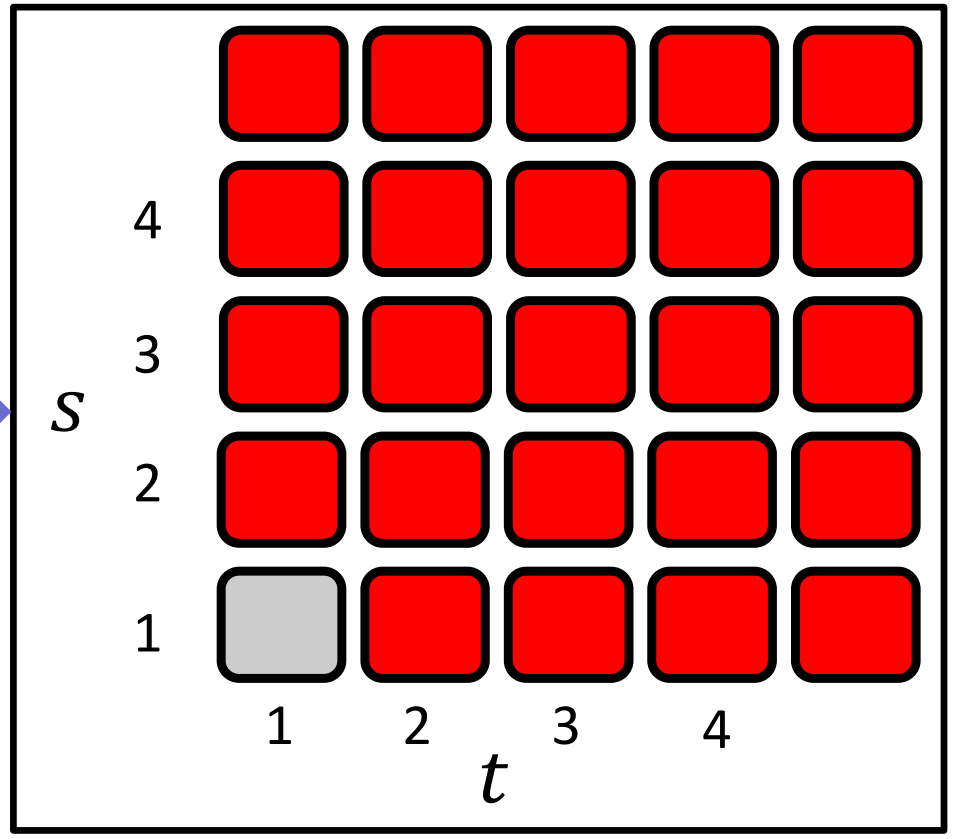
(P. P. Kulish and E. K. Sklyanin, J. Sov. Math. 19, 1596 (1982),
H.C. Öttinger and J. Honerkamp, Phys. Lett. A 88, 339 (1982),
M. Jimbo and T. Miwa, Lett. Math. Phys. 8, 529 (1984),
M. de Leeuw et al., SciPost Phys. 11, 069 (2021),
L. Corcoran and M. de Leeuw, SciPost Phys. Core 7, 045 (2024),
A. Hokkyo, Phys. Rev. B 114, 055108 (2026))

What this conjecture means?

Reshetikhin condition



Yang-Baxter equation



These two have the same information???

