Unified Hierarchy of Fluctuation-Response Identities & Bounds in Nonequilibrium Markovian Dynamics

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Main reference: Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)

Chun, Kwon, Park, JSL, arXiv:2601.16387

Kwon, Chun, Park, JSL, arXiv:2605.05038

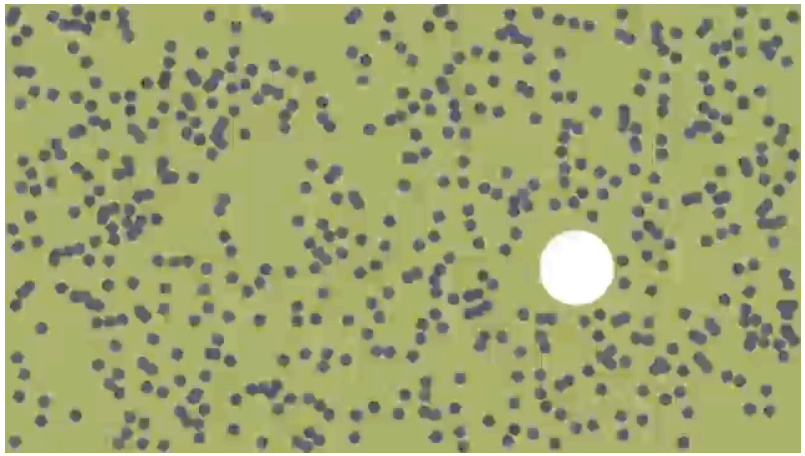
in collaboration with Hyunggyu Park (KIAS)

Hyun-Myung Chun (KIAS)

Euijoon Kwon (KIAS)

Fluctuation - Dissipation Relations

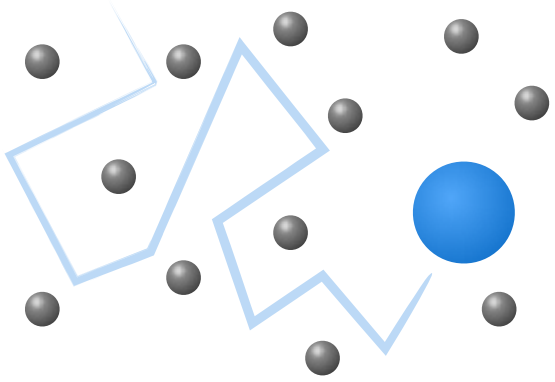
Brownian particle in a thermal environment



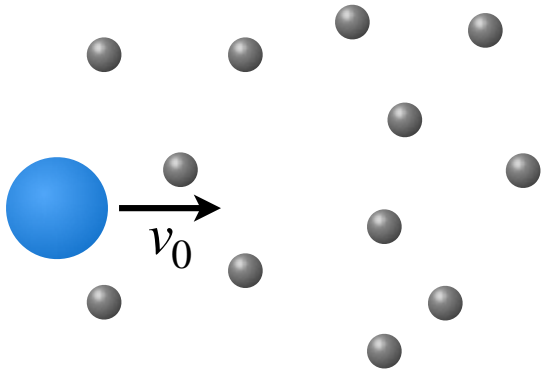
https://www.youtube.com/watch?v=ZNzoTGv_XiQ

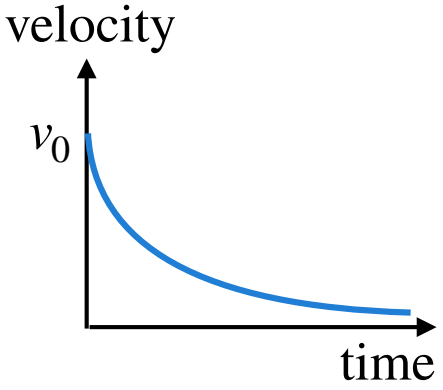
Effects of a thermal environment

1. Stochastic motion (fluctuation)



2. Dissipation

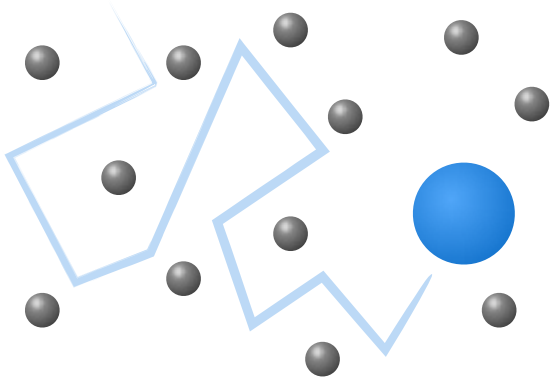




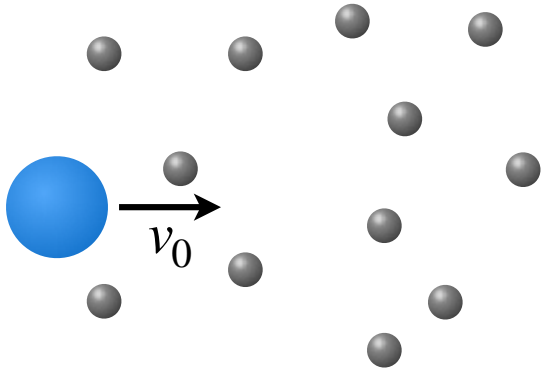
Fluctuation - Dissipation - Response Relations

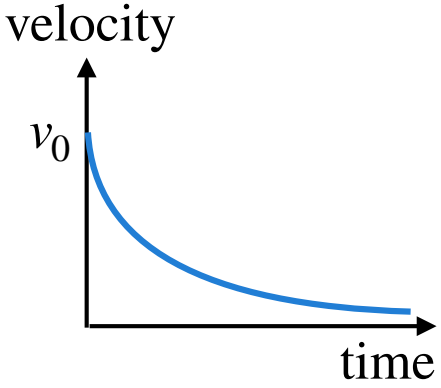
Effects of a thermal environment

1. Stochastic motion (fluctuation)

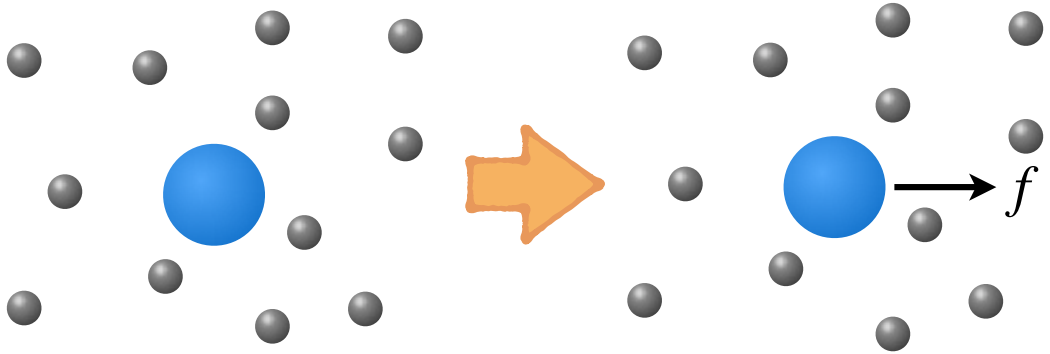


2. Dissipation





3. Response to a perturbation



$\langle v \rangle_{\mathrm{ss}} = 0$ $\qquad$ $\langle v \rangle_{\mathrm{ss}} = \mu f$

μ : mobility, determined from bath properties

Fluctuation - Dissipation - Response Relations

Effects of a thermal environment

1. Stochastic motion (fluctuation)

2. Dissipation

3. Response to a perturbation → These have the same origin.

There must be relations between them.

In equilibrium

Their mathematical relationships (linear response) are well established.

fluctuation

response $\mu = \dfrac{1}{k_B T}\int_0^\infty \langle v(t)v(0)\rangle_{eq} dt = \dfrac{1}{\gamma}$ dissipation

Green-Kubo

$\langle J \rangle = Lf \qquad L = \alpha \int_0^\infty \langle J(t)J(0)\rangle_{eq} dt$

For a free Brownian particle in Langevin dynamics

Fluctuation - Dissipation - Response Relations in Nonequilibrium

Previous studies

1. Harada-Sasa relation:

Harada and Sasa, PRL 95, 130602 (2005)

$$\langle J\rangle_0 = \gamma\left\{ v_s^2 + \int_{-\infty}^{\infty} \left[\tilde{C}(\omega) - 2TR'(\omega)\right]\frac{d\omega}{2\pi}\right\}$$

dissipation — fluctuation — response

→ (violation of FDT) ∝ (heat dissipation)

2. Nonequilibrium FDT Baiesi, Maes, Wynants, PRL 103, 010602 (2009)

$$(t > s)\qquad R_{QV}(t,s) = \frac{\delta\langle Q(t)\rangle}{\delta h_s} = \frac{\beta}{2}\frac{d}{ds}\langle V(s)Q(t)\rangle - \frac{1}{2}\langle [L^\dagger V(s)]Q(t)\rangle = \beta\frac{d}{ds}\langle V(s)Q(t)\rangle$$

average of $\frac{dV}{ds}$ at x_s

equilibrium limit

(equilibrium FDT)

$U \to U - h_s V$ (small potential perturbation)

$Q(t)$: observable at time t

3. TUR (Thermodynamic Uncertainty Relation)

$$\frac{\mathrm{Var}[\Theta_\tau]}{\langle\Theta_\tau\rangle^2}\langle\Delta S_{\mathrm{tot}}\rangle \geq 2k_{\mathrm{B}}$$

fluctuation — Dissipation

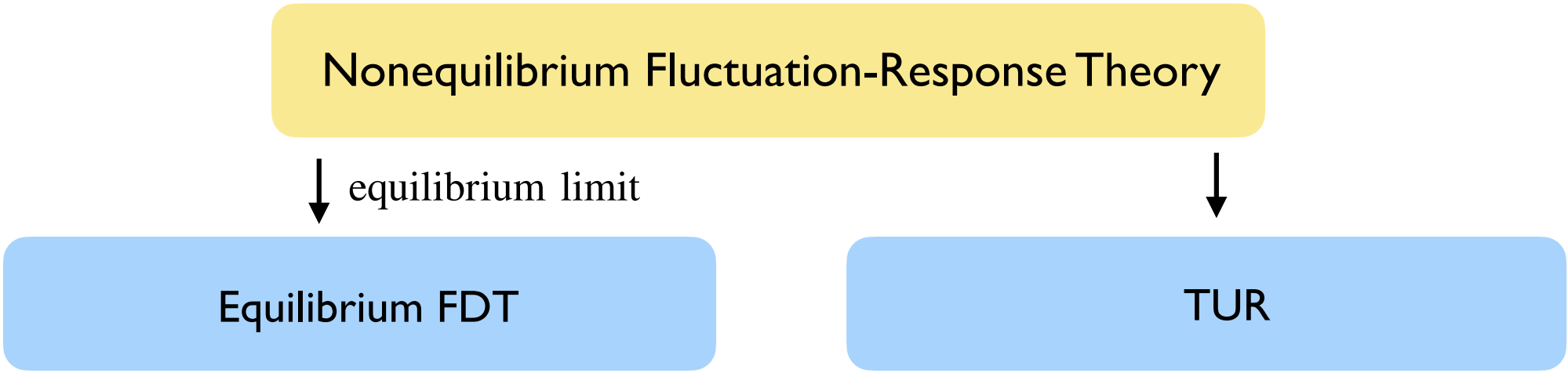
ΔS_{tot} : entropy production

Θ_t : observable at time t

→ These theories have been studied independently.
A clear link between them has not been found.

