The SL(K+3,C) Symmetry of Bosonic String Theory

Sheng-Hong Lai (NCTU)

arXiv: 1806.05033 with Jen-Chi Lee, Yi Yang

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Linear Relations in the Hard Scattering limit

- Symmetry :
  - Gauge theory : “Symmetry dictates interaction”
  - String theory : self-consistent condition => interactions=>symmetry=?
- QFT: Spontaneously broken symmetries are
  - Hidden at low-energy
  - Evident at high-energy
- String: Infinite symmetries are suggested by
  - Soft UV of hard string scatterings (Exponential fall off)
  - Field theory hard scatterings (Power law)
  - Infinite states with no free parameters=>Huge Symmetry Group or Relations among SSA!?
- Conjectures: Saddle-point approximation (high-energy, fixed angle limit) (Gross 1988 PRL)
  - Existence of *linear relations* among hard String Scattering Amplitudes of different string state.

Linear Relations in the Hard Scattering limit

- There are four leading order SSA in the decoupling of ZNS (Lee 1990 PRL) calculations at the mass level $M_2^2 = 4$ (Chan-Lee 2004)

$$T_{TTT} : T_{LLT} : T_{(LT)} : T_{[LT]} = 8 : 1 : -1 : -1$$

- A simple analogy from particle physics. The ratios of the nucleon-nucleon scattering processes

(a) $p + p \rightarrow \mathrm{d} + \pi^+$ (b) $p + n \rightarrow \mathrm{d} + \pi^0$ (c) $n + n \rightarrow \mathrm{d} + \pi^-$

$$T_a : T_b : T_c = 1 : \frac{1}{\sqrt{2}} : 1$$

Linear Relations in the Hard Scattering limit

- The linear relations can be generalized to higher mass levels: $M_2^2 = 2(N-1)$. One first note that for each fixed mass level N, states of leading order in the Hard string scattering limit are of the following form (CHLTY, 2006, PRL)

$$V_2 \to |N, 2m, q\rangle \equiv \left|\left(\alpha_{-1}^T\right)^{N-2m-2q}\left(\alpha_{-1}^L\right)^{2m}\left(\alpha_{-2}^L\right)^q\right| 0; k\rangle$$

- The ratios among hard SSA of different string states at each fixed mass level N are (Decoupling of ZNS) (to all string loops!) (CHLTY, 2006, PRL)

$$\frac{A_{st}^{N,2m,q}}{A_{st}^{N,0,0}} = \left(\frac{-1}{M_2}\right)^{2m+q}\left(\frac{1}{2}\right)^{m+q}(2m-1)!!$$

- At string-tree level, the above ratios can be explicitly calculated by saddle-point method.

- Gross conjectures were explicitly proved!!

Recurrence relations in Regge scattering limit

- SSA in High energy Fixed momentum transfer or Regge Regime (RR):

$$s \to \infty, \sqrt{t} = fixed (but\ \sqrt{t} \neq \infty)$$

- The SSA of one arbitrary state and three tachyons can be written as the "first Appell function"[Lee-Yang, 2014]

$$A_{st}^{(r_n^T, r_m^P, r_l^L)} \cong B\left(\frac{-t}{2} - 1, \frac{-s}{2} - 1\right)$$

$$\times \prod_{n=1} \left[(n-1)!\sqrt{-t}\right]^{r_n^T} \prod_{m=1} \left[(m-1)!\frac{\tilde{t}}{2M_2}\right]^{r_m^P} \prod_{l=1} \left[(l-1)!\frac{\tilde{t}'}{2M_2}\right]^{r_l^L}$$

$$\times F_1\left(-\frac{t}{2} - 1; -q_1; -r_1; -\frac{s}{2}; \frac{s}{\tilde{t}}, \frac{s}{\tilde{t}'}\right).$$

- Appell function can be define as

$$F_1(a; b_1, b_2; c; z_1, z_2) = \sum_{n_1, n_2 = 0}^{\infty} \frac{(a)_{n_1+n_2} (b_1)_{n_1} (b_2)_{n_2}}{(c)_{n_1+n_2}} \frac{z_1^{n_1}}{n_1!} \frac{z_2^{n_2}}{n_2!}$$

$$(a)_n = a \cdot (a+1) \cdots (a+n-1)$$

Recurrence relations in Regge scattering limit

- There are four fundamental recurrence relations which link the contiguous functions in the Appell case

$$\begin{aligned}&(a-b_1-b_2)\,F_1\left(a;b_1,b_2;c,x,y\right)-aF_1\left(a+1;b_1,b_2;c,x,y\right)\\&+b_1F_1\left(a;b_1+1,b_2;c,x,y\right)+b_2F_1\left(a;b_1,b_2+1;c,x,y\right)=0,\end{aligned}$$