Outline of Part II

Background

Main result

Proof outline

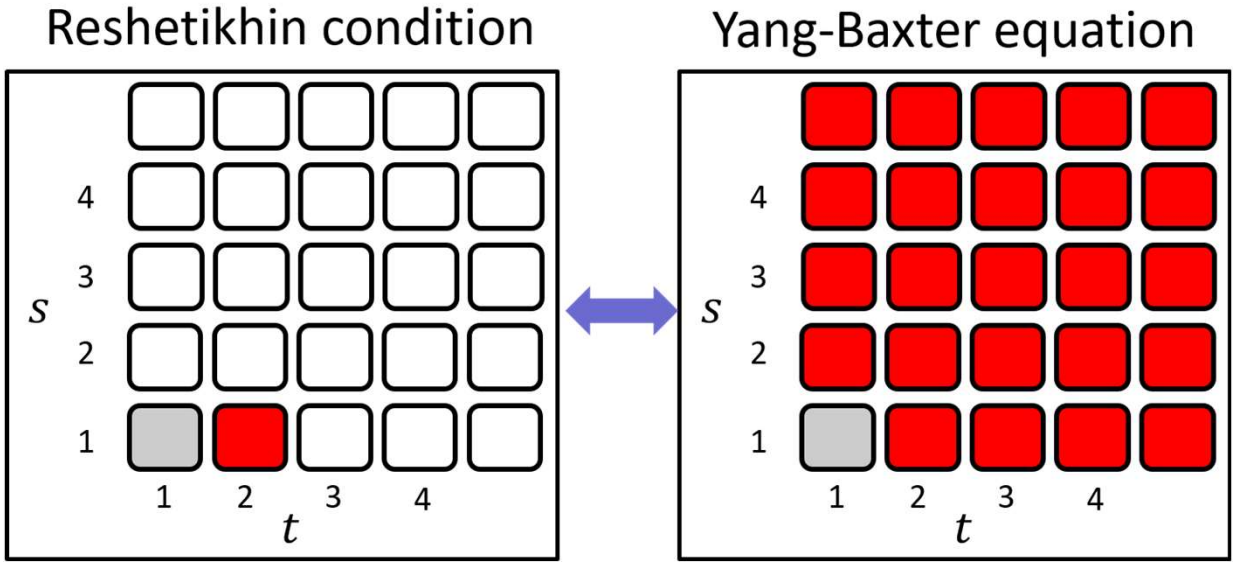
Reshetikhin and YBE are equivalent!

We solve this conjecture in the affirmative!

Theorem

If a Hamiltonian satisfies the Reshetikhin condition, then it accompanies R-matrix with YBE.

(M. Sanatani, N. Shiraishi, and F. Ishii, in preparation)