Goal of This Study

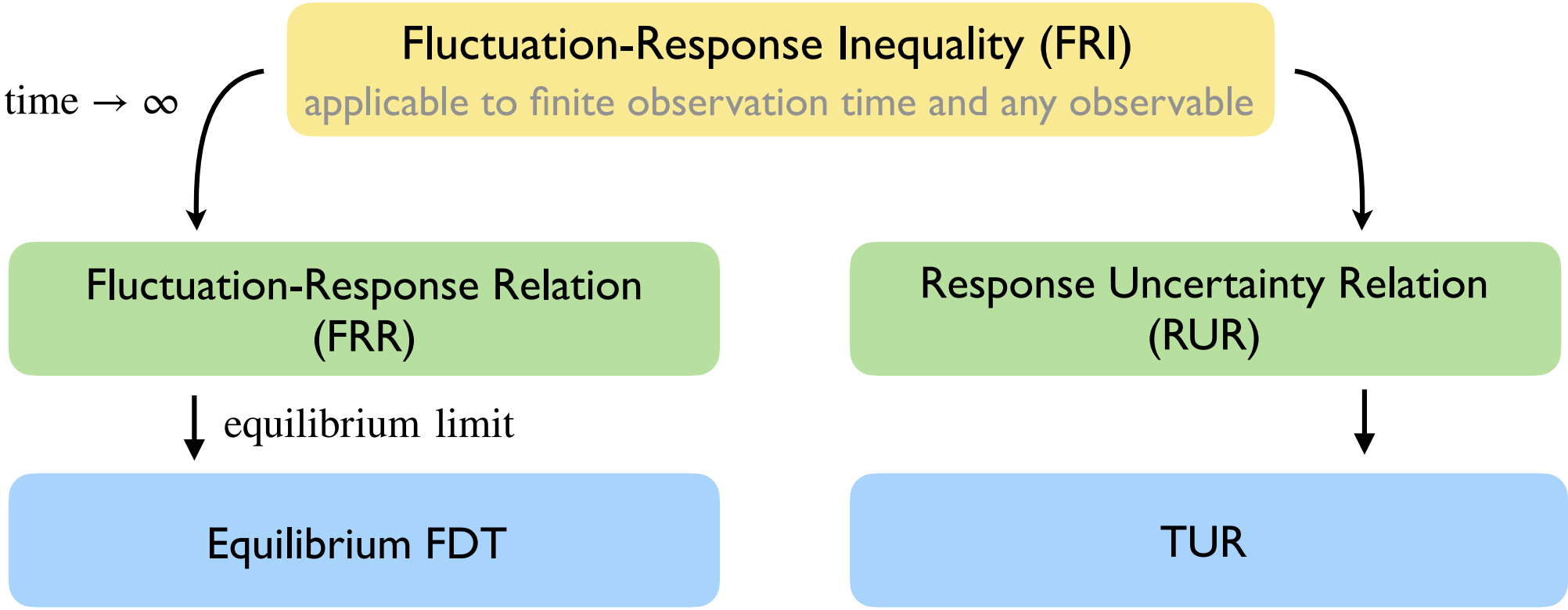
→ Establish such a unified theoretical framework in nonequilibrium systems



Summary of the Main Results

Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)
Chun, Kwon, Park, JSL, arXiv:2601.16387

1. Classical Stochastic System for both Markov-jump & Langevin systems



2. Extension to Open Quantum System

: Quantum Fluctuation-Response Inequality (QFRI)

Setup

Continuous-time Markov jump System

Master equation

$$\dot{p}_i(t) = \sum_{j(\neq i)} \Big(W_{ij} p_j(t) - W_{ji} p_i(t) \Big) \quad \Rightarrow \quad \lim_{t\to\infty} p_i(t) = \pi_i \text{ : steady-state probability}$$

Transition rate: $W_{ij} = \exp\left[B_{ij}(\epsilon) + \frac{F_{ij}(\eta)}{2} \right]$

$B_{ij} = B_{ji}$: symmetric part

$F_{ij} = -F_{ji}$: antisymmetric part

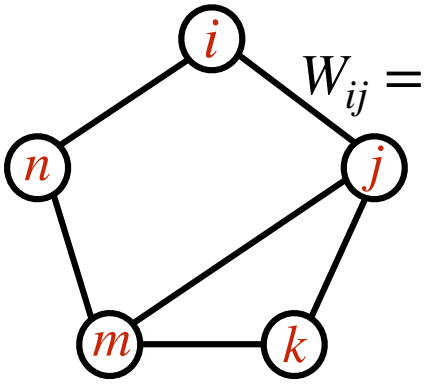
Observable $\Theta(\tau) = \int_0^\tau dt \left[\sum_i g_i h_i(t) + \sum_{i\neq j} \Lambda_{ij} \dot{N}_{ij}(t) \right]$

state-dependent current-like if $\Lambda_{ij} = -\Lambda_{ji}$

$h_i(t) = \delta_{s(t),i}$ $s(t)$: state of the system at time t

$N_{ij}(t)$: accumulated number of jumps $j \to i$

Two Ways of Perturbing W_{ij}



$$W_{ij} = \exp\left[B_{ij}(\epsilon) + \frac{F_{ij}(\eta)}{2}\right]$$

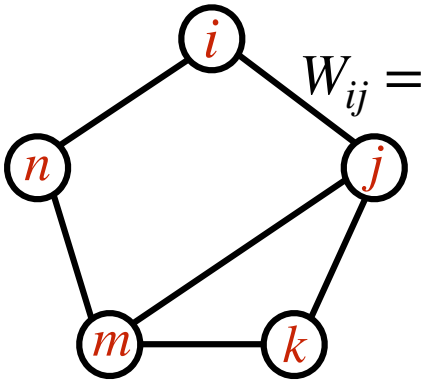
$B_{ij} = B_{ji}$: symmetric part

$F_{ij} = -F_{ji}$: antisymmetric part

1. Kinetic perturbation: Perturb B_{ij}

2. Entropic perturbation: Perturb F_{ij}

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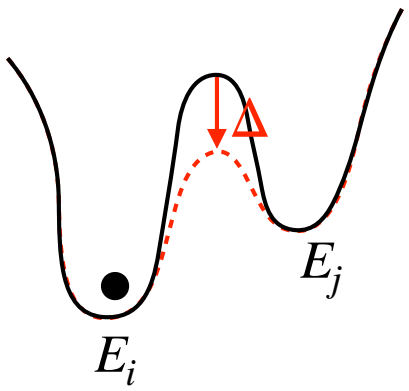
$F_{ij} = -F_{ji}$: antisymmetric part

1. Kinetic perturbation: Perturb B_{ij}

2. Entropic perturbation: Perturb F_{ij}

$B_{ij} \to B_{ij} + \Delta$

$$W_{ij} \to W_{ij} e^{\Delta}$$
$$W_{ji} \to W_{ij} e^{\Delta}$$



perturbing energy barrier changes the transition rate

Effect of B_{ij} perturbation in equilibrium

$$W_{ij}\pi_j^{\mathrm{eq}} - W_{ji}\pi_i^{\mathrm{eq}} = 0 \xrightarrow{\text{perturbation}} e^{\Delta} W_{ij}\pi_j^{\mathrm{eq}} - e^{\Delta} W_{ji}\pi_i^{\mathrm{eq}} = 0$$

detailed balance — same equilibrium state

Steady state does not change. $\to \partial_{B_{ij}}\langle\Theta\rangle = 0$

Only kinetic property varies.

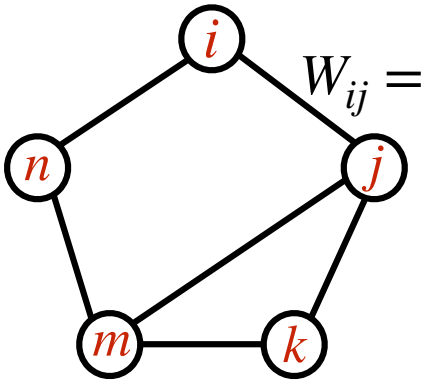
Not need to consider B_{ij} perturbation.

Effect of B_{ij} perturbation in nonequilibrium

B_{ij} perturbation changes the steady state. $\to \partial_{B_{ij}}\langle\Theta\rangle \neq 0$

Need to consider B_{ij} perturbation.

Two Ways of Perturbing W_{ij}



$$W_{ij} = \exp\left[B_{ij}(\epsilon) + \frac{F_{ij}(\eta)}{2}\right]$$

$B_{ij} = B_{ji}$: symmetric part

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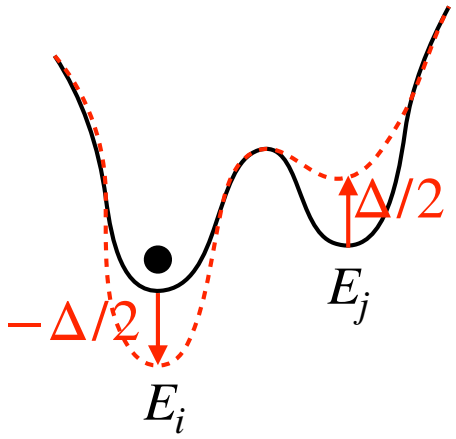
1. Kinetic perturbation: Perturb B_{ij}

2. Entropic perturbation: Perturb F_{ij}

$F_{ij} \to F_{ij} + \Delta$

$W_{ij} \to W_{ij} e^{\Delta/2}$

$W_{ji} \to W_{ij} e^{-\Delta/2}$



tilting energy barrier
(applying a driving force)

F_{ij} perturbation drives the system out of equilibrium.

→ entropy is produced

F_{ij} perturbation changes the steady state and current.

$\to \partial_{F_{ij}} \langle \Theta \rangle \neq 0$

Need to consider F_{ij} perturbation.

