Holographic Entanglement Entropy in Boundary Conformal Field Theory

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[En-Jui Chang, C.J. Chou, Yi Yang: 1805.06117]

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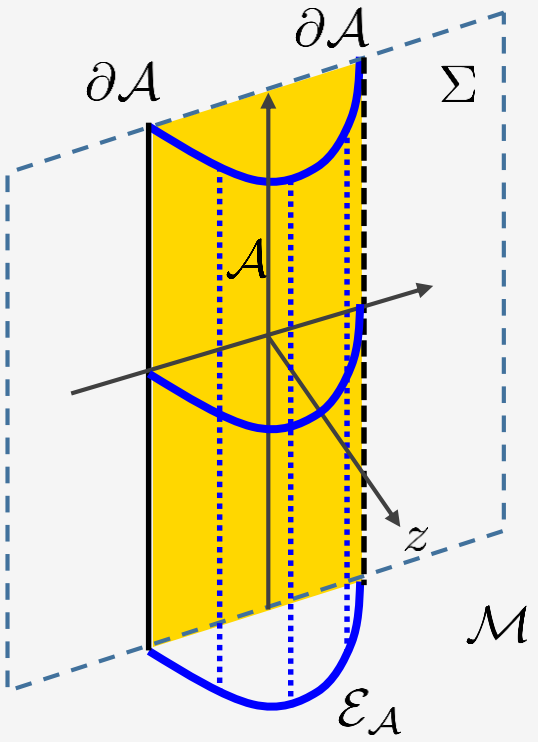
Outline

- Holographic Entanglement Entropy(HEE) of Static Time-Independent States in CFT
- Gravitational Setup Considering the Boundary in CFT
- HEE with Bulk Spacetime to be AdS-Schwarzschild Black Hole
- Phase Structure of HEE of BCFT
- Araki-Lieb Inequality and Holographic Entanglement Plateau
- Summary

Holographic Entanglement Entropy in QFT

- $(d+1)$-dim CFT defined on the codimension-1 static spacelike hypersurface.
- HEE of entanglement region $\mathcal{A}$ is calculated by Ryu-Takayanagi formula:

$$S_{\mathcal{A}} = \frac{\min(\text{Area}(\mathcal{E}_{\mathcal{A}}))}{4G_N^{(d+2)}}.$$



Reference:

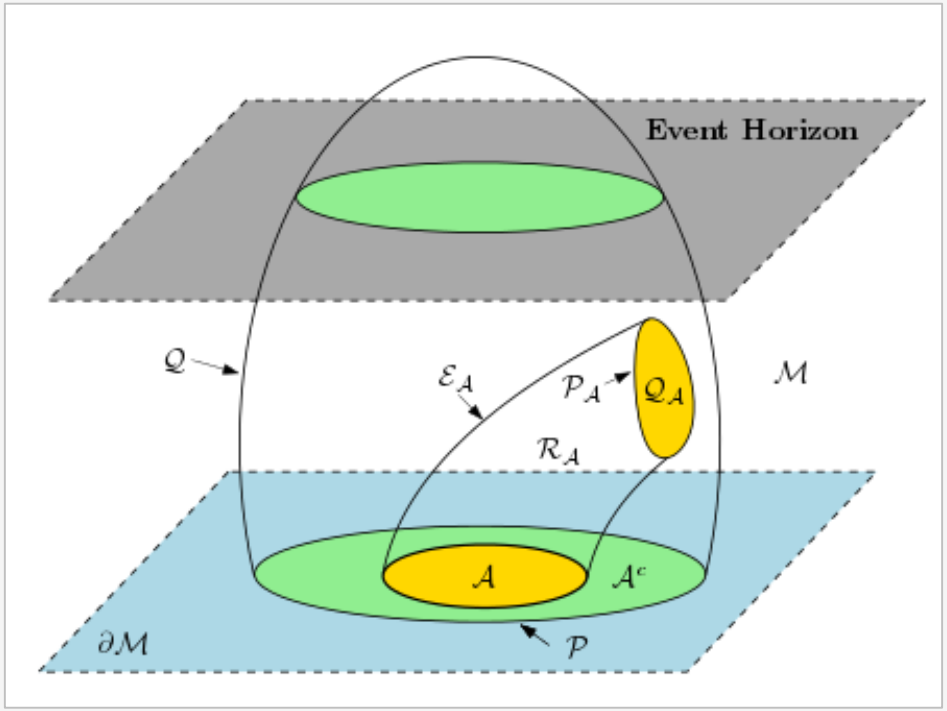
[Shinsei Ryu, Tadashi Takayanagi: hep-th/0603001], [Mukund Rangamani, Tadashi Takayanagi: 1609.01287]

