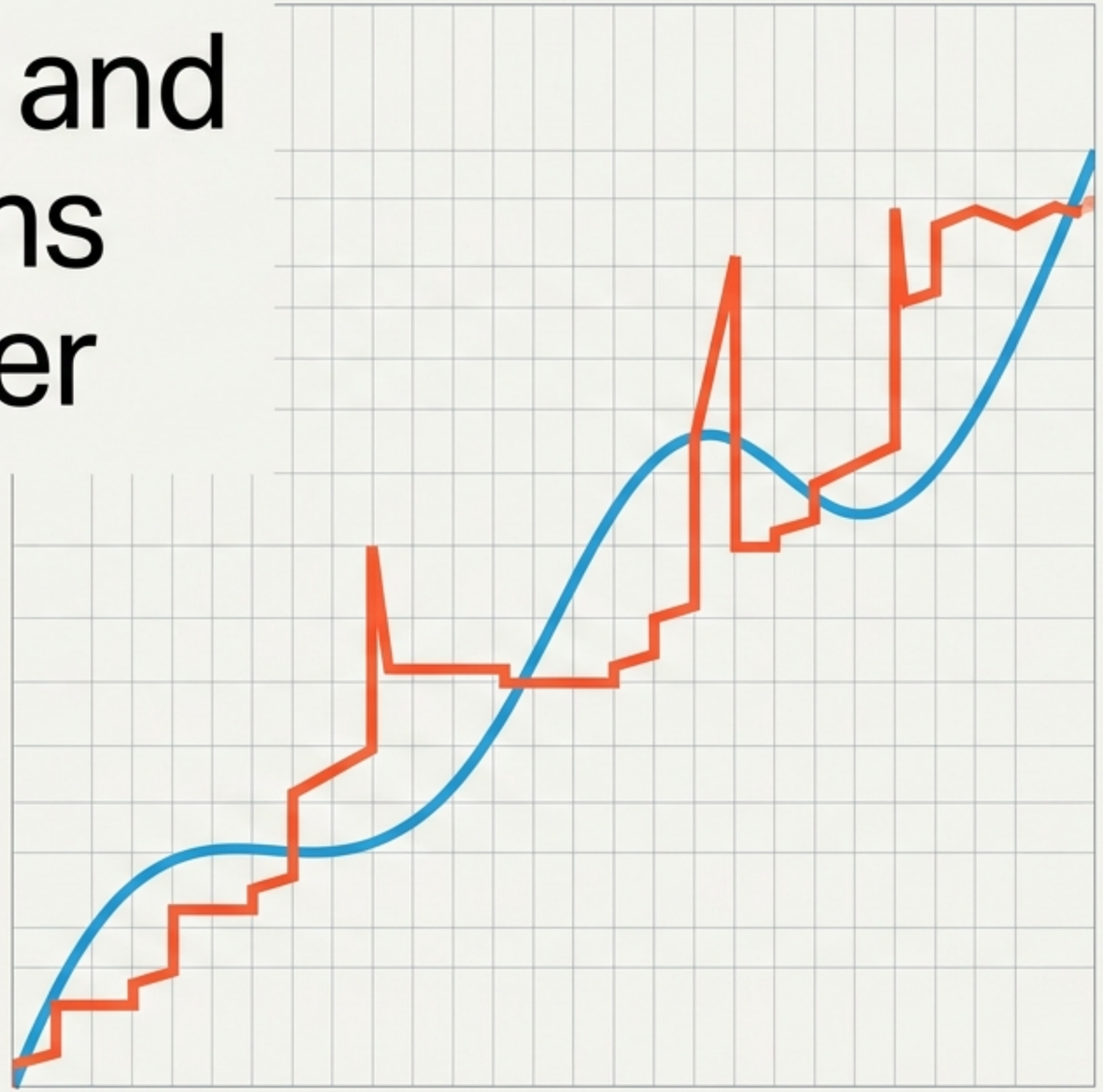
Energy Conversion and Fluctuation Relations in Lévy Active Matter

Authors: Yuanfei Huang and Jong-Min Park
(Asia Pacific Center for Theoretical Physics)

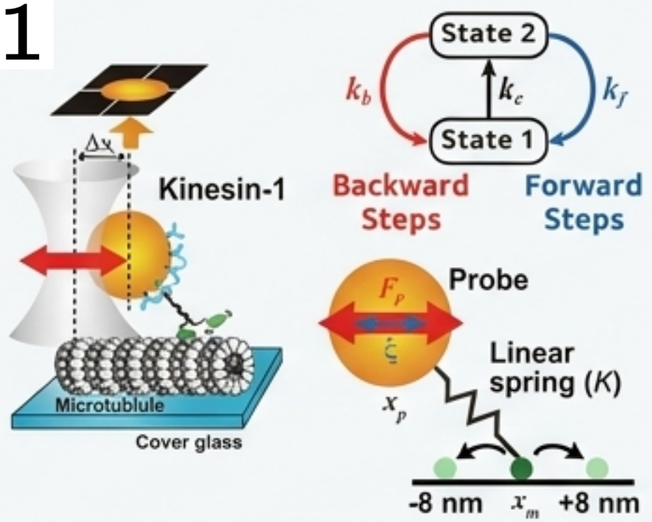


> **Core Premise**: Heavy-tailed active fluctuations act as a genuine thermodynamic resource, enabling energy conversion beyond classical equilibrium bounds.

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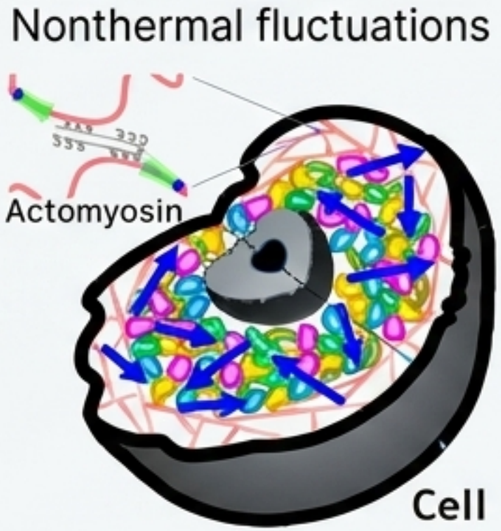
Beyond Gaussian Systems: The Physical Motivation

1



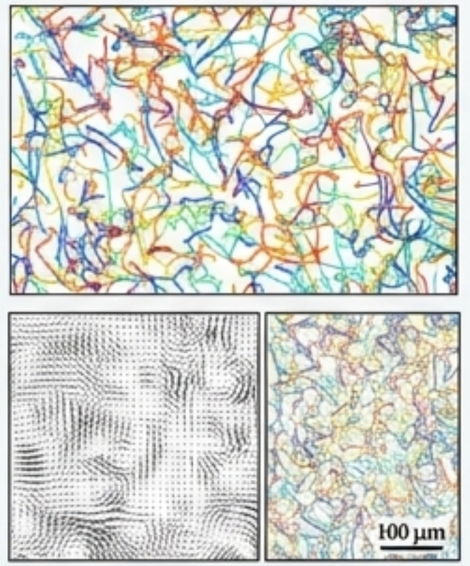
Kinesin-1 stepping dynamics

2



Actomyosin networks in living cells

3



Bacterial motion trajectories

Traditional Gaussian noise models assume purely Gaussian noise. However, complex biological and physical systems exhibit non-thermal, non-Gaussian fluctuations.