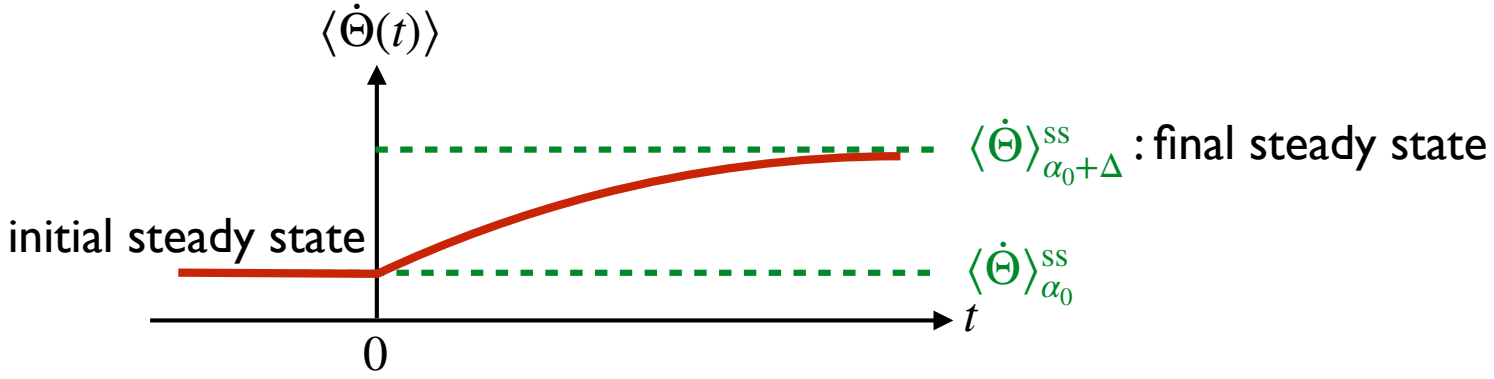
Response to the Perturbation

How observable responds to the B_{ij} or F_{ij} perturbation

Observable: $\Theta(\tau) = \int_0^\tau dt \underbrace{\left[\sum_i g_i h_i(t) + \sum_{i \neq j} \Lambda_{ij} \dot{N}_{ij}(t) \right]}_{= \dot{\Theta}(t)}$

Protocol of Perturbing α $\quad (\alpha = B_{ij}, F_{ij})$

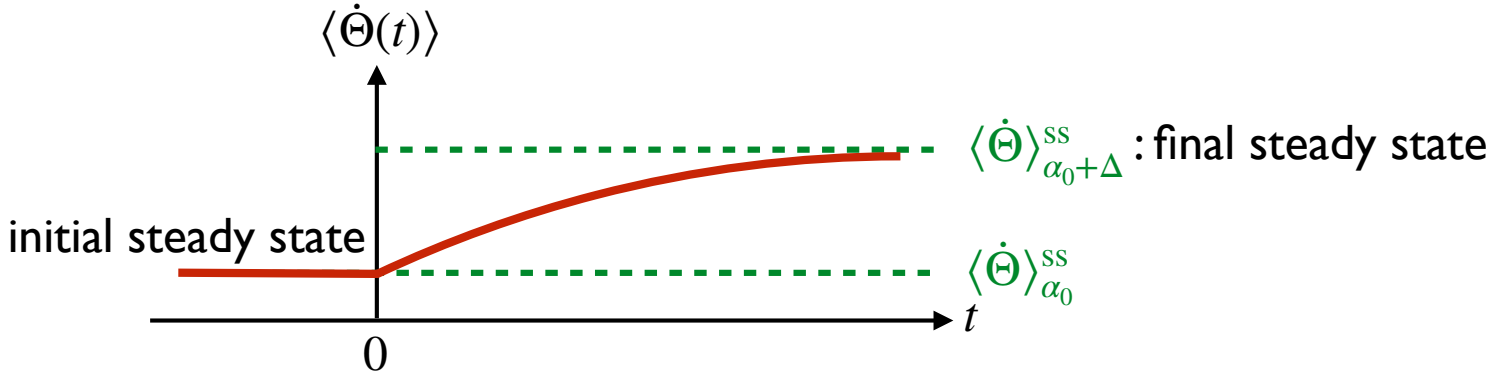
- initial condition: steady state with $\alpha(t) = \alpha_0 \quad (t < 0)$
- perturbing α at $t = 0$: $\alpha_0 \rightarrow \alpha_0 + \Delta \quad (t = 0)$
- relaxation to new steady state $t > 0$ with $\alpha(t) = \alpha_0 + \Delta \quad (0 < t)$



Response to the Perturbation

How observable responds to the B_{ij} or F_{ij} perturbation

Observable: $\Theta(\tau) = \int_0^\tau dt \left[\sum_i g_i h_i(t) + \sum_{i \neq j} \Lambda_{ij} \dot{N}_{ij}(t) \right] = \dot{\Theta}(t)$

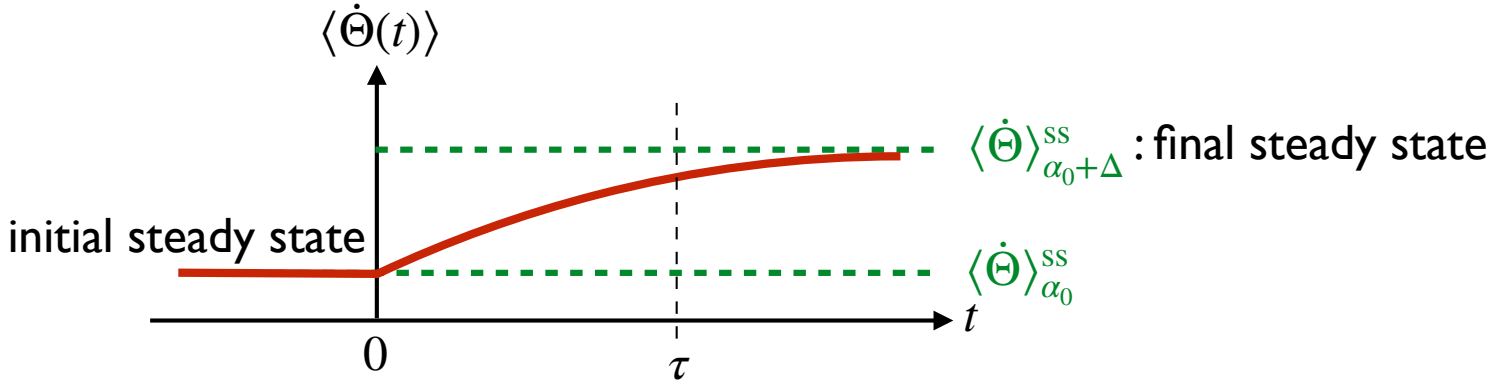


Static response: $\partial_\alpha \langle \dot{\Theta} \rangle^{\rm ss} = \lim_{\Delta \to 0} \dfrac{\langle \dot{\Theta} \rangle^{\rm ss}_{\alpha+\Delta} - \langle \dot{\Theta} \rangle^{\rm ss}_{\alpha}}{\Delta}$ (obtained from infinite-time measurement)

Response to the Perturbation

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Static response: $\partial_\alpha \langle \dot{\Theta} \rangle^{ss} = \lim_{\Delta \to 0} \dfrac{\langle \dot{\Theta} \rangle^{ss}_{\alpha+\Delta} - \langle \dot{\Theta} \rangle^{ss}_{\alpha}}{\Delta}$ (obtained from infinite-time measurement)

Dynamic response: $\partial_\alpha \langle \Theta(\tau) \rangle = \lim_{\Delta \to 0} \dfrac{\langle \Theta(\tau) \rangle_{\alpha+\Delta} - \langle \Theta(\tau) \rangle_{\alpha}}{\Delta}$

(obtained from finite-time measurement)

note : $\partial_\alpha \langle \dot{\Theta} \rangle^{ss} = \lim_{\tau \to \infty} \partial_\alpha \langle \Theta(\tau) \rangle / \tau$

Fluctuation-Response Inequalities (FRI)

Cramér-Rao inequality with multiple perturbation parameters (B_{ij} or F_{ij})

$$\sum_{\alpha,\beta} R_\alpha(\tau)[\mathbb{I}^{-1}(\tau)]_{\alpha,\beta} R_\beta(\tau) \leq \mathrm{Var}[\Theta(\tau)]$$

$R_\alpha(\tau) \equiv \partial_\alpha \langle \Theta(\tau) \rangle$: dynamic response $\quad (\alpha = B_{ij}, F_{ij})$

$[\mathbb{I}]_{\alpha,\beta} = -\langle \partial_\alpha \partial_\beta \ln \mathscr{P}[\Gamma] \rangle$: Fisher information matrix

1. B_{ij} perturbation $\forall(i,j)$

$[\mathbb{I}]_{B_{ij},B_{i'j'}}(\tau) = \tau \delta_{ii'} \delta_{jj'} a_{ij}$: diagonal

$(a_{ij} = W_{ij}\pi_j + W_{ji}\pi_i)$

$$\sum_{i<j} \frac{R^2_{B_{ij}}(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

2. F_{ij} perturbation $\forall(i,j)$

$[\mathbb{I}]_{F_{ij},F_{i'j'}}(\tau) = \frac{1}{4}\tau \delta_{ii'} \delta_{jj'} a_{ij}$: diagonal

$$\sum_{i<j} \frac{4R^2_{F_{ij}}(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

FRIs are valid for finite-time observations and arbitrary observables!