Homology Constraint of HEE

$$S_{EE}^{\mathcal{A}} = \min_{X} \frac{Area\left(\mathcal{E}_{\mathcal{A}}\right)}{4G_N^{(d+2)}},$$

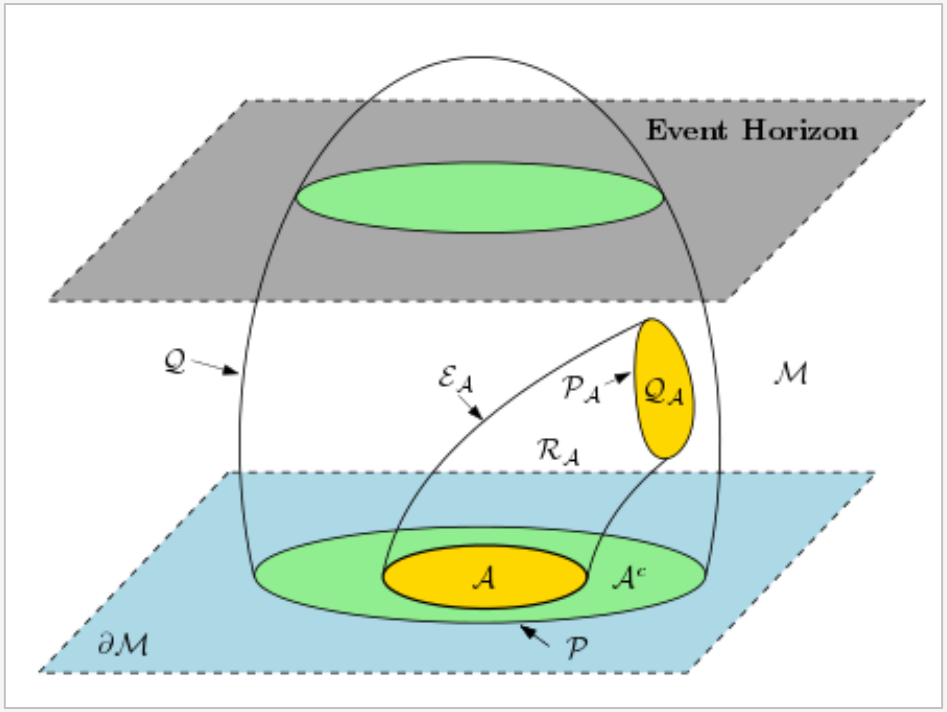
$$X = \Big\{ \mathcal{E}_{\mathcal{A}} \Big| \ \mathcal{E}_{\mathcal{A}}|_{\partial\mathcal{M}} = \partial\mathcal{A}, \mathcal{E}_{\mathcal{A}}|_{Q} = \mathcal{P}_{\mathcal{A}};$$

$$\exists \mathcal{R}_{\mathcal{A}} \subset \mathcal{M}, \partial\mathcal{R}_{\mathcal{A}} = \mathcal{E}_{\mathcal{A}} \cup \mathcal{A} \cup Q_{\mathcal{A}} \Big\}$$



Action of the Bulk Spacetime with Boundary Terms

- $S = S_{\mathcal{M}} + S_{\partial\mathcal{M}} + S_{\mathcal{Q}} + S_{\mathcal{P}}$:
- $S_{\mathcal{M}} = \int_{\mathcal{M}} \sqrt{-g}(R - 2\Lambda_{\mathcal{M}})$,
- $S_{\mathcal{Q}} = \int_{\mathcal{Q}} \sqrt{-h}(R_{\mathcal{Q}} - 2\Lambda_{\mathcal{Q}} + 2K)$,
- $S_{\partial\mathcal{M}} = 2\int_{\partial\mathcal{M}} \sqrt{-\gamma}K'$,
- $S_{\mathcal{P}} = 2\int_{\mathcal{P}} \sqrt{-\sigma}\theta$;



where $h_{ab} = g_{ab} - n^{\mathcal{Q}}_a n^{\mathcal{Q}}_b$, $\gamma_{ab} = g_{ab} - n^{\mathcal{M}}_a n^{\mathcal{M}}_b$, $\theta = \cos^{-1}(n^{\mathcal{Q}} \cdot n^{\mathcal{M}})$.