The Challenge: How do we accurately model systems characterized by anomalous diffusion, heavy-tailed distributions, and discontinuous jump processes?

[1]T. Ariga el. PRL 127, 178101 (2021)
[2]T. Ariga. Biophysical Reviews (2024) 16:605–612
[3]E. Lauga and R. E. Goldstein, Phys. Today 65, 30 (2012).

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Standard stochastic thermodynamics assumes smooth diffusion, but active matter jumps.

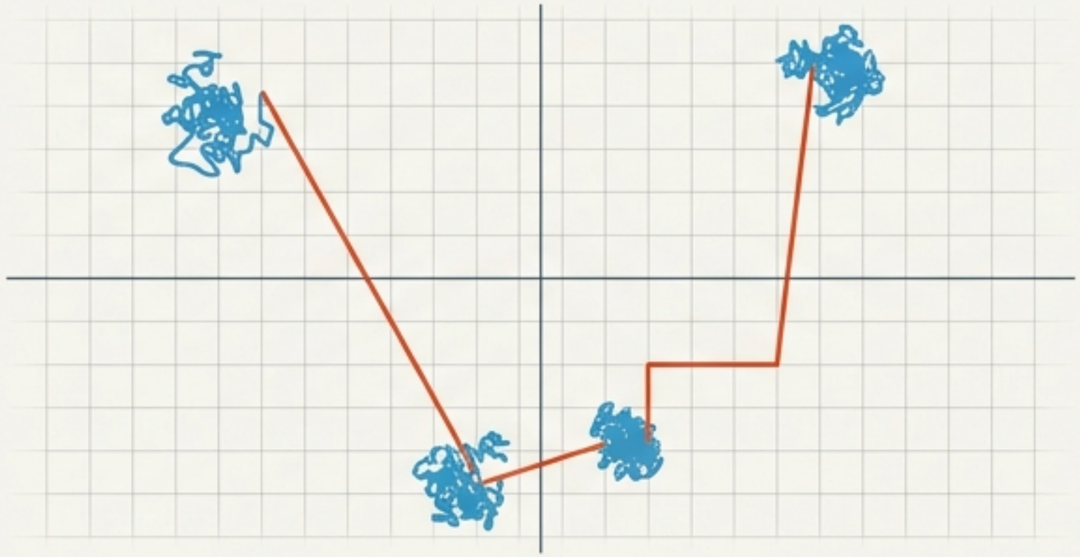
Standard trajectory-level definitions of heat, work, and entropy production become incomplete when faced with the heavy-tailed statistics of real biological and synthetic active systems.

The Gaussian Assumption



Continuous Gaussian Noise. Successfully describes near-equilibrium systems and weakly driven active matter.

The Active Reality



Strongly Intermittent Dynamics. Found in bacterial motility, molecular motors, and amoeboid motion. Characterized by rare, large fluctuations lacking finite second moments.

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1

System Total Force

$F(\mathbf{x};\lambda_t) = -\nabla U(\mathbf{x};\lambda_t) + \mathbf{f}_{\mathrm{nc}}(\mathbf{x})$

Total external force containing conservative and nonconservative forces modulated by protocol λ_t.

3

Active Environment

Independent pure-jump Lévy white noise characterized by Lévy measure $\nu(dz)$. Encodes compound Poisson processes, bacterial run-and-tumble, and molecular motor discrete stepping.

$$\dot{x}(t) = \frac{F(x;\lambda_t)}{\gamma} + \xi_{\mathrm{th}}(t) + \xi_{\mathrm{act}}(t)$$

2

Thermal Environment

Gaussian white noise.
Zero mean, covariance
$\langle \xi_{\mathrm{th},i}(t)\xi_{\mathrm{th},j}(t')\rangle = 2D_{\mathrm{th}}\delta_{ij}\delta(t-t')$.

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The Nonlocal Lévy-Fokker-Planck Equation

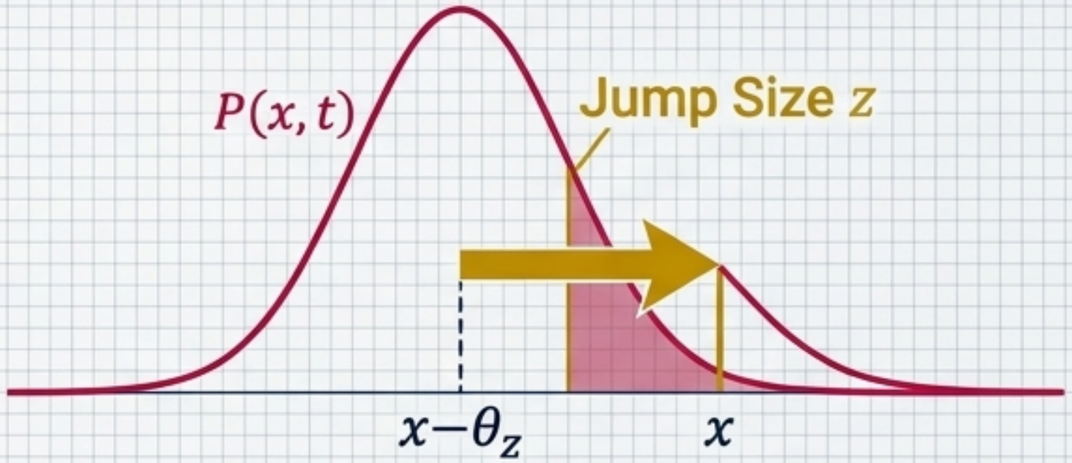
$$\partial_t P(x,t) = -\nabla \cdot [V(x,t)P(x,t)]$$

$$V(x,t) = \frac{F(x;\lambda_t)}{\gamma} - D_{th}\nabla \ln P(x,t) - S_L(x,t)$$

The Lévy Score (S_L)

Y Huang, C Liu, X Zhou
SIAM Journal on Numerical Analysis 64 (3), 656-684

$$S_L(x,t) \equiv -\int_0^1 d\theta \int \nu(dz)\, z \left[\frac{P(x-\theta z,t)}{P(x,t)}\right]$$



Unlike local thermal diffusion, S_L depends on displaced positions. It acts as a nonlocal entropic force quantifying jump-induced probability redistribution.