Dynamic response: $\partial_\alpha \langle \Theta(\tau) \rangle = \lim_{\Delta \to 0} \frac{\langle \Theta(\tau) \rangle_{\alpha+\Delta} - \langle \Theta(\tau) \rangle_\alpha}{\Delta}$

Fluctuation-Response Inequalities (FRI)

$$\sum_{i<j} \frac{R^2_{B_{ij}}(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

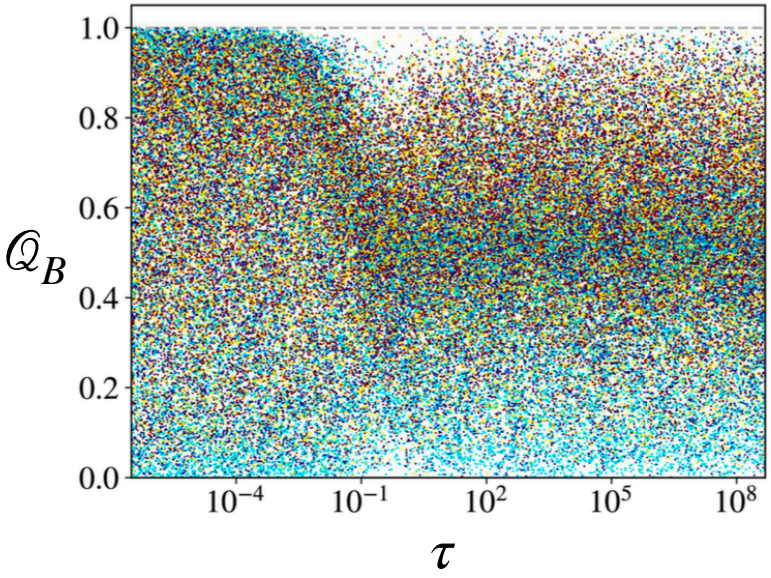
$$\rightarrow \mathcal{Q}_B \leq 1$$

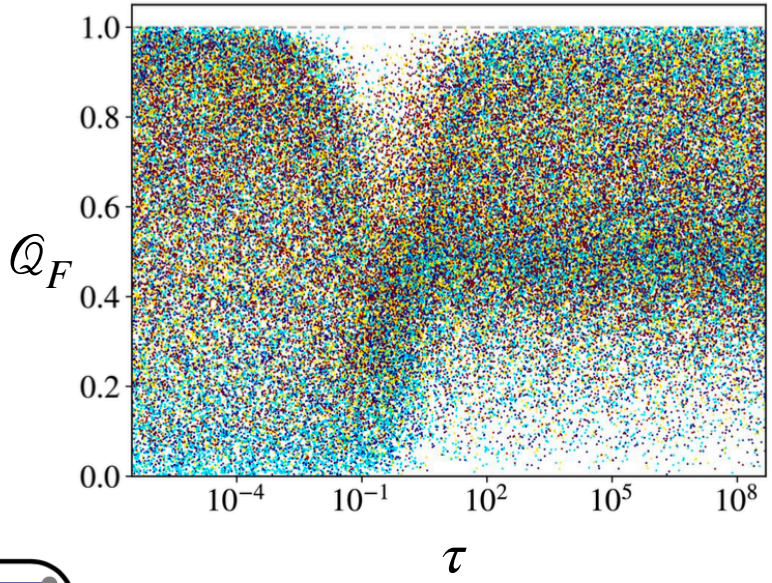
$$\mathcal{Q}_B \equiv \sum_{i<j} \frac{R^2_{B_{ij}}(\tau)}{\tau a_{ij} \mathrm{Var}\{\Theta(\tau)\}}$$

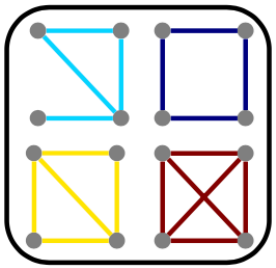
$$\sum_{i<j} \frac{4R^2_{F_{ij}}(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

$$\rightarrow \mathcal{Q}_F \leq 1$$

$$\mathcal{Q}_F \equiv \sum_{i<j} \frac{4R^2_{F_{ij}}(\tau)}{\tau a_{ij} \mathrm{Var}[\Theta(\tau)]}$$







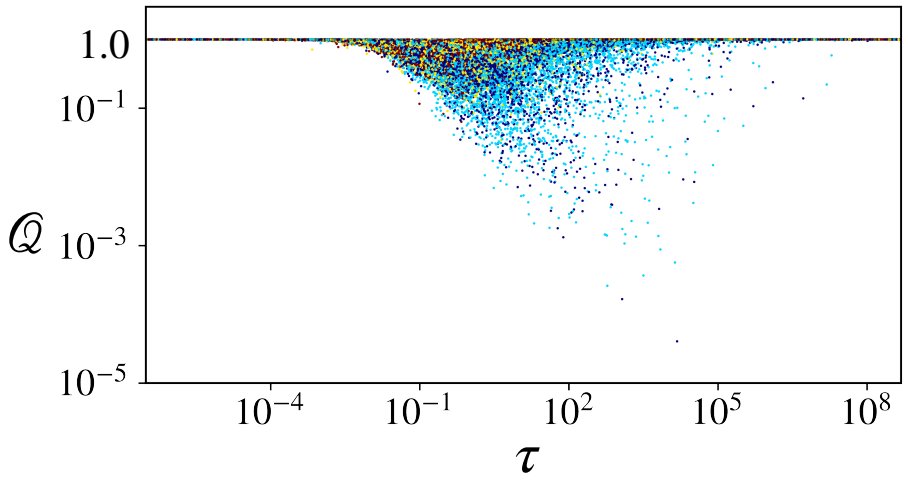
FRI → FRR

$$\sum_{i<j} \frac{R_{B_{ij}}^2(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)] \qquad \sum_{i<j} \frac{4R_{F_{ij}}^2(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

relation btw B_{ij} and F_{ij} responses

$$\frac{R_{B_{ij}}(\tau)}{R_{F_{ij}}(\tau)} = \frac{2J_{ij}}{a_{ij}} \qquad a_{ij} = W_{ij}\pi_j + W_{ji}\pi_i \qquad J_{ij} = W_{ij}\pi_j - W_{ji}\pi_i$$

$$\sum_{i<j} \frac{a_{ij}R_{B_{ij}}^2(\tau)}{\tau J_{ij}^2} \leq \mathrm{Var}[\Theta(\tau)] \longrightarrow \mathcal{Q} \equiv \sum_{i<j} \frac{a_{ij}R_{B_{ij}}^2(\tau)}{\tau J_{ij}^2 \mathrm{Var}[\Theta(\tau)]}$$



(Θ : current-like observable)

FRI → FRR

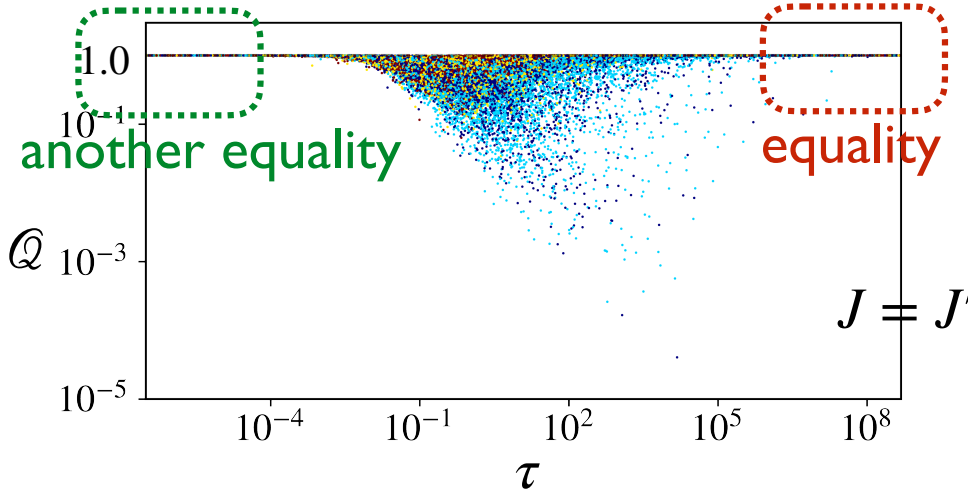
$$\sum_{i<j} \frac{R_{B_{ij}}^2(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)] \qquad \sum_{i<j} \frac{4R_{F_{ij}}^2(\tau)}{\tau a_{ij}} \leq \mathrm{Var}[\Theta(\tau)]$$

relation btw B_{ij} and F_{ij} responses

$$\frac{R_{B_{ij}}(\tau)}{R_{F_{ij}}(\tau)} = \frac{2J_{ij}}{a_{ij}} \qquad a_{ij} = W_{ij}\pi_j + W_{ji}\pi_i \qquad J_{ij} = W_{ij}\pi_j - W_{ji}\pi_i$$

dynamic

$$\sum_{i<j} \frac{a_{ij}R_{B_{ij}}^2(\tau)}{\tau J_{ij}^2} \leq \mathrm{Var}[\Theta(\tau)] \longrightarrow \mathcal{Q} \equiv \sum_{i<j} \frac{a_{ij}R_{B_{ij}}^2(\tau)}{\tau J_{ij}^2 \mathrm{Var}[\Theta(\tau)]}$$



(Θ : current-like observable)

static

$$\to \lim_{\tau\to\infty} \sum_{i<j} \frac{a_{ij}R_{B_{ij}}^2(\tau)/\tau^2}{J_{ij}^2} = \lim_{\tau\to\infty} \mathrm{Var}[\Theta(\tau)]/\tau$$

$J = J'$

Fluctuation-Response Relation (infinite time)

$$\sum_{i<j} \frac{a_{ij}}{J_{ij}^2} \partial_{B_{ij}} \langle J \rangle^{\mathrm{ss}} \partial_{B_{ij}} \langle J' \rangle^{\mathrm{ss}} = \mathrm{Cov}(J, J')$$

Aslyamov et al., PRL 134, 157101 (2025)

FRI → FRR

1. Classical Stochastic System

Fluctuation-Response Inequality (FRI)

applicable to finite-time observation and any observable

time → ∞

Fluctuation-Response Relation (FRR)

Aslyamov et al., PRL 134, 157101 (2025)

FRI → R-TKUR

1. Classical Stochastic System

Fluctuation-Response Inequality (FRI)

applicable to finite-time observation and any observable

time → ∞

Fluctuation-Response Relation (FRR)

Aslyamov et al., PRL 134, 157101 (2025)

Response Thermodynamic Kinetic Uncertainty Relation (R-TKUR)

FRI → R-TKUR

Fluctuation-Response Inequality

$$\sum_{i<j} \frac{a_{ij} R_{B_{ij}^2(\tau)}}{\tau J_{ij}^2} \leq \mathrm{Var}[\Theta(\tau)] \rightarrow \frac{R_\epsilon^2(\tau)}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{b_{\max}^2}{2} \Sigma_{\mathrm{ps}} \leq \frac{b_{\max}^2}{2} \min\{\Sigma, 2A\}$$

Cauchy-Schwartz

Σ_{ps} : pseudo EP $(\Sigma_{\mathrm{ps}} \leq \Sigma, 2A)$

Σ : total EP

A : total dynamical activity

Transition rate: $W_{ij} = \exp\left[B_{ij}(\epsilon) + \frac{F_{ij}(\eta)}{2}\right]$

Response: $R_\epsilon(\tau) \equiv \partial_\epsilon \langle \Theta(\tau) \rangle = \sum_{i<j} \partial_\epsilon B_{ij} \cdot \partial_{B_{ij}} \langle \Theta(\tau) \rangle = \sum_{i<j} b_{ij} R_{B_{ij}}$ $(b_{ij} \equiv \partial_\epsilon B_{ij})$