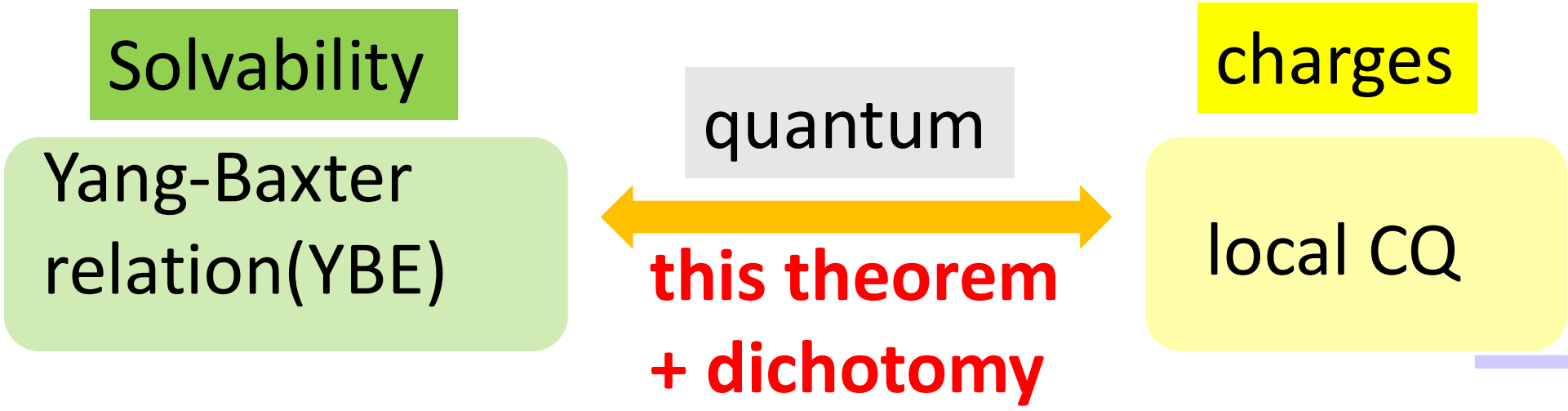


Quantum Liouville-Arnold theorem

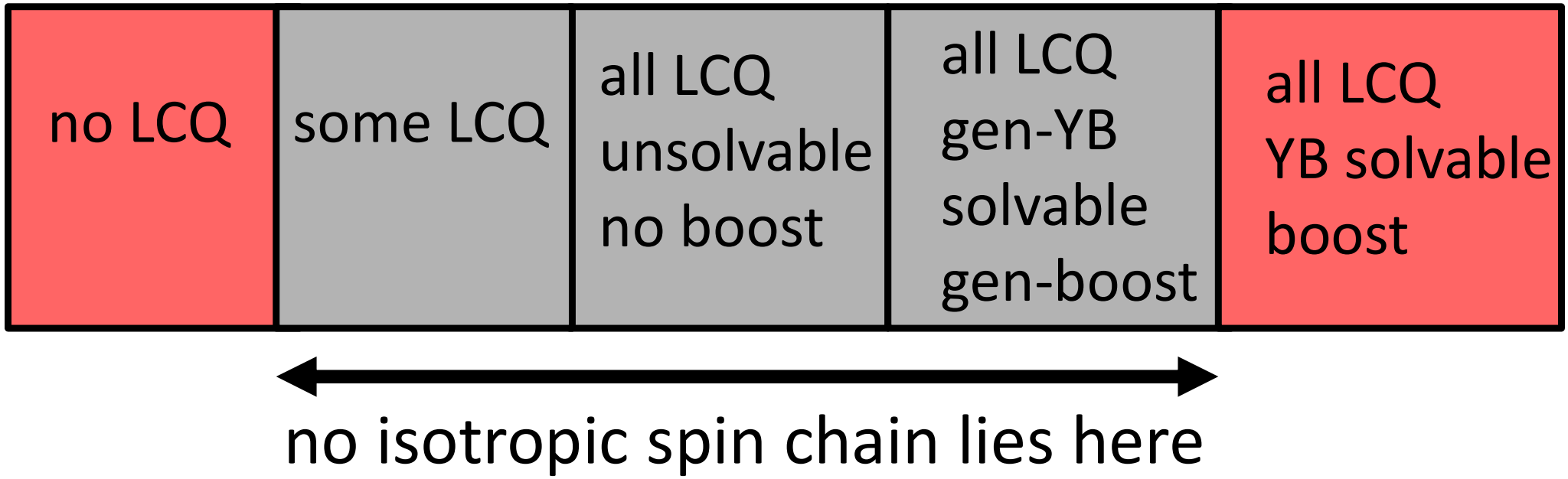
Combining with the dichotomy theorem, we have

Corollary: In isotropic spin chains, these two are equivalent:

1. Solvable by the **Yang-Baxter equation**.
2. Having **all local conserved quantities**.



Sharp dichotomy in isotropic spin chain

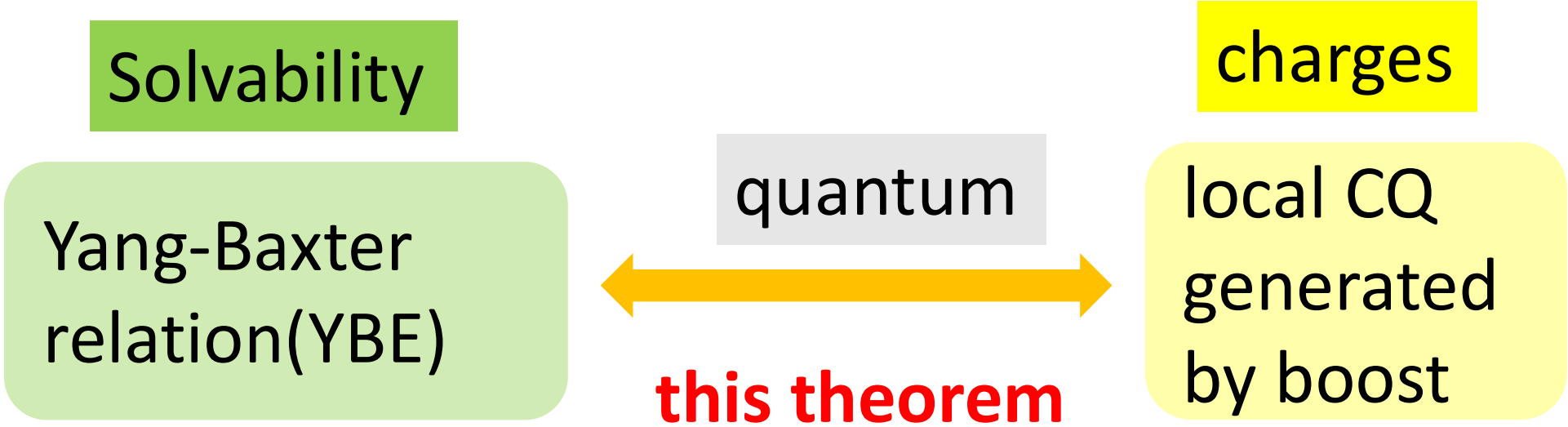


Some integrable models (e.g., Hubbard model) admit only generalized boost and generalized (non-difference-form) YBE.

Quantum Liouville-Arnold theorem (general but slightly weak)

Corollary: These two are equivalent.

1. Solvable by the **Yang-Baxter equation**.
2. Having **all local conserved quantities** generated by the boost operator $B = \sum_j j\, h_{j,j+1}$.



Advantage for numerical search of integrable models

Previous search of integrable models

Search of parametrized R-matrix satisfying the YBE

$$R_{12}(s)R_{13}(s+t)R_{23}(t) = R_{23}(t)R_{13}(s+t)R_{12}(s)$$

(S. Majumder, and E. Sobko, arXiv:2503.10469)

Search space is very large. Hard task

Search based on our theorem

Direct search of Hamiltonian h satisfying

$$[[h_{12}, h_{23}], h_{12} + h_{23}] = X_{23} - X_{12}$$

Tractable search!

Various fundamental implications

- The first-order YBE (Reshetikhin condition) and full YBE have the same information. (What structure lies behind it?)

- This theorem explains why the YBE has such a privilege status in quantum spin chains.

- **Solvability is experimentally verifiable** by measuring a single tractable observable (global heat current).