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Generalized Second Law via Probability Current

$$\dot{S}_{tot} = \dot{S}_{sys} + \dot{S}_m + \dot{S}_{act} \geq 0$$

Y Huang, C Liu, B Miao, X Zhou
Physical Review Letters 136 (6), 068302

System Entropy

$$\dot{S}_{sys}(t) = -\int dx\, \partial_t P(\boldsymbol{x}, t) \ln P(\boldsymbol{x}, t)$$

Medium Entropy ($\dot{S}_m$)

$$\dot{S}_m(t) = \int dx \left[\frac{\boldsymbol{F} \cdot \boldsymbol{J}}{k_B T}\right] = -\frac{\langle \dot{Q} \rangle}{k_B T}$$

The conventional environmental contribution.

Active Entropy ($\dot{S}_{act}$)

$$\dot{S}_{act}(t) = -\int dx \left[\frac{\boldsymbol{S}_L \cdot \boldsymbol{J}}{D_{th}}\right]$$

Entropy induced by the nonlocal Lévy score.

Key Insight: Total entropy is nonnegative, vanishing only when the local mean velocity / probability current is zero.

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Redefining the First Law

$$dU = dW + dQ$$

The Marcus Product ($\diamond$): Preserves the ordinary chain rule for functions of jump processes, extending trajectory-level energetics to discontinuous dynamics.

External Work

$$dW \equiv \partial_\lambda U d\lambda(t) + f_{\mathrm{nc}} \diamond dx(t)$$

Energy Exchange

$$dQ \equiv \gamma\left[-\dot{x} + \xi_{\mathrm{th}} + \xi_{\mathrm{act}}\right] \diamond dx$$

By combining Marcus calculus with external, thermal, and active energy sources, we secure a mathematically consistent First Law of Thermodynamics for Lévy active dynamics.

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Generalized Crooks Fluctuation Theorem

$$\frac{P_F[\Sigma]}{P_R[-\Sigma]} = \exp\left(-\frac{\Delta F}{k_B T} + \Sigma\right)$$

$$\Delta F = -k_B \mathcal{T} \ln \frac{\hat{Z}_0}{Z_0}$$

$$\Sigma \equiv \frac{W}{k_B T} - \int \frac{F}{k_B T} \diamond \xi_{\text{act}}(t)\, dt + \sum \log\left(\frac{\nu[\phi(t) - \phi(t^-)]}{\nu[\phi(t^-) - \phi(t)]}\right)$$

Classical Work

Standard thermodynamic work extracted during the protocol.

Energetic Coupling

The Marcus Integral ($\diamond$) capturing the coupling between deterministic forces and active Lévy jumps.

Jump Asymmetry

Accounts for the spatial asymmetry of the heavy-tailed jump statistics (ν), directly driving active entropy.

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Deriving fluctuation relations requires a novel conditioning path-integral approach.

Step 1

The Barrier: Constructing an explicit Onsager-Machlup functional for jump-diffusion processes is generally intractable. Standard **trajectory time-reversals fail.**

Step 2

The Solution: Develop a novel path-integral representation by explicitly conditioning on the discrete jump events.

Step 3

The Result: A precise mathematical log-ratio comparing forward paths $P_F[\phi(t)|\phi(0)]$ against time-reversed paths $P_R[\hat{\phi}(t)|\hat{\phi}(0)]$.

By assuming both forward and reverse processes initialize from their respective thermal equilibrium states (Boltzmann distributions), we can derive the exact fluctuation theorems governing the Lévy active matter.

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Generalized Fluctuation Relations

Generalized Crooks Fluctuation Theorem

$$\frac{P_F(\Sigma)}{P_R(-\Sigma)} = \exp\left[-\frac{\Delta F}{k_B T} + \Sigma\right]$$

Where the trajectory functional is:

$$\Sigma[\varphi] \equiv \frac{W}{k_B T} + A[\varphi]$$

Generalized Jarzynski Equality

$$\left\langle \exp\left[-\frac{W}{k_B T} - A\right]\right\rangle = e^{-\frac{\Delta F}{k_B T}}$$

In the absence of active jumps, the gold terms vanish, reducing exactly to standard relations. Heavy-tailed pure-jump active fluctuations fundamentally rewire equilibrium free-energy connections to non-equilibrium work.

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Two Faces of Active Entropy

Probability-Current-Based (ΔS_{act})

- **Origin:** Derived via local probability current velocity field $\boldsymbol{V}(\boldsymbol{x}, \boldsymbol{t})$.
- **Second Law:**
  $\Delta S_{tot} = \Delta S_{sys} + \Delta S_m + \Delta S_{act} \geq 0.$

Advantage: Provides a tighter bound in near-steady-state (quasistationary) conditions.

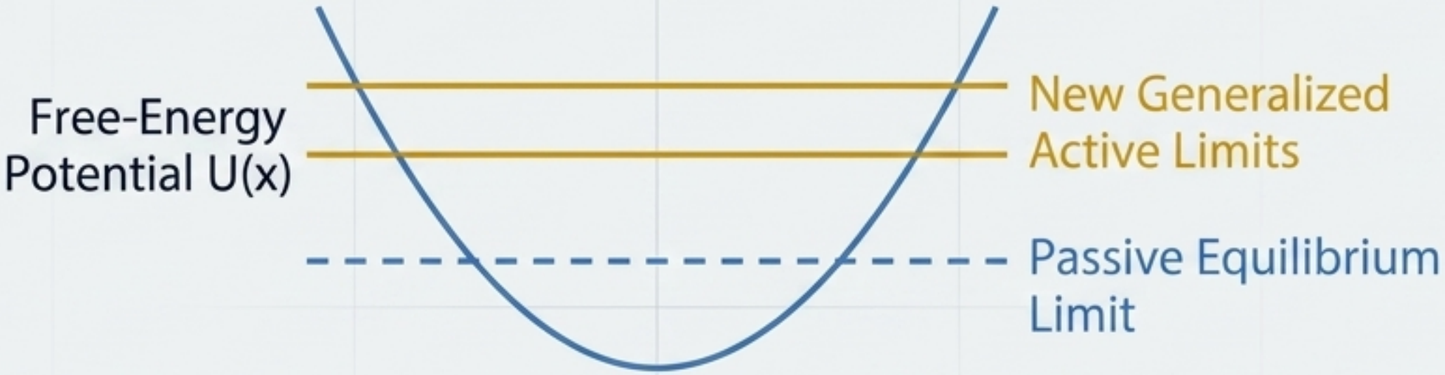
Irreversibility-Based ($\langle A \rangle$)