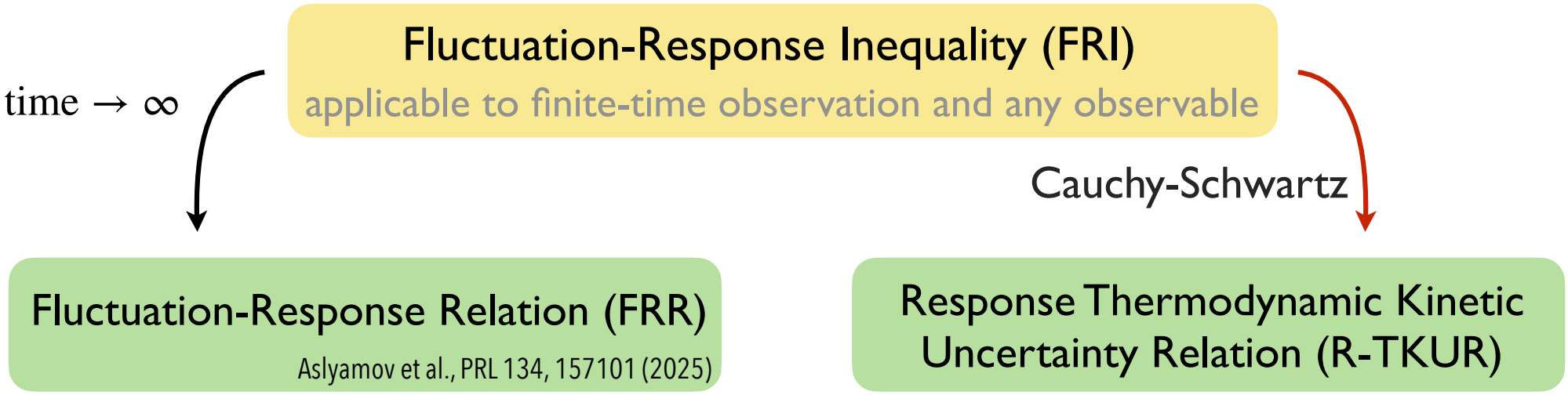
$b_{\max} = \max\{|b_{ij}|\}$

Response Thermodynamic Kinetic Uncertainty Relation (R-TKUR)

$$\frac{R_\epsilon^2(\tau)}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{b_{\max}^2}{2} \min\{\Sigma, 2A\}$$

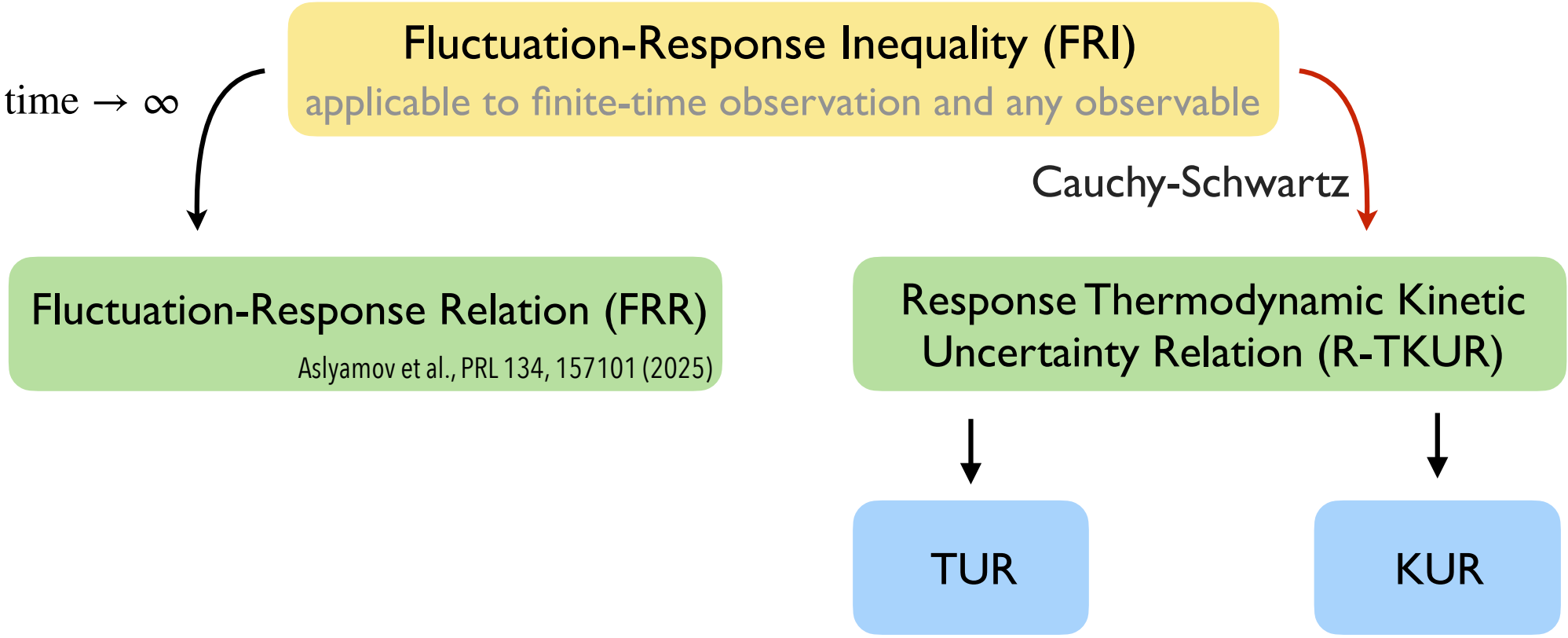
FRI → R-TKUR

1. Classical Stochastic System



R-TKUR → TUR & KUR

1. Classical Stochastic System



R-TKUR → TUR & KUR

Response Thermodynamic Kinetic Uncertainty Relation (R-TKUR)

$$\frac{R_\epsilon^2(\tau)}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{b_{\max}^2}{2}\min\{\Sigma,\ 2A\}$$

Σ : total EP

A : total dynamical activity

$b_{ij} = b = b_{\max} \quad \forall\ ij$: uniform perturbation

$\rightarrow R_\epsilon(\tau) = b\langle\Theta(\tau)\rangle$

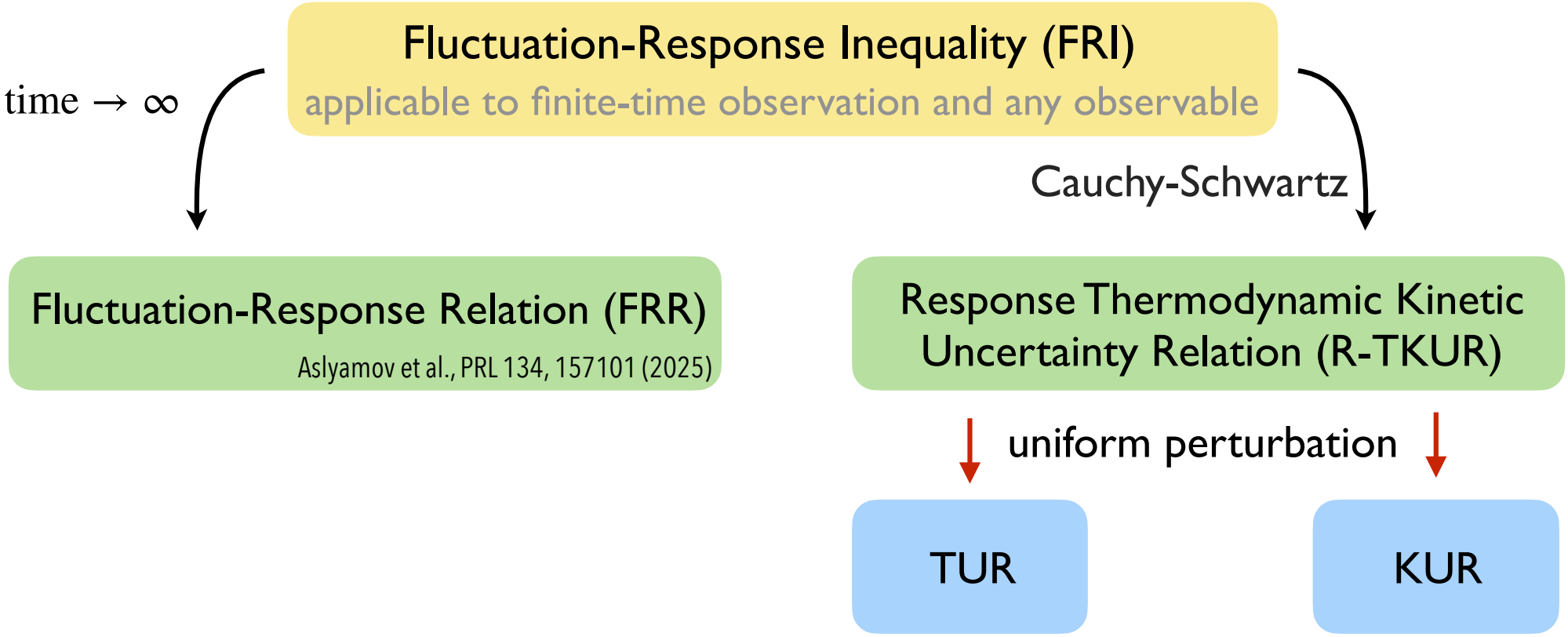
Thermodynamic, Kinetic Uncertainty Relation (TUR, KUR)

$$\rightarrow \frac{\langle\Theta(\tau)\rangle^2}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{1}{2}\min\{\Sigma,\ 2A\}$$

TUR KUR

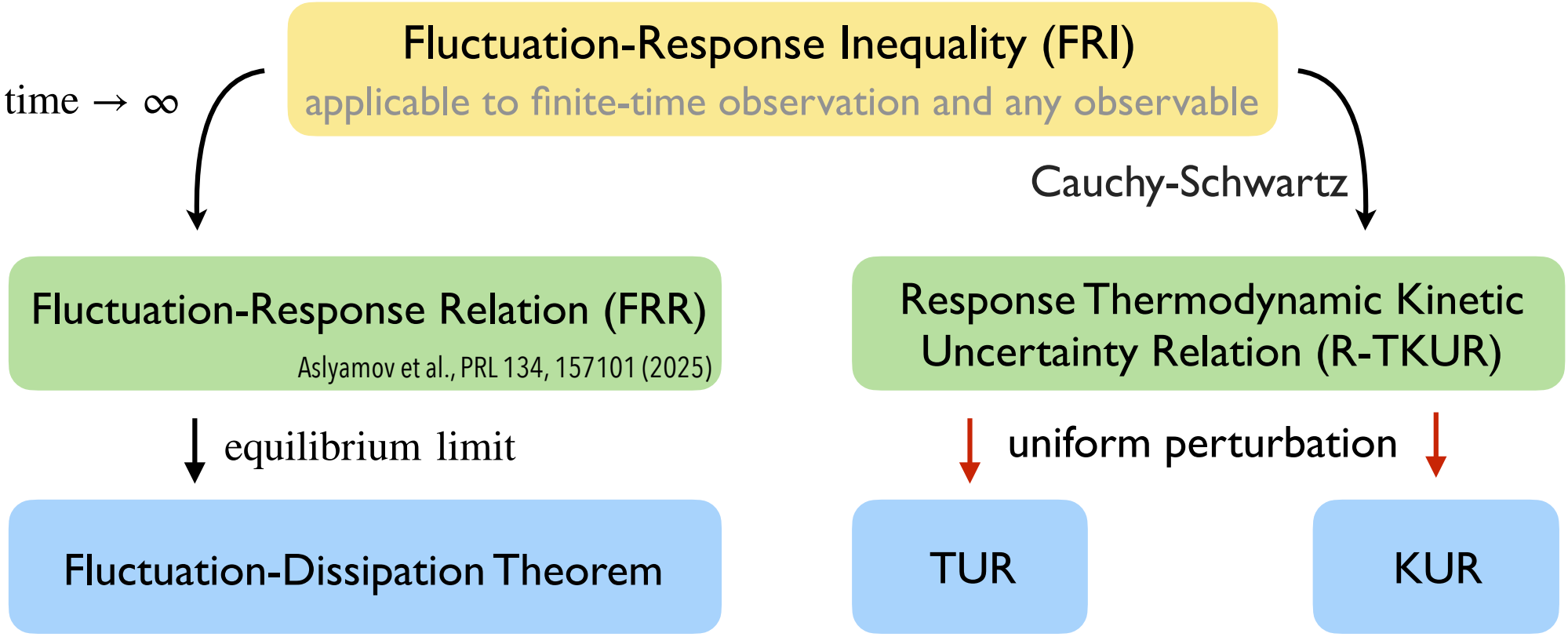
R-TKUR → TUR & KUR

1. Classical Stochastic System



FRR → FDT (Equilibrium)