Outline of Part II

Background

Main result

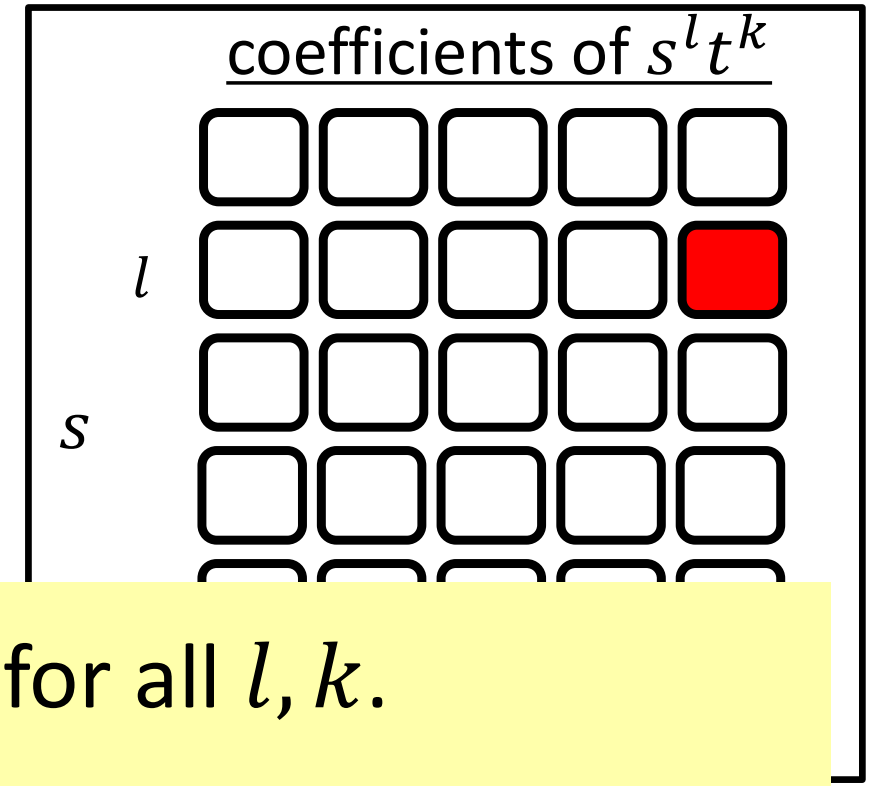
Proof outline

Proof strategy

For a general R, YBE-type relation has corrections:

$$R_{12}(s)R_{13}(s+t)R_{23}(t)-R_{23}(t)R_{13}(s+t)R_{12}(s) = \sum_{k,l} F_{123}^{l,k} s^l t^k$$

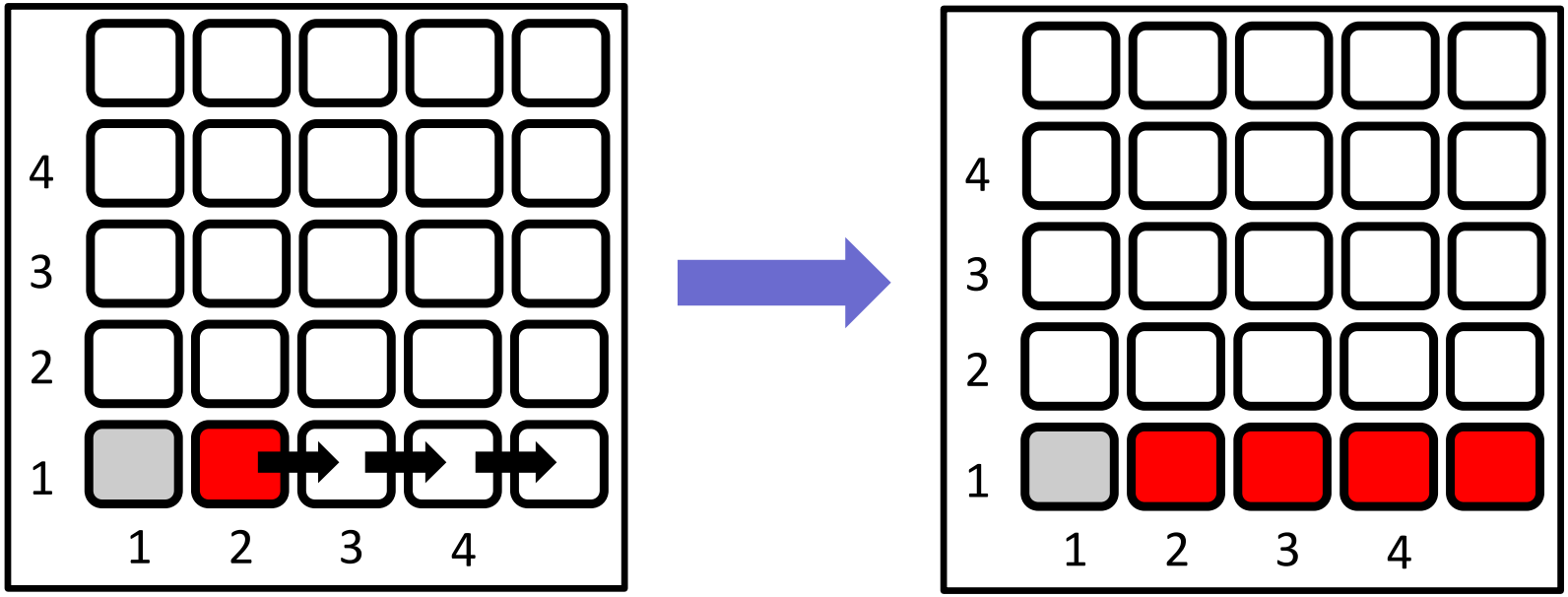
$$F_{123}^{l,k} = 0 \quad \Longleftrightarrow$$