- **Origin:** Derived from path-probability trajectory irreversibility and force fields.
- **Second Law:**
  $\Delta S_{irr} = \Delta S_{sys} + \Delta S_m + \langle A \rangle \geq 0.$

Advantage: More computationally tractable; provides a tighter bound when the system exhibits low irreversibility (e.g., finite-time short protocols).

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Thermodynamic Consequences I: Work Extraction



Active fluctuations act as a thermodynamic resource, permitting work extraction beyond passive equilibrium limits.

Generalized Clausius Inequalities

Bound 1 (Irreversibility)

$$\langle W_{\text{out}} \rangle \leq -\Delta F_{\text{neq}} + k_B T \langle A \rangle$$

Generalized Clausius Inequalities

Bound 2 (Probability-Current)

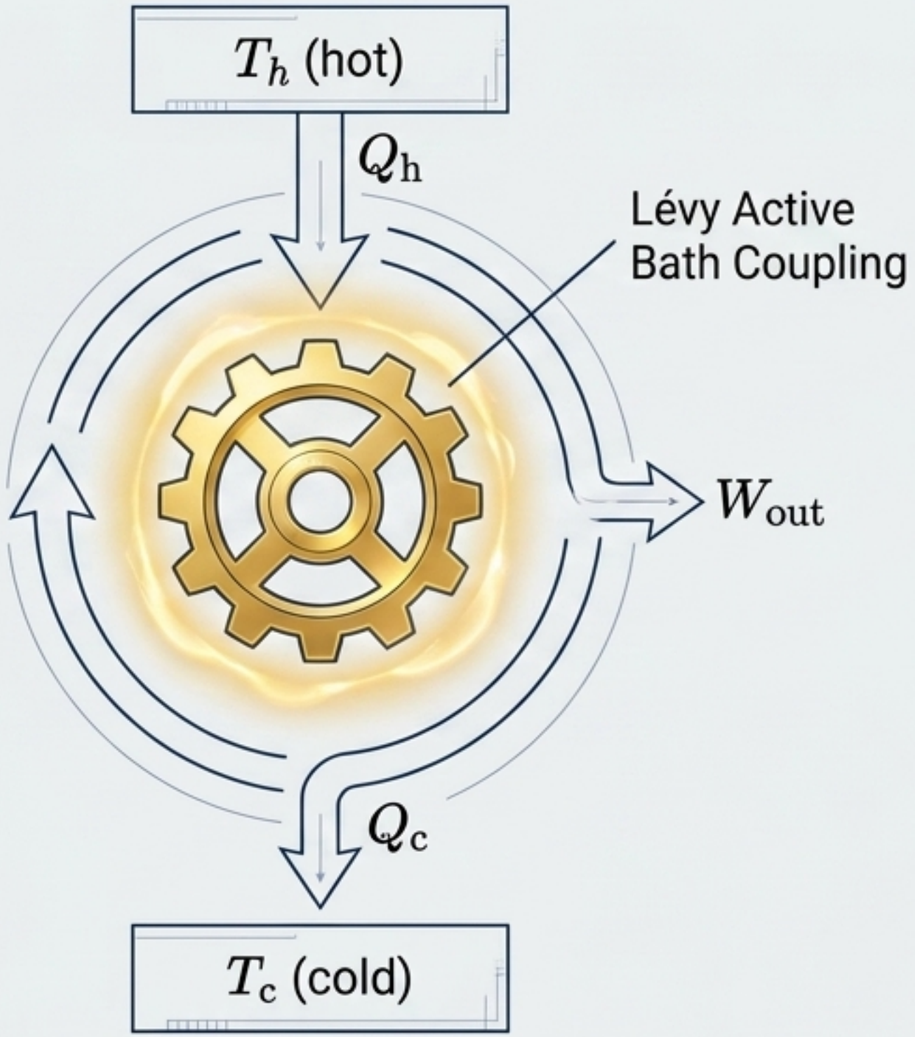
$$\langle W_{\text{out}} \rangle \leq -\Delta F_{\text{neq}} + k_B T \Delta S_{\text{act}}$$

$$\mathcal{F}_{\text{neq}} \equiv \langle U \rangle - k_B \mathcal{T} S_{\text{sys}}$$

A positive value for either active entropy metric reduces the effective free-energy cost. A system coupled to a Lévy bath can output more work than passively possible.

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Thermodynamic Consequences II: Cyclic Energy Converters



Apparent Efficiency Bound

$$\eta_\mathrm{appr} \equiv \frac{\langle W_\mathrm{out}\rangle}{\langle Q_\mathrm{h}\rangle}, \qquad \eta_\mathrm{appr} \leq \eta_C + \left[\frac{k_B T_\mathrm{c}}{\langle Q_\mathrm{h}\rangle}\right]\langle A\rangle$$

Insight: Apparent efficiency exceeding passive Carnot (η_C) does not violate the second law—it is actively fueled by $\langle A\rangle$.