Boundary Condition of BCFT

- Varying $S_{\mathcal{M}}$ with g^{ab} (E.O.M. for $\mathcal{M}$):

$$0 = R_{ab} - \frac{1}{2} R g_{ab} + \Lambda_{\mathcal{M}} g_{ab},$$

- Varying $S_{\mathcal{Q}}$ with h^{ab} (Neumann B.C.) [Tadashi Takayanagi, 1105.5165]:

$$R_{\mathcal{Q}ab} + 2K_{ab} - \left(\frac{1}{2} R_{\mathcal{Q}} + K - \Lambda_{\mathcal{Q}} \right) h_{ab} = 0,$$

- Taking trace of the equation above(Traceless B.C.)
  [Chong-Sun Chu, Rong-Xin Miao, Wu-Zhong Guo, 1701.07202]:

$$(d-1)\left(R_{\mathcal{Q}} + 2K\right) - 2\left(d+1\right)\Lambda_{\mathcal{Q}} = 0.$$

Finding the Hypersurface Q

- Metric of Bulk Spacetime $\mathcal{M}$ (AdS-Schwarzschild BH, Poincaré patch):

$$ds^2_{\mathcal{M}} = \frac{l^2_{AdS}}{z^2}\left(-g\left(z\right)dt^2 + \frac{dz^2}{g\left(z\right)} + \sum_{i=1}^{d} dx_i^2\right); \quad g\left(z\right) = 1 - \frac{z^{d+1}}{z_h^{d+1}},$$

- Traceless Boundary Condition:

$$\left(d-1\right)\left(R_{\mathcal{Q}} + 2K\right) - 2\left(d+1\right)\Lambda_{\mathcal{Q}} = 0.$$

- Solution of $\mathcal{Q}$: $n^{\mathcal{Q}a} = \left(\frac{\partial}{\partial x^1}\right)^a$,

$$ds^2_{\mathcal{Q}} = \frac{l^2_{AdS}}{z^2}\left(-h\left(z\right)dt^2 + \frac{dz^2}{h\left(z\right)} + \sum_{i=2}^{d} dx_i^2\right); \quad h\left(z\right) = 1 - \frac{z^d}{z_h^d},$$

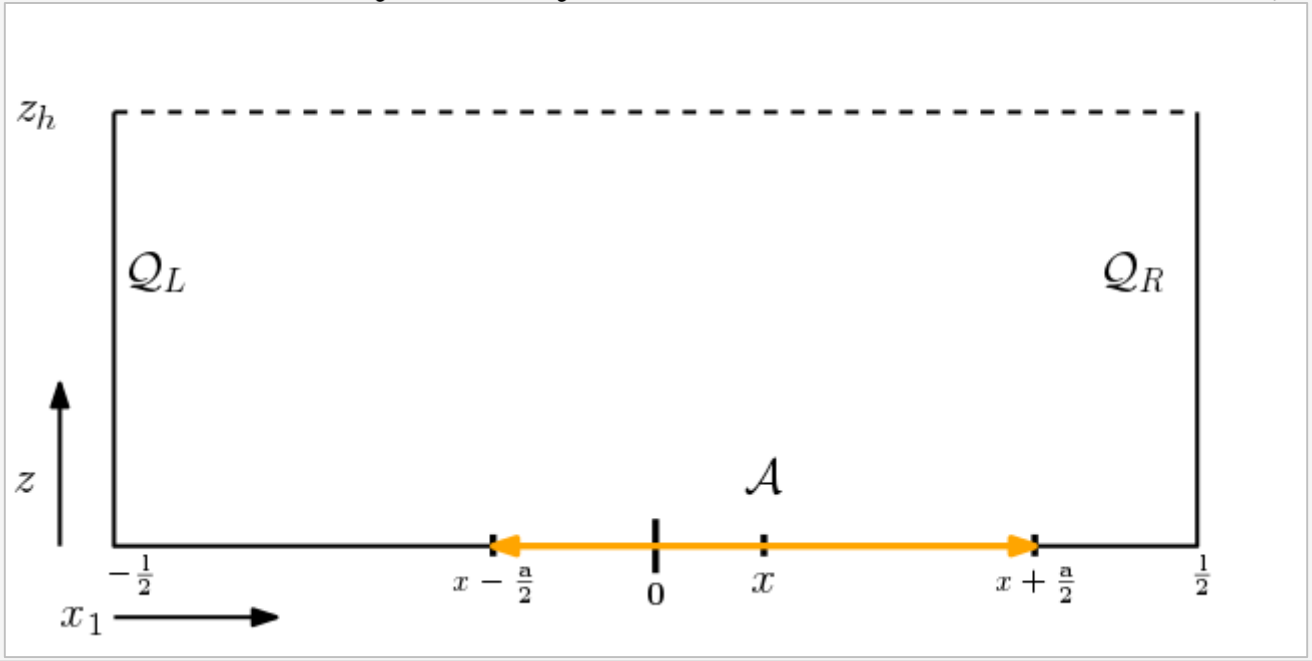
$$R_{\mathcal{Q}} = -\frac{d(d+1)}{l^2_{AdS}}, K_{ab} = 0, \Lambda_{\mathcal{Q}} = -\frac{d(d-1)}{2l^2_{AdS}},$$