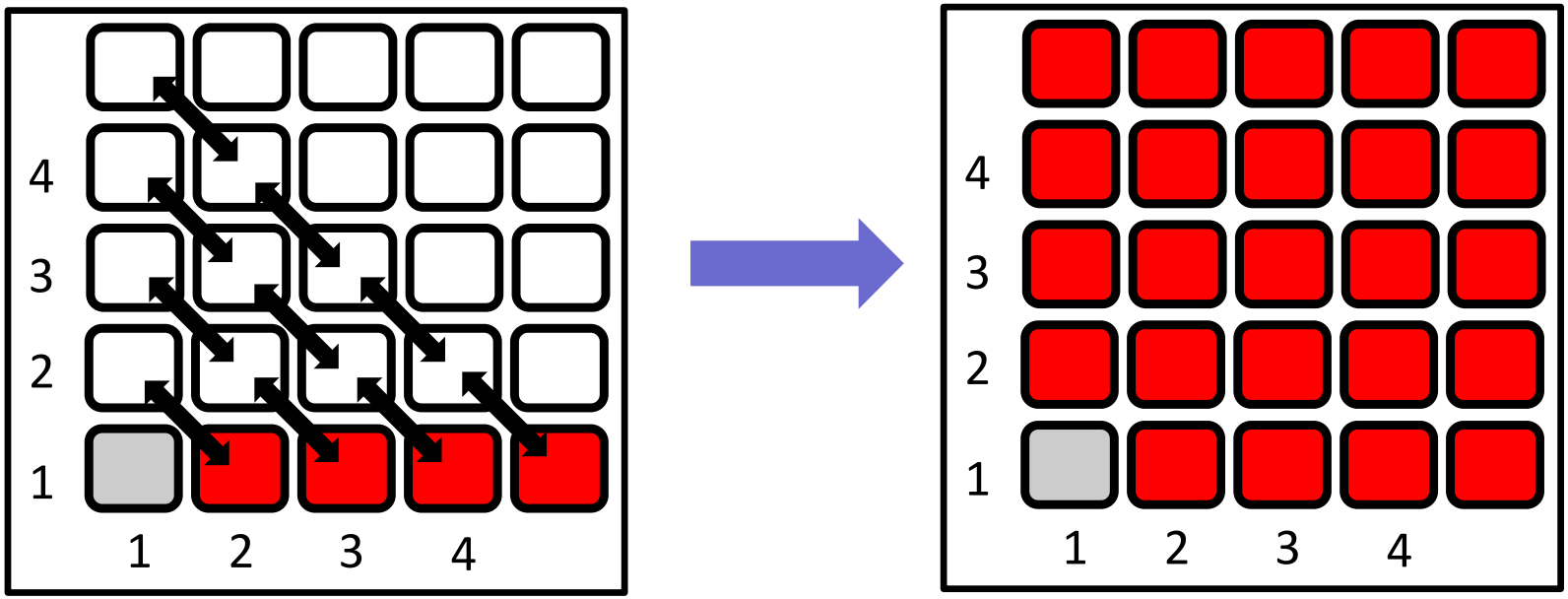
Our goal is to prove $F_{123}^{l,k} = 0$ for all l, k.

Proof strategy

step 1



step 2



Proof strategy

Through step 1, all $\check{R}^{(n)}$ in

$$\Pi R(s) = I + sh + s^2\check{R}^{(2)} + s^3\check{R}^{(3)} + \cdots$$

are determined so that $F_{123}^{1,k} = 0$ for all k.
(i.e., the reconstruction of R is completed)

In step 2, we ensure that this R is consistent by showing

$$F_{123}^{l,k} \propto F_{123}^{1,l+k-1}$$

In this talk, we explain only step 1.