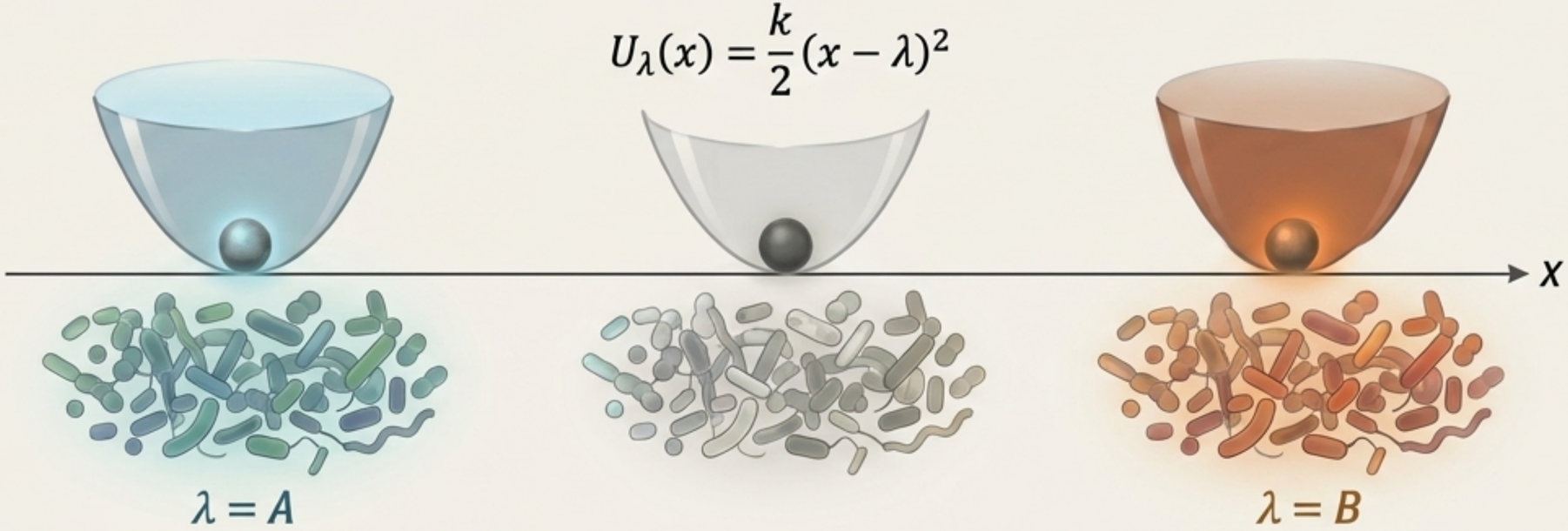
Power-Efficiency Trade-Off

$$P \leq \frac{P_\mathrm{C}}{1-\eta_\mathrm{C}}\left[\eta_\mathrm{C} - \eta_\mathrm{appr} + \frac{k_\mathrm{B} T_\mathrm{c}\langle A\rangle}{\langle Q_\mathrm{h}\rangle}\right]$$

Insight: Passive engines must drop power to zero to reach η_C. Lévy active fluctuations enlarge the accessible operating regime, allowing finite power at higher efficiencies.

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Numerical Validation A: Active Brownian particle in a moving harmonic potential.



Forward Protocol ($A \to B$)

The trap moves at velocity $+u$. The active jumps counteract the trap (headwind), explicitly breaking spatial symmetry.

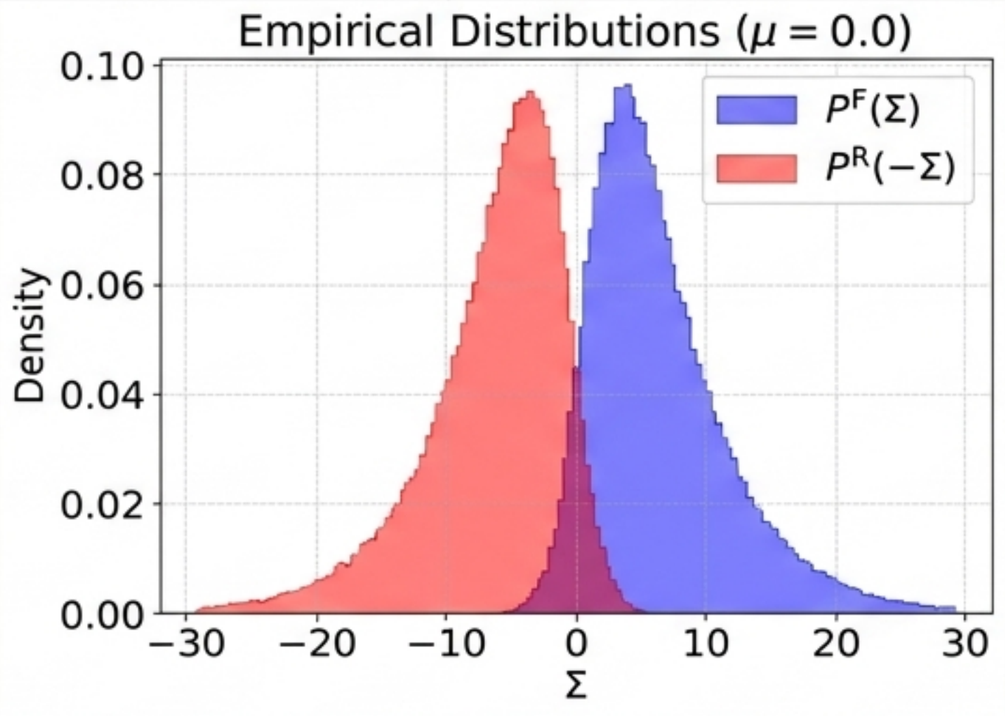
Reverse Protocol ($B \to A$)

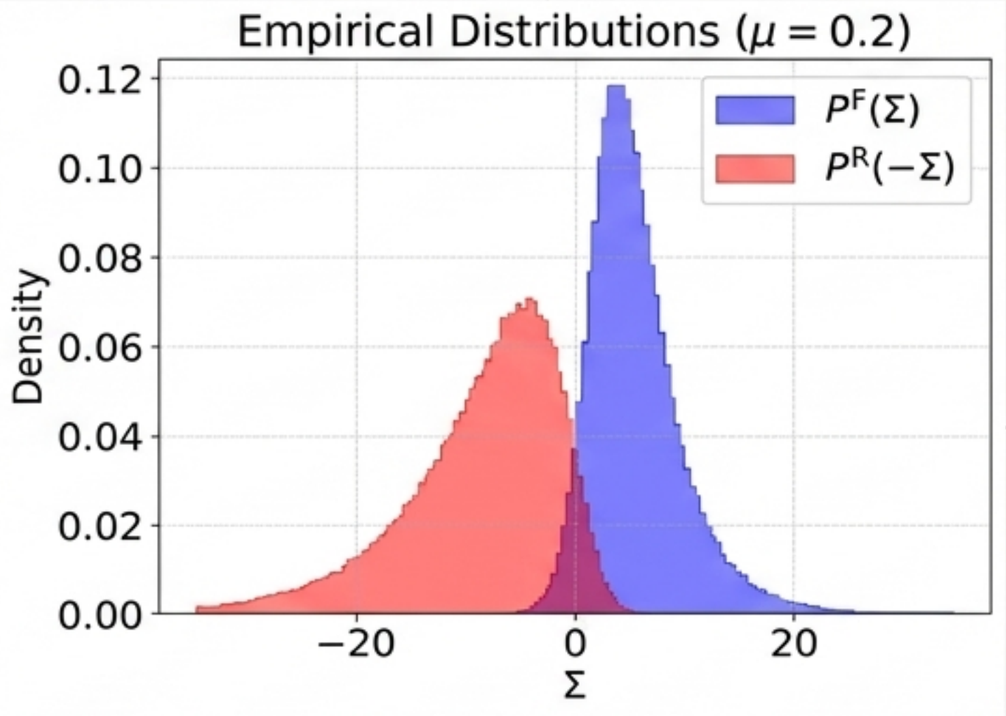
The trap moves at velocity $-u$. The active jumps assist the trap (tailwind).