1. Classical Stochastic System



FRR → FDT (Equilibrium)

Fluctuation-Response Relation

$$\mathrm{Cov}(J, J') = \sum_{i>j} \frac{4}{a_{ij}} \partial_{F_{ij}} \langle J \rangle^{\mathrm{ss}} \partial_{F_{ij}} \langle J' \rangle^{\mathrm{ss}} \xrightarrow{J = J'} \underbrace{\mathrm{Var}(J)}_{\text{fluctuation}} = \sum_{i>j} \frac{4}{a_{ij}} \underbrace{\left(\partial_{F_{ij}} \langle J \rangle^{\mathrm{ss}}\right)^2}_{\text{response}}$$

In equilibrium, fluctuation ∝ response

Empirical edge current: $j_{ij}(t) = \dot{N}_{ij}(t) - \dot{N}_{ji}(t) \quad \rightarrow \langle j_{ij}(t) \rangle = W_{ij} p_j(t) - W_{ji} p_i(t) = J_{ij}$

$N_{ij}(t)$: accumulated number of jumbs $j \rightarrow i$

Edge response: $L_{(ij),(mn)} \equiv \dfrac{dJ_{ij}}{dF_{mn}}$: how J_{ij} responds to external driving F_{mn} perturbation

Edge current fluctuation: $C_{(ij),(mn)} \equiv \langle\langle j_{ij}, j_{mn} \rangle\rangle = \mathrm{Cov}[j_{ij}, j_{mn}]$

$$\mathrm{Var}(J) = \mathbf{\Lambda}^{\mathrm{T}} \mathbb{C} \mathbf{\Lambda} = \sum_{i>j} \frac{4}{a_{ij}} \left(\partial_{F_{ij}} \langle J \rangle^{\mathrm{ss}}\right)^2 = \mathbf{\Lambda}^{\mathrm{T}} 2\mathbb{L} \mathbf{\Lambda} \longrightarrow \mathbb{C} = 2\mathbb{L}$$

: FDT in equilibrium

$J(t) = \sum_{i \neq j} \Lambda_{ij} \dot{N}_{ij}(t)$: current-like observable

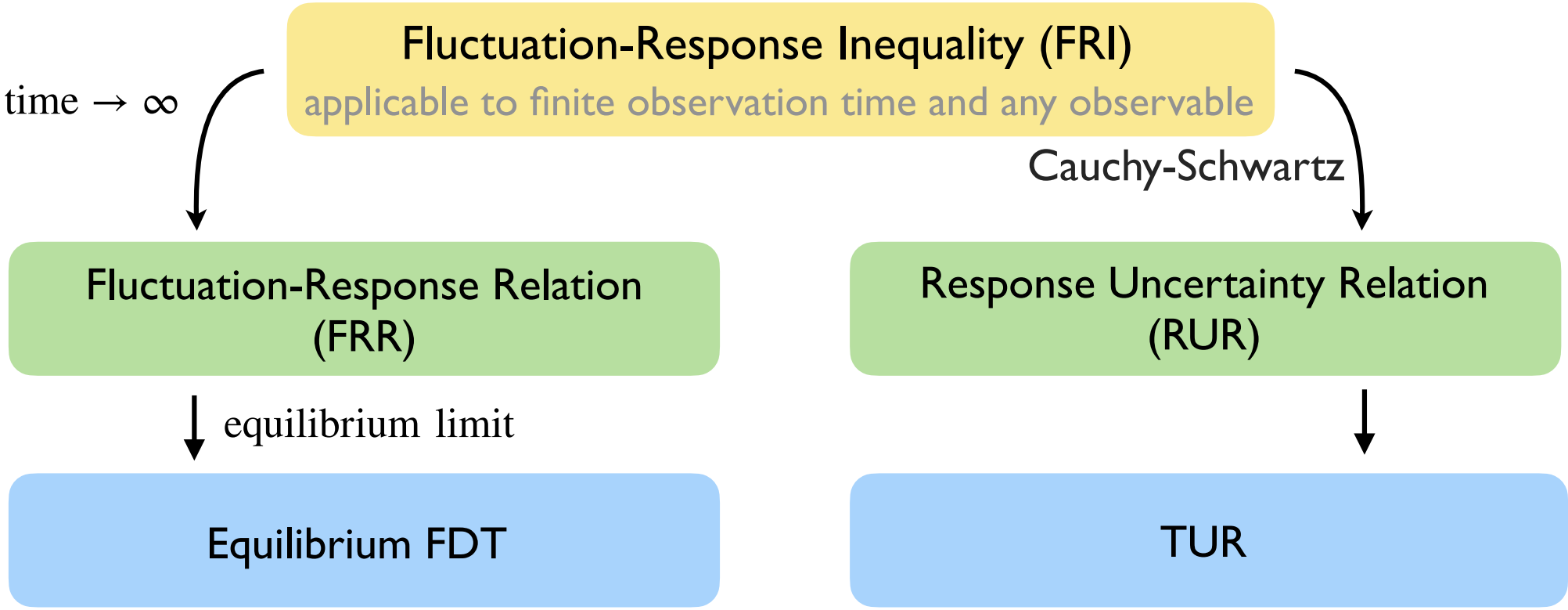
Aslyamov et al., PRL 134, 157101 (2025)

Vroylandt et al., JStatMech 054002 (2019)

Summary of the Main Results

Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)
Chun, Kwon, Park, JSL, arXiv:2601.16387

1. Classical Stochastic System for both Markov-jump & Langevin systems



Setup for Overdamped Langevin Systems

Langevin equation

$$\dot{x}_t = \mu(x_t)F(x_t) + \sqrt{2\mu(x_t)T(x_t)} \circledast \xi_t$$

$\circledast$: anti-Ito product

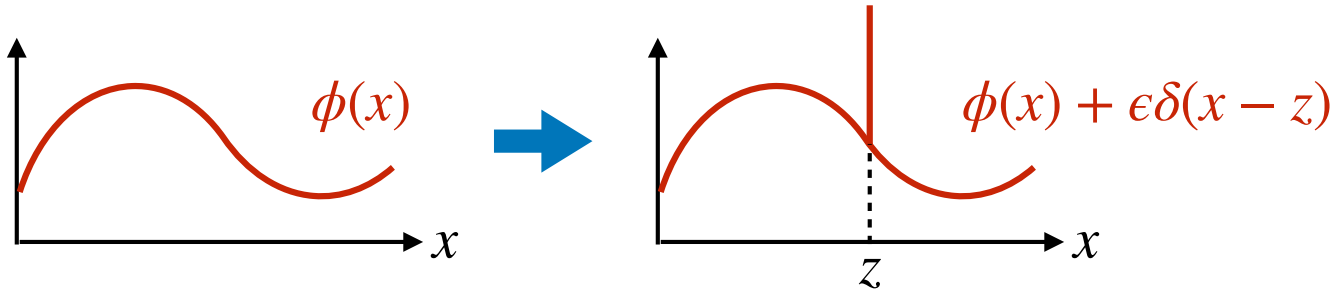
$\mu(x)$: mobility

$F(x)$: external force

$T(x)$: temperature

Time-averaged observable: $\Theta(\tau) = \frac{1}{\tau}\int_0^\tau [f(x_t) + \dot{x}_t \circ g(x_t)]dt$

Local perturbation at $x = z$: $\phi(x) \to \phi(x) + \epsilon\delta(x - z)$ $\quad \phi \in \{\mu, F, T\}$



$$\langle\Theta\rangle_{\rm ss} \to \langle\Theta\rangle_{\rm ss} + \epsilon\frac{\delta\langle\Theta\rangle_{\rm ss}}{\delta\phi(z)}$$

Response to the perturbation at $x = z$: $\frac{\delta\langle\Theta\rangle_{\rm ss}}{\delta\phi(z)}$

Hierarchy for Langevin Systems

Chun, Kwon, Park, JSL, arXiv:2601.16387

Fluctuation-Response Inequality (FRI)

applicable to finite observation time and any observable

$$\mathrm{Var}[\Theta(\tau)] \geq \int dz \int_0^\tau ds \, \frac{2p(z,s)D(z)}{[\tilde{N}_\phi(z,s)]^2} \left(\frac{\delta \langle \Theta(\tau) \rangle}{\delta \phi(z,s)} \right)^2$$

time $\to \infty$

Cauchy – Schwarz

Fluctuation-Response Relation (FRR)

$$C_{\Theta_1,\Theta_2} = \int \frac{2\pi(z)D(z)}{N_{\phi_1}(z)N_{\phi_2}(z)} \frac{\delta \langle \Theta_1 \rangle_{\mathrm{ss}}}{\delta \phi_1(z)} \frac{\delta \langle \Theta_2 \rangle_{\mathrm{ss}}}{\delta \phi_2(z)} dz$$

$$C_{\Theta_1,\Theta_2} \equiv \lim_{\tau \to \infty} \tau \mathrm{Cov}\{\Theta_1(\tau), \Theta_2(\tau)\}$$

equilibrium limit

Equilibrium FDT

$$C^{\mathrm{eq}}_{J(x),J(y)} = 2T \left[\frac{\delta \langle J(x) \rangle}{\delta F(y)} \right]_{\mathrm{eq}}$$

Response Uncertainty Relation (RUR)

$$\frac{[\delta_\phi \langle \Theta(\tau) \rangle]^2}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{\psi^2_{\max}}{2} \int dz \int_0^\tau ds \, \frac{[\tilde{N}_\phi(z,s)]^2}{p(z,s)D(z)}$$