$$\begin{aligned}&cF_1\left(a;b_1,b_2;c,x,y\right)+c\left(x-1\right)F_1\left(a;b_1+1,b_2;c,x,y\right)\\&-\left(c-a\right)xF_1\left(a;b_1+1,b_2;c+1,x,y\right)=0,\end{aligned}$$

$$\begin{aligned}&cF_1\left(a;b_1,b_2;c,x,y\right)-\left(c-a\right)F_1\left(a;b_1,b_2;c+1,x,y\right)\\&-aF_1\left(a+1;b_1,b_2;c+1,x,y\right)=0,\end{aligned}$$

$$\begin{aligned}&cF_1\left(a;b_1,b_2;c,x,y\right)+c\left(y-1\right)F_1\left(a;b_1,b_2+1;c,x,y\right)\\&-\left(c-a\right)yF_1\left(a;b_1,b_2+1;c+1,x,y\right)=0.\end{aligned}$$

- **Regge stringy symmetries or recurrence relations can be used to reduce the number of independent Regge SSA from ∞ down to 1 [Lee-Yang, 2014]**

- In the Regge limit, the string scattering amplitudes with the Appell function F_1 are associsted with the symmetry group SL(5,C).

The D-type Lauricella SSA

- We calculated the 26D open bosonic SSA of three tachyons and one arbitrary string state for **all** energies.

- We discovered that these SSA can be expressed in terms of the D-type Lauricella functions with associated SL(K+3,C) symmetry. (Lai, Lee, Yang, 2016, JHEP)

- The results of the hard scattering limit, the Regge scattering limit can be rederived from the D-type Lauricella SSA.

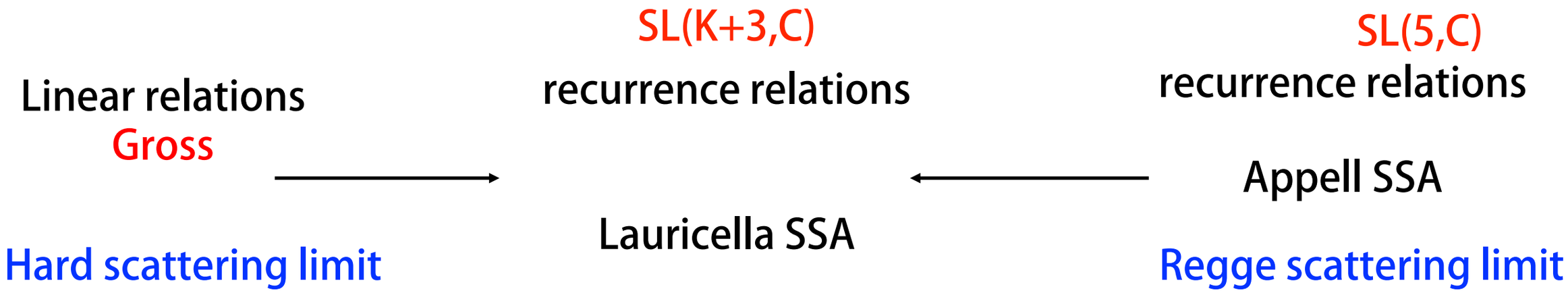
$$A_{st}^{(p_n;q_m;r_l)} = \frac{\Gamma(\frac{-s}{2}-1+N)\Gamma(\frac{-t}{2}-1)}{\Gamma(\frac{u}{2}+2)} \prod_{n=1} \left[-(n-1)!k_3^T\right]^{p_n} \prod_{m=1} \left[-(m-1)!k_3^P\right]^{q_m} \prod_{l=1} \left[-(l-1)!k_3^L\right]^{r_l}$$
$$\cdot F_D^{(K)} \left(\begin{array}{c} -\frac{t}{2}-1; \{-p_1\}^1, ..., \{-p_n\}^n, \{-q_1\}^1, ..., \{-q_m\}^m, \{-r_1\}^1, ..., \{-r_l\}^l; \frac{s}{2}+2-N; \\ \left[z_1^T\right], ..., \left[z_n^T\right], \left[z_1^P\right], ..., \left[z_m^P\right], \left[z_1^L\right], ..., \left[z_l^L\right] \end{array} \right)$$

The recurrence relations of D-type Lauricella SSA

- The linear relations can be used to reduced the number of independent hard SSA from ∞ down to 1. (ZNS)

- The Regge stringy symmetries or recurrence relations can be used to reduce the number of independent Regge SSA from ∞ down to 1.