The system is prepared in passive equilibrium prior to active driving. Thermal noise is Gaussian.
Active driving is modeled as Poisson shot noise with jump amplitudes drawn from a Gaussian distribution.

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Simulation data confirms the microscopic validity of the generalized Crooks theorem.

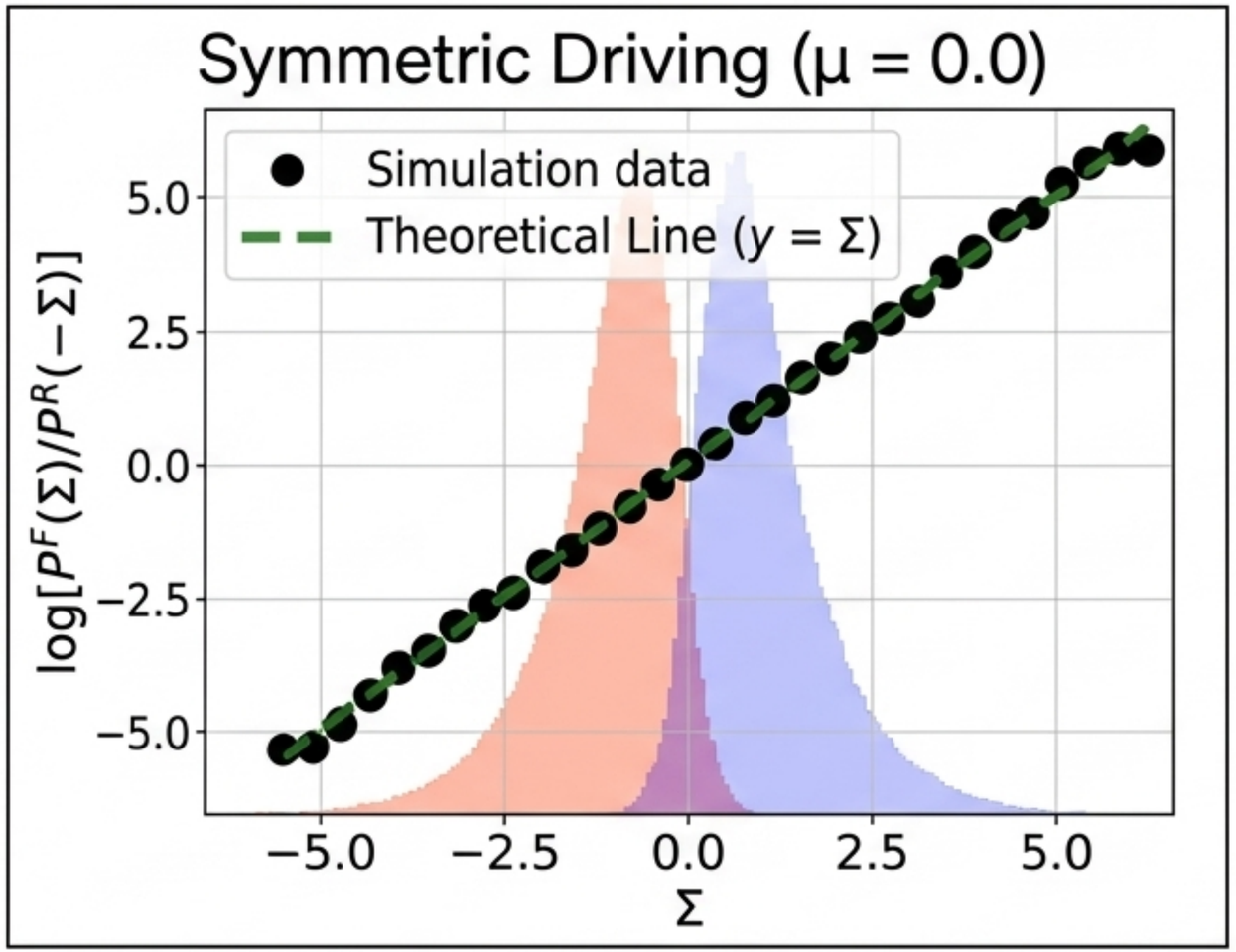


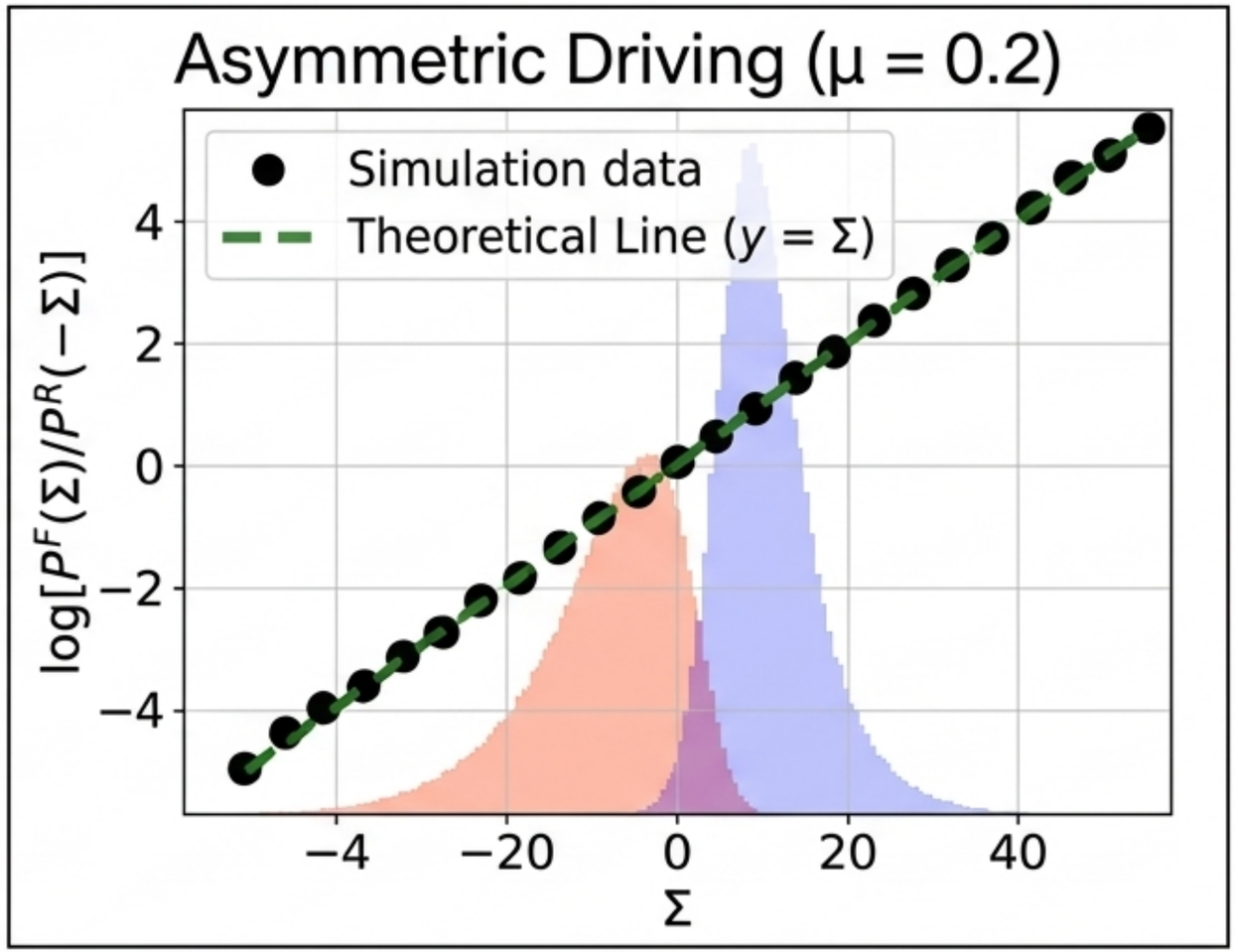


- Evaluating the empirical PDFs over 2×10^5 stochastic trajectories perfectly raio$\left\langle \ln \left[\frac{P_{\mathrm{F}}(\Sigma)}{P^{\mathrm{F}}(-\Sigma)} \right] \right\rangle = \Sigma.$
- Conclusion: The exact overlap confirms that non-Gaussian fluctuations are not a breakdown of thermodynamics, but a universal, calculable non-equilibrium resource for energy conversion.

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Validating the Generalized Crooks Relation



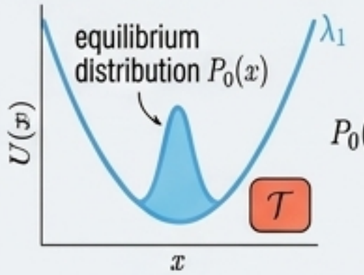


Conclusion: The log-ratio of forward to reversed probability distributions flawlessly aligns with the theoretical line ($\ln[P_F/P_R] = \Sigma$). This perfect intersection confirms the framework's exact microscopic validity, even when spatial symmetry is broken to trigger active entropy production.