Crash course on transfer-matrix charges and boost charges

We have two charges; Q^T and Q^B:

Transfer-matrix charges

$$Q^{T;n} = \left. \frac{d^{n-1}}{dt^{n-1}} \ln T(t) \right|_{t=0}.$$

Boost charges

$$Q^{B;n+1} := [B, Q^{B;n}].$$

with $B := \sum_i i h_{i,i+1}$.

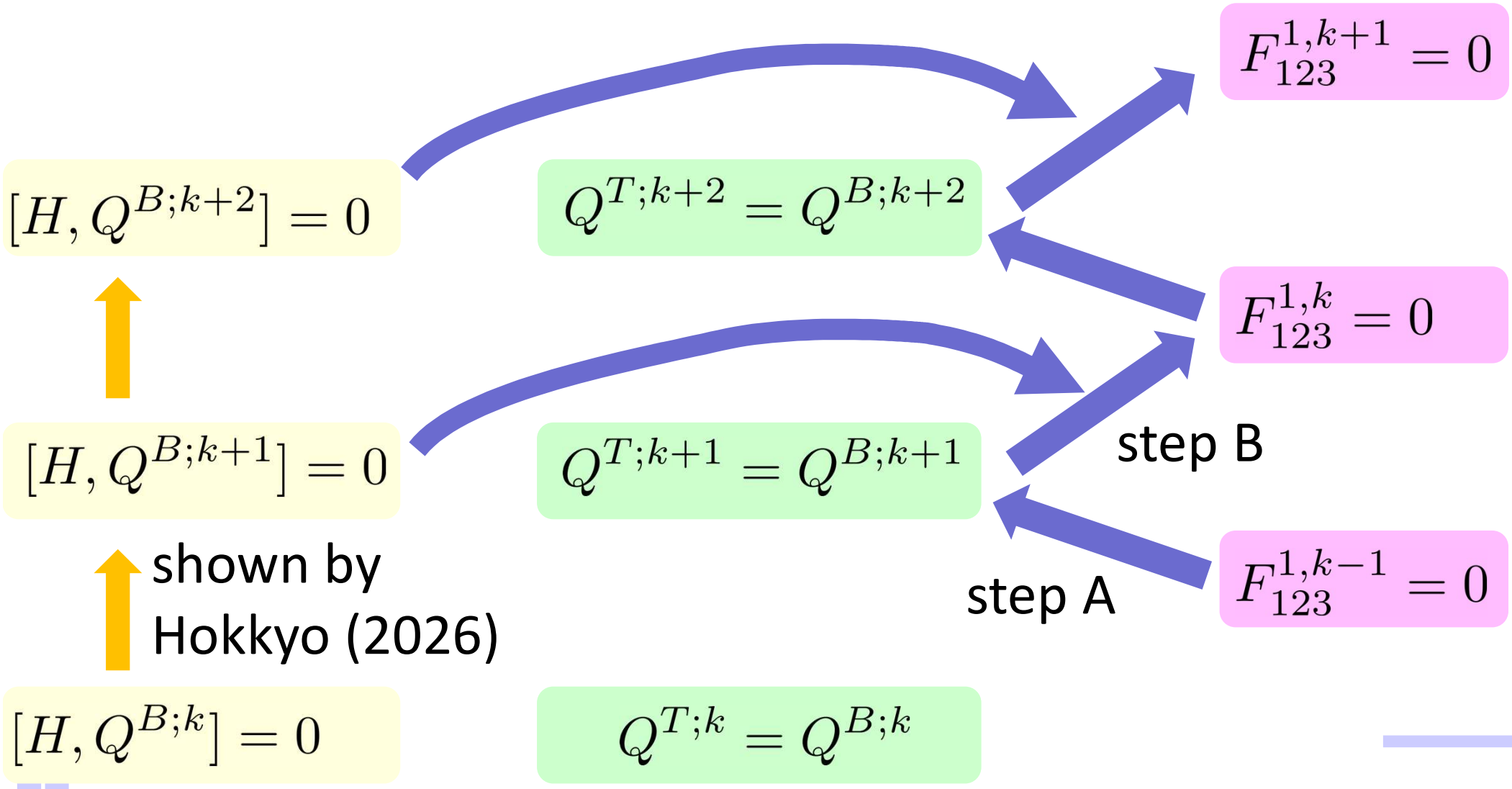
With YBE, we can show $Q^{T;n} = Q^{B;n}$ recursively.