- To solve the Lauricella SSA, we will first generalize the 2+2 recurrence relations of the Appell functions to the K+2 recurrence relations of the D-type Lauricella functions.



The recurrence relations of D-type Lauricella SSA

- The K+2 recurrence relations of the D-type Lauricella functions.

$$
\begin{aligned}
& cF_D^{(K)}\left(a;b_1,...,b_K;c;x_1,...,x_K\right) \\
& -(c-a)\,F_D^{(K)}\left(a;b_1,...,b_K;c+1;x_1,...,x_K\right) \\
& -aF_D^{(K)}\left(a+1;b_1,...,b_K;c+1;x_1,...,x_K\right)=0,
\end{aligned}
$$

$$
\begin{aligned}
& \left(a-\sum_i b_i\right)F_D^{(K)}\left(a;b_1,...,b_K;c;x_1,...,x_K\right) \\
& -aF_D^{(K)}\left(a+1;b_1,...,b_K;c;x_1,...,x_K\right) \\
& +b_1F_D^{(K)}\left(a;b_1+1,...,b_K;c;x_1,...,x_K\right) \\
& +... \\
& +b_KF_D^{(K)}\left(a;b_1,...,b_K+1;c;x_1,...,x_K\right)=0,
\end{aligned}
$$

$$
\begin{aligned}
& cF_D^{(K)}\left(a;b_1,...,b_m,...,b_K;c;x_1,...,x_m,...,x_K\right) \\
& +c(x_m-1)F_D^{(K)}\left(a;b_1,...,b_m+1,...,b_K;c;x_1,...,x_m,...,x_K\right) \\
& +(a-c)x_mF_D^{(K)}\left(a;b_1,...,b_m+1,...,b_K;c+1;x_1,...,x_m,...,x_K\right)=0
\end{aligned}
$$

The extended LSSA

- To calculate the group representation of the LSSA for general K, we first define the extended LSSA

$$f_{ac}^{b_1 \dots b_K}(\alpha; \beta_1, \dots \beta_K; \gamma; x_1, \dots, x_K)$$
$$= B(\gamma - \alpha, \alpha) F_D^{(K)}(\alpha; \beta_1, \dots \beta_K; \gamma; x_1, \dots, x_K) a^{\alpha} b_1^{\beta_1} \dots b_K^{\beta_K} c^{\gamma}.$$

- By setting the a = 1 = c, the extended LSSA can reduce to LSSA

$$A_{st}^{(r_n^T, r_m^P, r_l^L)} = f_{11}^{-R_n^T, -R_m^P, -R_l^L}\left(-\frac{t}{2} - 1; -R_n^T, -R_m^P, -R_l^L; \frac{u}{2} + 2 - N; \tilde{Z}_n^T, \tilde{Z}_m^P, \tilde{Z}_l^L\right).$$

The Operators

- We first introduce the $(K+3)^2-1$ generators of SL(K + 3, C) group

- $E^{\alpha}=a\left(\sum_j x_j\partial_j+a\partial_a\right), E^{\beta_k}=b_k\left(x_k\partial_j+b_k\partial_{b_k}\right), E^{\gamma}=c\left(\sum_j\left(1-x_j\right)\partial_{x_j}+c\partial_c-a\partial_a-\sum_j b_j\,\partial_{b_j}\right),$
$E^{\alpha\gamma}=\mathrm{ac}\left(\sum_j\left(1-x_j\right)\partial_{x_j}-a\partial_a\right), E^{\beta_k\gamma}=b_kc\left[\left(x_k-1\right)\partial_{x_k}+b_k\partial_{b_k}\right], E^{\alpha\beta_k\gamma}=ab_kc\partial_{x_k},$

$$E_{\alpha}=\frac{1}{a}\left[\sum_j x_j\left(1-x_j\right)\partial_{x_j}+c\partial_c-a\partial_a-\sum_j x_jb_j\,\partial_{b_j}\right],$$

$$E_{\beta_k}=\frac{1}{b_k}\left[x_k(1-x_k)+x_k\sum_{j\neq k}x_j\left(1-x_j\right)\partial_{x_j}+c\partial_c-x_ka\partial_a-\sum_j b_j\partial_{b_j}\right],$$

$$E_{\gamma}=-\frac{1}{c}\left(\sum_j x_j\partial_{x_j}+c\partial_c-1\right), E_{\alpha\gamma}=\frac{1}{ac}\left[\sum_j x_j\left(1-x_j\right)\partial_{x_j}+c\partial_c-1-\sum_j x_jb_j\partial_{b_j}\right],$$

$$E_{\beta_k\gamma}=\frac{1}{b_kc}\left[x_k(x_k-1)\partial_{x_k}+\sum_{j\neq k}\left(x_j-1\right)x_j\partial_{x_j}+x_ka\partial_a-c\partial_c+1\right],$$

$$E_{\alpha\beta_k\gamma}=\frac{1}{ab_kc}\left[\sum_j x_j\left(x_j-1\right)\partial_{x_j}-c\partial_c+x_ka\partial_a+\sum_j x_jb_j\partial_{b_j}-x_k+1\right],$$