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Numerical Validation B: Bounding Extractable Work

Stage I. Passive Preparation ($\lambda = \lambda_1$)

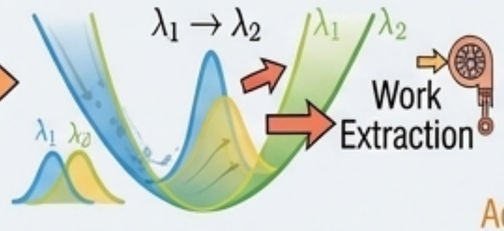


$$\lambda = \lambda_1$$

$$P_0(x) = Z_0^{-1} \exp\left[-\frac{\lambda_1 x^2}{2k_B T}\right]$$

Active Reservoir Absent $\mathcal{T}$
Equilibration with Thermal Bath

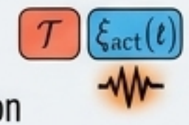
Stage II. Active Decompression ($\lambda_1 \to \lambda_2$)



$$\lambda_1 \to \lambda_2$$
$$0 \le t \le T_1$$
$$\lambda_2 < \lambda_1$$

Active Drive On
Work Extracted
Non-equilibrium Evolution

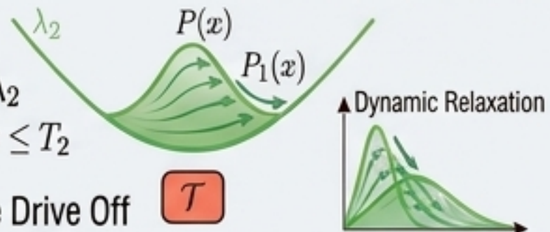
Active Drive



THERMODYNAMIC CYCLE

Stage III: Passive Relaxation

$$\lambda = \lambda_2 \quad T_1 < t \le T_2$$



$$\lambda = \lambda_2$$
$$T_1 < t \le T_2$$

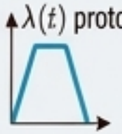
Active Drive Off
System Relaxing to $P_1(x)$

$$P_1(x) = Z_1^{-1} \exp\left[-\frac{\lambda_{ss} x^2}{2k_B T}\right]$$

Stage IV. Reverse Compression & Final Relaxation ($\lambda_2 \to \lambda_1$)



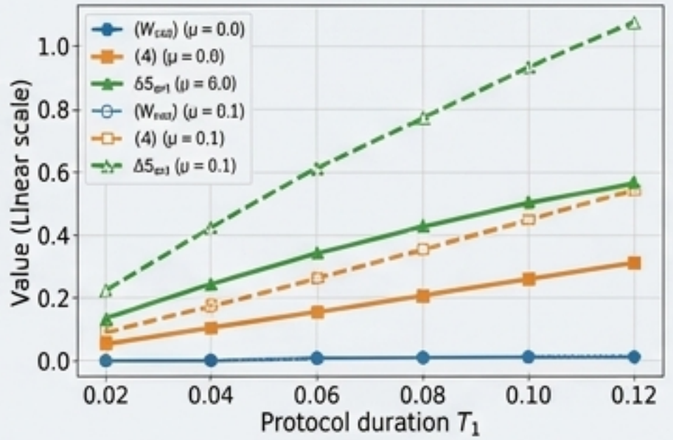
λ_1-ret $\quad \lambda_2 \to \lambda_1$
T_1
Reverse Drive
Compression
(no active noise)