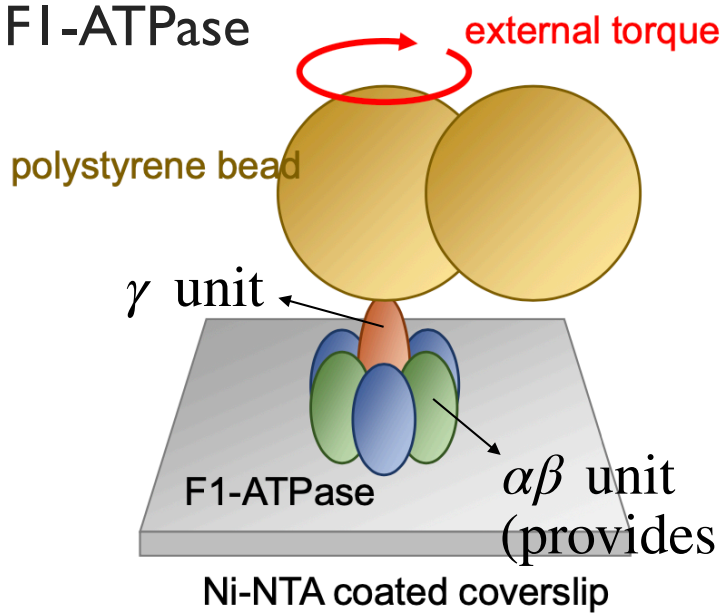
uniform perturbation

TUR

$$\frac{\langle (1 + \tau \partial_\tau) J(\tau) \rangle^2}{\mathrm{Var}[J(\tau)]} \leq \Sigma_\tau / 2$$

Numerical Confirmation: Giant-Diffusion Phenomenon

diffusion coefficient can be enhanced by several orders of magnitude under the periodic potential



Langevin equation for a duplex bead

$$\dot{\theta}_t = \mu[-\partial_\theta U(\theta)|_{\theta=\theta_t} + F] + \sqrt{2\mu T}\xi_t$$

Hayashi et al., PRL 114, 248101 (2015)

θ : angle of the bead

F : constant torque

$U(\theta) = U_0 \cos(3\theta)$: potential with period 120°

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Observable

$$\bar{\omega}(\tau) = \tau^{-1} \int_0^\tau \dot{\theta}_t dt$$

time-averaged angular velocity

Diffusion coefficient

$$D_\infty = \lim_{\tau\to\infty} \tau \mathrm{Var}[\bar{\omega}(\tau)]/2$$

Perturbations	Responses	Response Uncertainty Relation
$F \mapsto F + \varepsilon h(t)$	$\delta_F \langle\bar{\omega}\rangle_{\mathrm{ss}}$	$D_\infty \geq \dfrac{T(\partial_F \omega)^2}{\mu}$
$\ln\mu \mapsto \ln\mu + \varepsilon h(t)$	$\delta_{\ln\mu} \langle\bar{\omega}\rangle_{\mathrm{ss}}$	$D_\infty \geq \dfrac{(\partial_{\ln\mu} \omega)^2}{\sigma}$
$T \mapsto T + \varepsilon h(t)$	$\delta_T \langle\bar{\omega}\rangle_{\mathrm{ss}}$	$D_\infty \geq \dfrac{T(\partial_T \omega)^2}{\mu I}$

Numerical Confirmation: Giant-Diffusion Phenomenon

diffusion coefficient can be enhanced by several orders of magnitude under the periodic potential

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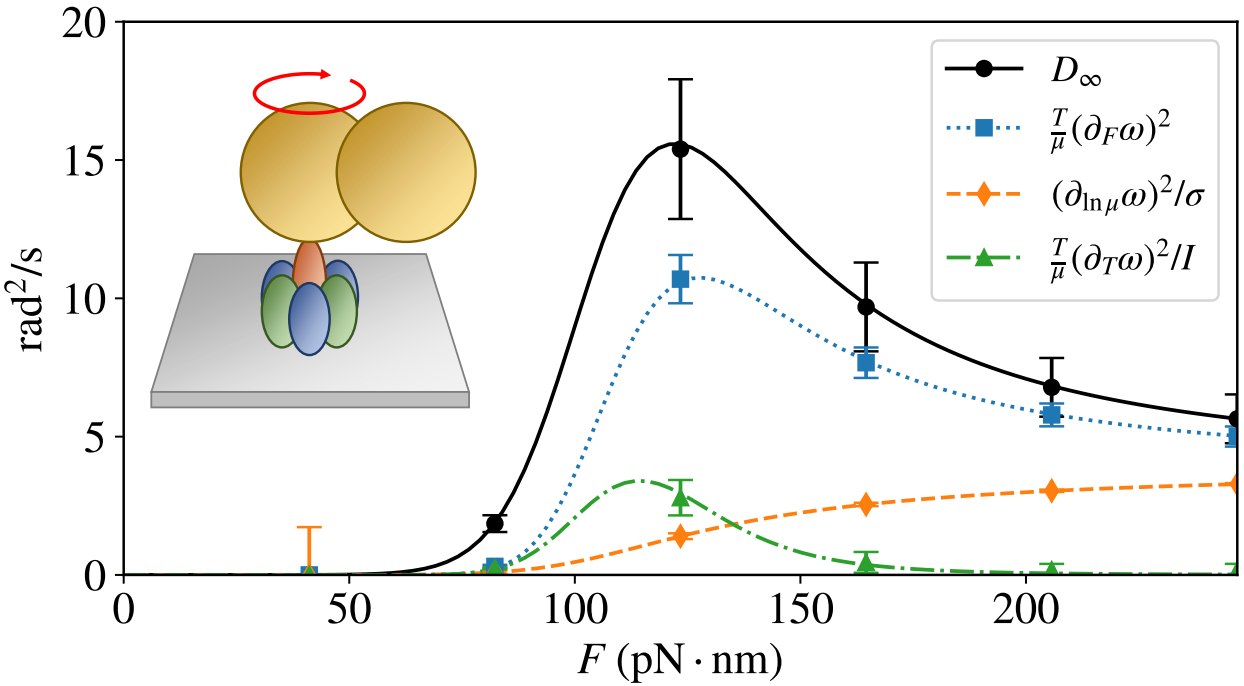
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$$\bar{\omega}(\tau) = \tau^{-1} \int_0^\tau \dot{\theta}_t dt$$

time-averaged angular velocity

Diffusion coefficient

$$D_\infty = \lim_{\tau\to\infty} \tau \mathrm{Var}[\bar{\omega}(\tau)]/2$$



Response Uncertainty Relation

$$D_\infty \geq \frac{T(\partial_F \omega)^2}{\mu}$$

$$D_\infty \geq \frac{(\partial_{\ln \mu} \omega)^2}{\sigma}$$

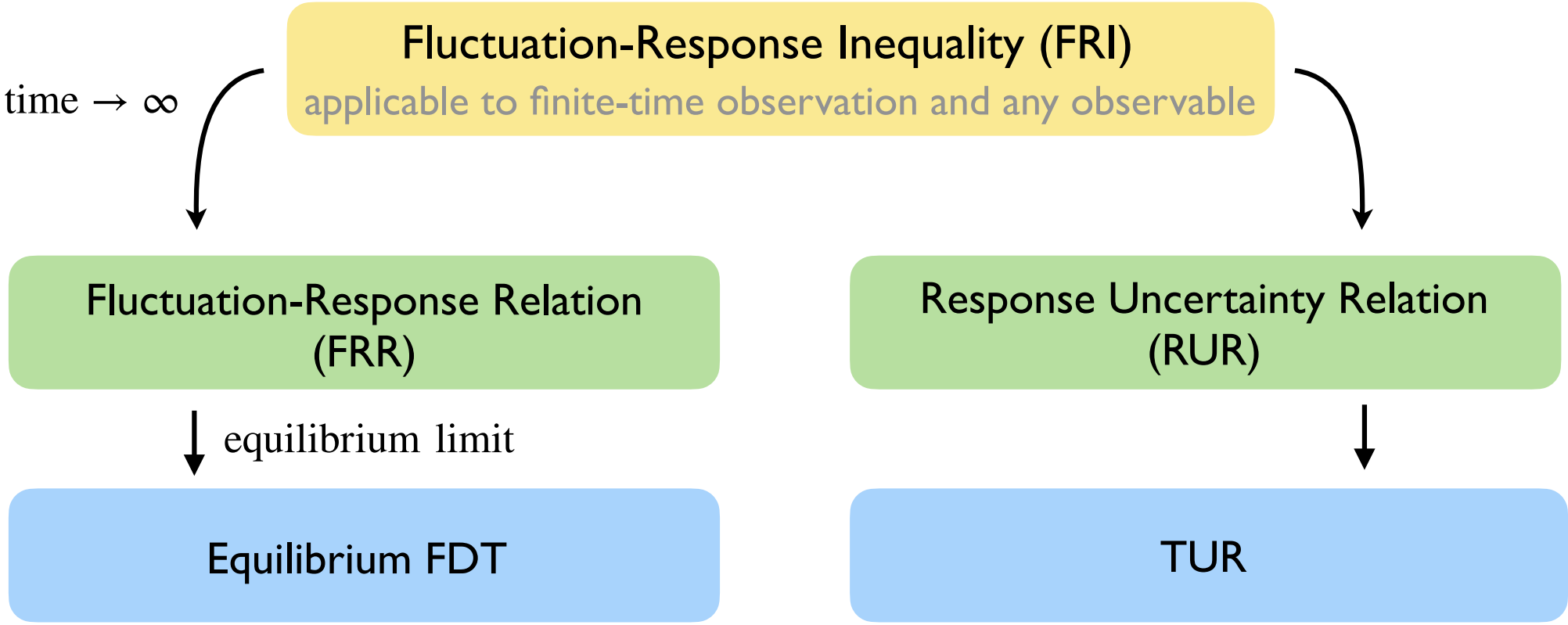
$$D_\infty \geq \frac{T(\partial_T \omega)^2}{\mu I}$$

Summary of the Main Results

Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)
Chun, Kwon, Park, JSL, arXiv:2601.16387

1. Classical Stochastic System

for both Markov-jump & Langevin systems



2. Extension to Open Quantum System

: Quantum Fluctuation-Response Inequality (QFRI)

Quantum Fluctuation-Response Inequality (QFRI)

Setup

Lindblad master equation:

$$\dot{\rho}(t) = -i[H, \rho(t)] + \sum_{k=1}^{K} \underbrace{\left(L_k \rho(t) L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho(t) - \frac{1}{2} \rho(t) L_k^\dagger L_k \right)}_{\mathscr{D}(L_k)\rho(t)} \equiv \mathscr{L}\rho(t)$$