(proof outline) YBE suggests $[B, \ln T(t)] = \frac{d}{dt} \ln T(t)$.

Using this, we can show $[B, Q^{T;n}] = Q^{T;n+1}$.

Recursive proof of step 1

We do not have YBE initially, but we reconstruct it as



Proof of step A

Suppose $F_{123}^{1,n-1} = 0$. Then, YBE with corrections reads

$$[B, \ln T(t)] = \frac{d}{dt} \ln T(t) + O(t^n).$$

$$\begin{aligned} Q^{T;n+1} &= \left. \frac{d^n}{dt^n} \ln T(t) \right|_{t=0} = \left. \frac{d^{n-1}}{dt^{n-1}} \left([B, \ln T(t)] + O(t^n) \right) \right|_{t=0} \\ &\cdots = \left. \underbrace{[B, \cdots [B}_{n-1 \text{ copies}}, \frac{d}{dt} \ln T(t)] \cdots] + O(t) \right|_{t=0} \\ &= Q^{B;n+1} \end{aligned}$$

Conservation of Q^B (shown by Hokkyo)

We show it recursively (i.e., $Q^{B;m}$ implies $Q^{B;m+1}$).

Using the Jacobi identity, we have

$$[H,[H,Q^{B;m+1}]] = [H,[H,[B,Q^{B;m}]]] = [H,[Q^{B;m},[B,H]]]$$
$$= -[Q^{B;m},[Q^3,H]] - [Q^3,[H,Q^{B;m}]] = 0.$$

Let $|\psi_i\rangle$ be eigenstates of H with eigenvalue E_i.

$$\langle\psi_i|\,[H,[H,Q^{B;m+1}]]\,|\psi_j\rangle = (E_i - E_j)^2\,\langle\psi_i|Q^{B;m+1}|\psi_j\rangle\,.$$

Thus, $Q^{B;m+1}$ has no off-diagonal elements
$\rightarrow [H,Q^{B;m+1}] = 0$

(A. Hokkyo, Phys. Rev. B 114, 055108 (2026). The present proof is based on N. Shiraishi and M. Yamaguchi, Phys. Rev. B 113, L241111 (2026))

Proof of step B

We evaluate t^n terms in $[T(t), H]$ in two ways.

(i) Suppose $F_{123}^{1,n-1} = 0$. Similarly to step A, YBE with corrections reads

$$([T(t), H])_{t^n} \propto \sum_i \check{F}_{i+2,i+1,i}^{1,n}$$

(ii) Since

$$([T(t), H$$

$$= [(\text{polynomial of } Q^{D,2}, \ldots, Q^{D,n+1}), H] = 0.$$

Since $\check{F}$ is a telescopic sum, we can erase it by choosing proper $\check{R}^{(n)}$ as $\check{F}_{123} = \check{R}_{23}^{(n)} - \check{R}_{12}^{(n)}$.

Summary of Part II

- Reshetikhin condition (a simple Hamiltonian condition) is equivalent to the Yang-Baxter equation.

- This bridges solvability and local conserved quantities: the quantum Liouville-Arnold theorem.

- Solvability is experimentally verifiable (global heat current conservation).

Summary

- We establish the all-or-zero dichotomy in
  ① S=1/2 chains with NN and NNN interaction
  ② NN isotropic spin chain.

- We establish that the Reshetikhin condition is equivalent to the Yang-Baxter integrability.

- Solvability and local conserved quantities are connected.

END