Final Relaxation to P_0

$\lambda \to \lambda_1$ at $T_2 + T_1$, then fixed $\lambda = \lambda_1$

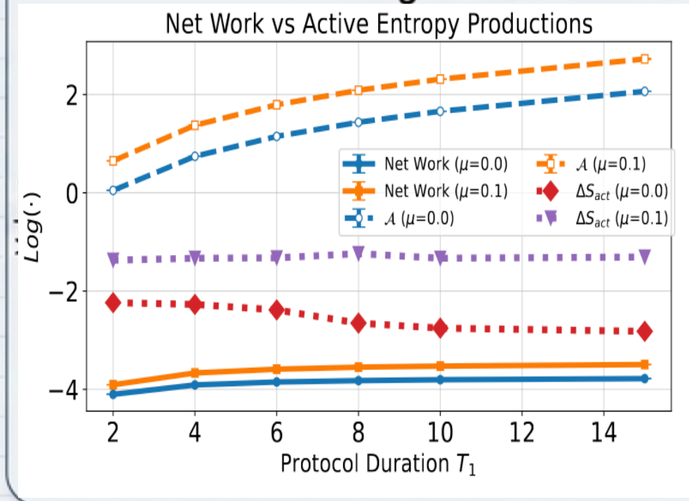
Short-Time Protocol



Net work $\langle W_{\text{net}} \rangle$ is strictly bounded by both metrics.

$\langle A \rangle$ provides the tighter, closer bound when far from stationarity.

Long-Time Protocol

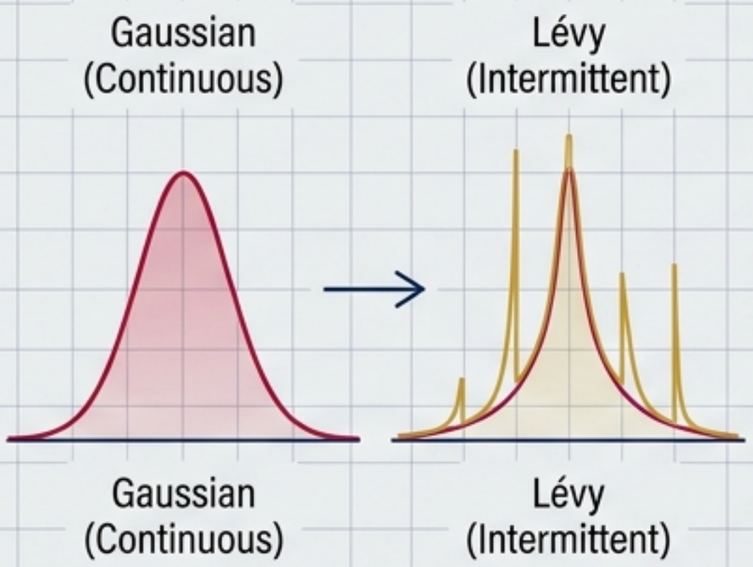


Approaching quasi-static operation, the probability-current metric ΔS_{act} becomes the tighter constraint.

Extractable work is governed purely by non-equilibrium activity.

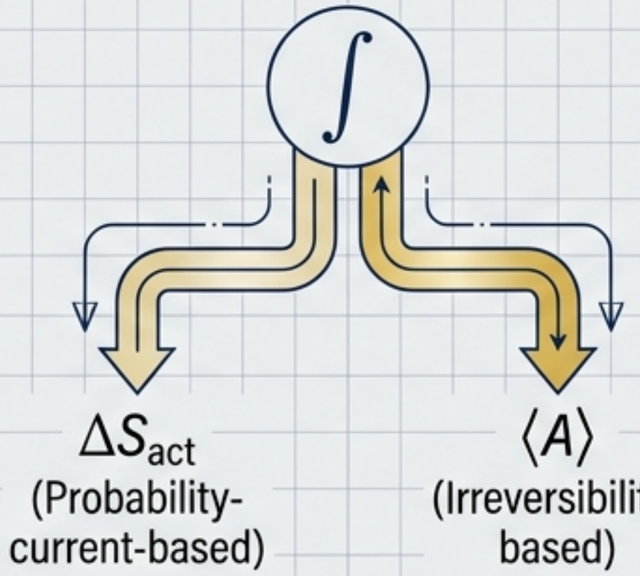
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The New Paradigm: Heavy-Tailed Dynamics



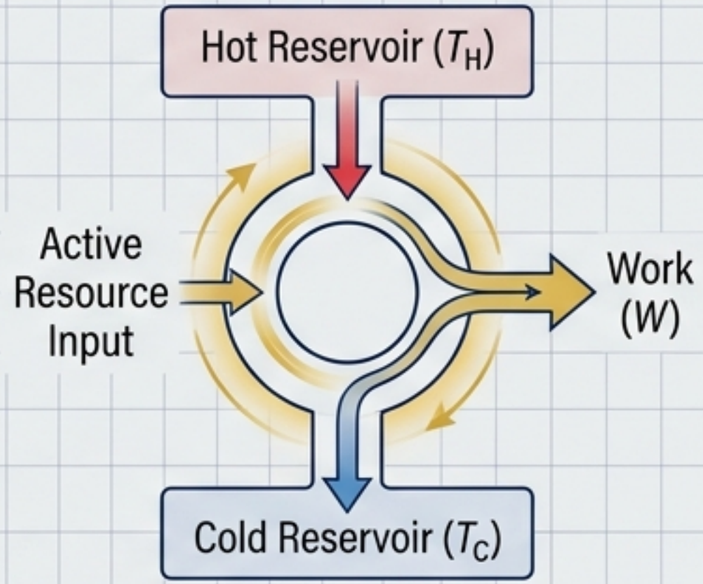
Standard stochastic thermodynamics relies on continuous Gaussian noise, failing in strongly intermittent active matter.

Mathematical Extension: Trajectory-Level Energetics



Marcus stochastic integrals resolve discontinuous trajectories, generating two new measures of active entropy production.

Thermodynamic Impact: Generalized Bounds



Active entropy serves as a thermodynamic resource, modifying Crooks/Jarzynski relations and pushing cyclic converters beyond passive apparent-efficiency bounds.

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