$E_{\beta_p}^{\beta_k}=\frac{b_k}{b_p}\left[\left(x_k-x_p\right)\partial_{x_k}+b_k\partial_{b_k}\right],(k\neq p), J_{\alpha}=a\partial_a, J_{\beta_k}=b_k\partial_{b_k}, J_{\gamma}=c\partial_c$

The Operators

- There are 1 E_α , K E_{β_k} , 1 E_γ ,1 $E_{\alpha\gamma}$,K $E_{\beta_k\gamma}$ and K $E_{\alpha\beta_k\gamma}$ which sum up to $3K+3$ lowering generators.

- There are also $3K+3$ raising operators.

- In addition, there are $K(K-1)$ $E^{\beta_k}_{\beta_p}$ and $K+2$ J , the Cartan subalgebra.

- In sum, the total number of generators are

$$2(3K+3)+K(K+1)+K+2=(K+3)^2-1$$

The Operators

- There are K+2 fundamental recurrence relations among $F_D^{(K)}(\alpha;\ \beta;\ \gamma;\ x)$ or the Lauricella functions.

- The K+2 equations together with K+2 equations for the operations are equivalent to the Cartan subalgebra and the simple root system of SL(K + 3, C) with rank K+2. With the Cartan subalgebra and the simple roots, one can easily write down the whole Lie algebra of the SL(K + 3, C) group. So one can construct the Lie algebra from the recurrence relations and vice vers.

- Recurrence relations among LSSA and reduce the number of independent LSSA from ∞ down to 1.

SL(K+3) algebra

- We can define

$$E^{\alpha} = E_{12}, E_{\alpha} = E_{21}, E^{\beta_k} = E_{k+3,3}, E_{\beta} = E_{3,k+3},$$
$$E^{\gamma} = E_{31}, E_{\gamma} = E_{13}, E^{\alpha\gamma} = E_{32}, E_{\alpha\gamma} = E_{23},$$
$$E^{\beta_k\gamma} = -E_{k+3,1}, E_{\beta_k\gamma} = -E_{1,k+3}, E_{\alpha\beta_k\gamma} = -E_{k+3,2},$$
$$E_{\alpha\beta_k\gamma} = -E_{2,k+3}, J_{\alpha}^{'} = \frac{1}{2}(E_{11} - E_{22}), J_{\beta_k}^{'} = \frac{1}{2}\left(E_{k+3,k+3} - E_{3,3}\right), J_{\gamma}^{'} = \frac{1}{2}(E_{33} - E_{11}).$$

- Then we can find the Lie algebra commutation relations of SL(k+3,C)

$$\left[E_{ij}, E_{kl}\right] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$$

Summary

- The SSA of three tachyons and one arbitrary string state of 26D open bosonic string theory can be expressed in terms of the Lauricella function for all energy.

- The LSSA can reproduce the linear relations/recurrence relations(Appell functions) in the hard/Regge limit respectively.

- We have to introduce the extended LSSA $f_{ac}^{b_1 \cdots b_K}$ to discuss the SL(K+3, C) group algebra.

- The K+2 recurrence relations among the LSSA can be used to reproduce the Cartan subalgebra and simple root system of the SL(K+3, C) group with rank K+2.

- The SL(K + 3, C) group can be used to solve all the LSSA and express them in terms of one amplitude

Thank you for your listening!

Recurrence relations in Regge scattering limit

- SSA in High energy Fixed momentum transfer or Regge Regime (RR):

$$s \to \infty, \sqrt{t} = fixed (but\ \sqrt{t} \neq \infty)$$

- The vertex two is the arbitrary state which can be written as

$$V_2 \to \left|r_n^T, r_m^P, r_l^L\right\rangle = \prod_{n>0} \left(\alpha_{-n}^T\right)^{r_n^T} \prod_{m>0} \left(\alpha_{-m}^P\right)^{r_m^P} \prod_{l>0} \left(\alpha_{-l}^L\right)^{r_l^L} |0, k\rangle.$$