Jump operator: $L_k(\theta_k) = L_k^{\theta_k} = e^{\theta_k/2} L_k$ K : number of jump operators L_k

Observable: $\Theta(\tau) = \sum_{k=1}^{K} \Lambda_k N_k(\tau)$ $N(\tau)$: accumlated number of jumps via channel k

: results are valid for arbitrary types of observable

Protocol of Perturbing θ_k

- initial condition: steady state with $\theta_k(t) = \theta_k \quad (t < 0)$
- perturbing θ_k at $t = 0$: $\theta_k \to \theta_k + \Delta \quad (t = 0)$
- relaxation to new steady state $t > 0$ with $\theta_k(t) = \theta_k + \Delta \quad (0 < t)$

Quantum Fluctuation-Response Inequality (QFRI)

Quantum FRI

Quantum Cramér-Rao inequality with multiple perturbation parameters θ_k

$$\sum_{\alpha,\beta} R_{\theta_\alpha}(\tau)[\mathbb{I}_Q^{-1}(\tau)]_{\theta_\alpha\theta_\beta}R_{\theta_\beta}(\tau) \le \mathrm{Var}[\Theta(\tau)] \qquad (\alpha,\beta = 1,\cdots,K)$$

$R_{\theta_\alpha}(\tau) \equiv \partial_{\theta_\alpha}\langle\Theta(\tau)\rangle$: dynamic response

$[\mathbb{I}_Q(\tau)]_{\theta_\alpha,\theta_\beta}$: Quantum Fisher information matrix

$= \tau\delta_{\alpha\beta}a_\alpha \qquad a_\alpha = \mathrm{tr}[L_\alpha^{\theta_k}\rho(L_\alpha^{\theta_k})^\dagger]$: traffic

Quantum FRI

$$\sum_k \frac{R_{\theta_k}^2(\tau)}{\tau a_k} \le \mathrm{Var}\{\Theta(\tau)\}$$

$\theta_k = \theta_k(\epsilon)$ → Cauchy-Schwartz

Quantum R-KUR

$$\frac{R_\epsilon^2(\tau)}{\mathrm{Var}\{\Theta(\tau)\}} \le \tau(\Delta\theta_{\max})^2\dot{A}$$

FRI valid for finite-time observations

$\Delta\theta_{\max} = \max_k\{\,|d_\epsilon\theta_k|\,\}$

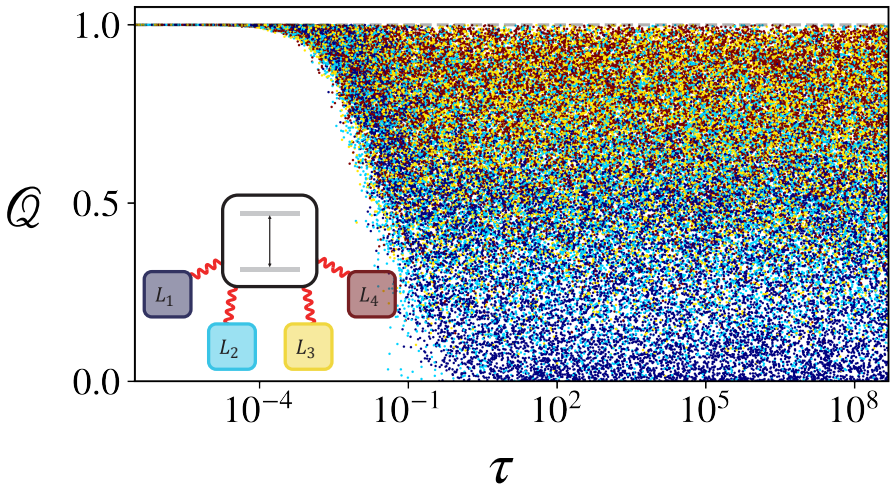
$\dot{A} = \sum_k \alpha_k$: dynamical activity

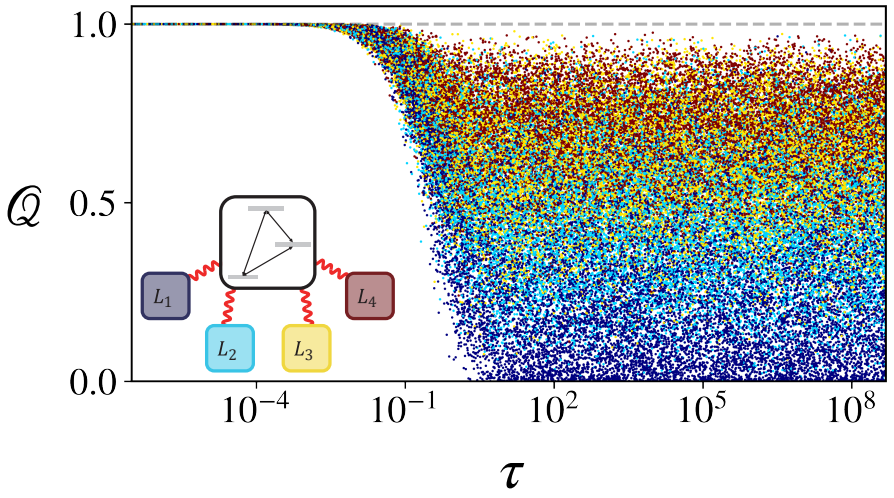
Quantum Fluctuation-Response Inequality (QFRI)

Numerical Confirmation

Quantum FRI

$$\sum_k \frac{R_{\theta_k}^2(\tau)}{\tau a_k} \leq \mathrm{Var}\{\Theta(\tau)\} \quad \rightarrow \mathcal{Q} \equiv \sum_k \frac{R_{\theta_k}^2(\tau)}{\tau a_k \mathrm{Var}\{\Theta(\tau)\}} \leq 1$$





Q does not converge to 1 (equality not satisfied) in the large τ regime.

Summary

1. We find unified hierarchy of various fluctuation-response theories (FRT) for both Markov-jump & Langevin.

Chun, Kwon, Park, JSL, arXiv:2601.16387 Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)

2. We extend the theory to Open Quantum System

and found the Quantum Fluctuation-Response Inequality (QFRI).

Chun, Kwon, Park, JSL, arXiv:2601.16387

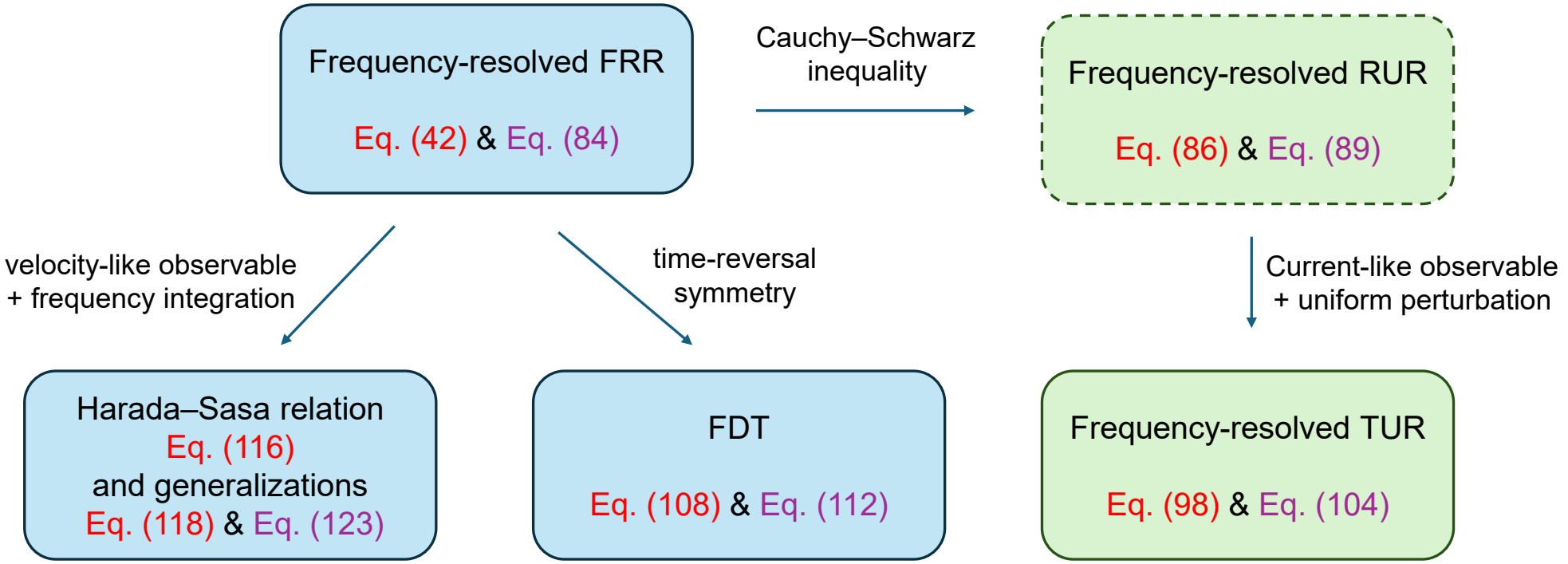
Final Remark

Recently, we also found the same hierarchy for FRT in the frequency domain. Kwon, Chun, Park, JSL, arXiv:2605.05038

Dechant, PRL 136, 207101 (2026)

$$C_{\boldsymbol{\theta},\boldsymbol{\theta}^\tau}(\omega) = \sum_{k,l=1}^{N} \int d\boldsymbol{z}\, R^{\boldsymbol{\theta}}_{\phi_k(\boldsymbol{z})}(\omega) \left[A^{\phi}(\boldsymbol{z})^{-1}\right]_{kl} \left[R^{\boldsymbol{\theta}}_{\phi_l(\boldsymbol{z})}(\omega)\right]^\dagger$$

$$\left[R^{\boldsymbol{\theta}}_{\boldsymbol{F}\mapsto\boldsymbol{F}+\epsilon\boldsymbol{\psi}}(\omega)\right]^\dagger C_{\boldsymbol{\theta},\boldsymbol{\theta}^\tau}(\omega)^{-1} R^{\boldsymbol{\theta}}_{\boldsymbol{F}\mapsto\boldsymbol{F}+\epsilon\boldsymbol{\psi}}(\omega) \leq \sum_{k=1}^{N} \left\langle \frac{\mu_k |\psi_k|^2}{2T_k} \right\rangle_{\mathrm{ss}}$$



→ E. Kwon's talk (11:40-12:00 Wed)

Collaborators



Prof. Hyunggyu Park (KIAS)



Dr. Hyun-Myung Chun (KIAS)



Euijoon Kwon (KIAS)

Main reference: Kwon, Chun, Park, JSL, PRL 135, 097101 (2025)

Chun, Kwon, Park, JSL, arXiv:2601.16387

Kwon, Chun, Park, JSL, arXiv:2605.05038