- The SSA of one arbitrary state and three tachyons can be written as the "first Appell function"[Lee-Yang, 2014]

$$A_{st}^{(r_n^T, r_m^P, r_l^L)}$$

$$\cong B\left(\frac{-t}{2} - 1, \frac{-s}{2} - 1\right) \prod_{n=1} \left[(n-1)!\sqrt{-t}\right]^{r_n^T} \prod_{m=1} \left[(m-1)! \frac{\tilde{t}}{2M_2}\right]^{r_m^P} \prod_{l=1} \left[(l-1)! \frac{\tilde{t}'}{2M_2}\right]^{r_l^L}$$

$$\times F_1\left(-\frac{t}{2} - 1; -q_1; -r_1; -\frac{s}{2}; \frac{s}{\tilde{t}}, \frac{s}{\tilde{t}'}\right).$$

- Appell function can be define as $F_1(a; b_1, b_2; c; z_1, z_2) = \sum_{n_1, n_2 = 0}^{\infty} \frac{(a)_{n_1+n_2} (b_1)_{n_1} (b_2)_{n_2}}{(c)_{n_1+n_2}} \frac{z_1^{n_1}}{n_1!} \frac{z_2^{n_2}}{n_2!}$

$$(a)_n = a \cdot (a+1) \cdots (a+n-1)$$

The D-type Lauricella SSA

- The D-type Lauricella function is one of the four extensions of the Gauss hypergeometric function to K variables and is defined as

$$F_D^{(K)}\left(a; b_1, \ldots, b_K; c; x_1, \ldots, x_K\right)$$
$$= \sum_{n_1, \cdots, n_K} \frac{(a)_{n_1+\cdots+n_K}}{(c)_{n_1+\cdots+n_K}} \frac{(b_1)_{n_1} \cdots (b_K)_{n_K}}{n_1! \cdots n_K!} x_1^{n_1} \cdots x_K^{n_K}$$

- The associated symmetry of D-type Lauricella function is SL(K+3,C) symmetry. (Miller 1977)

The Operators

- We first introduce the $(K+3)^2-1$ generators of SL(K + 3, C) group

- $E^{\alpha}=a\left(\sum_j x_j\partial_j+a\partial_a\right), E^{\beta_k}=b_k\left(x_k\partial_j+b_k\partial_{b_k}\right), E^{\gamma}=c\left(\sum_j\left(1-x_j\right)\partial_{x_j}+c\partial_c-a\partial_a-\sum_j b_j\,\partial_{b_j}\right),$
  $E^{\alpha\gamma}=\mathrm{ac}\left(\sum_j\left(1-x_j\right)\partial_{x_j}-a\partial_a\right), E^{\beta_k\gamma}=b_kc\left[\left(x_k-1\right)\partial_{x_k}+b_k\partial_{b_k}\right], E^{\alpha\beta_k\gamma}=ab_kc\partial_{x_k},$

$$E_{\alpha}=\frac{1}{a}\left[\sum_j x_j\left(1-x_j\right)\partial_{x_j}+c\partial_c-a\partial_a-\sum_j x_jb_j\,\partial_{b_j}\right],$$

$$E_{\beta_k}=\frac{1}{b_k}\left[x_k(1-x_k)+x_k\sum_{j\neq k}x_j\left(1-x_j\right)\partial_{x_j}+c\partial_c-x_ka\partial_a-\sum_j b_j\partial_{b_j}\right],$$

$$E_{\gamma}=-\frac{1}{c}\left(\sum_j x_j\partial_{x_j}+c\partial_c-1\right), E_{\alpha\gamma}=\frac{1}{ac}\left[\sum_j x_j\left(1-x_j\right)\partial_{x_j}+c\partial_c-1-\sum_j x_jb_j\partial_{b_j}\right],$$

$$E_{\beta_k\gamma}=\frac{1}{b_kc}\left[x_k(x_k-1)\partial_{x_k}+\sum_{j\neq k}\left(x_j-1\right)x_j\partial_{x_j}+x_ka\partial_a-c\partial_c+1\right],$$

$$E_{\alpha\beta_k\gamma}=\frac{1}{ab_kc}\left[\sum_j x_j\left(x_j-1\right)\partial_{x_j}-c\partial_c+x_ka\partial_a+\sum_j x_jb_j\partial_{b_j}-x_k+1\right],$$

$$E_{\beta_p}^{\beta_k}=\frac{b_k}{b_p}\left[\left(x_k-x_p\right)\partial_{x_k}+b_k\partial_{b_k}\right],(k\neq p), J_{\alpha}=a\partial_a, J_{\beta_k}=b_k\partial_{b_k}, J_{\gamma}=c\partial_c$$