Subspace: Infinite Strip

- $\mathcal{A} \subset \partial\mathcal{M}$ as an infinite long strip,

$$x_1 \in \left[x - \frac{a}{2}, x + \frac{a}{2}\right], x_i \in \mathbb{R}^{d-1} \text{ for } i = 2, \cdots d,$$

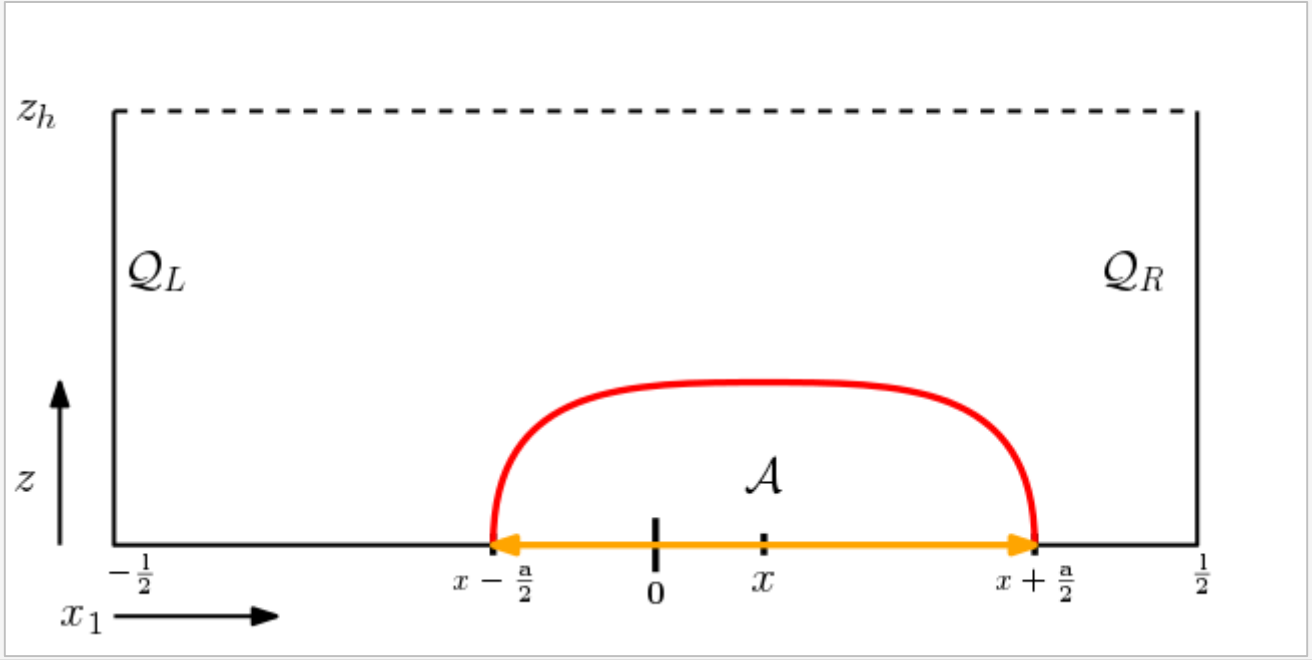
- $(d-1)$-dim translation symmetry in the directions x_i for $i = 2, \cdots d$.



The Ansatz for the Minimal Surface

- B.C. for the Minimal Surface $\mathcal{E}_{\mathcal{A}}$:

$$z = z\left(x_1\right), \quad z\left(x \pm \frac{a}{2}\right) = 0, \quad z\left(x\right) = z_0, \quad z'\left(x\right) = 0.$$



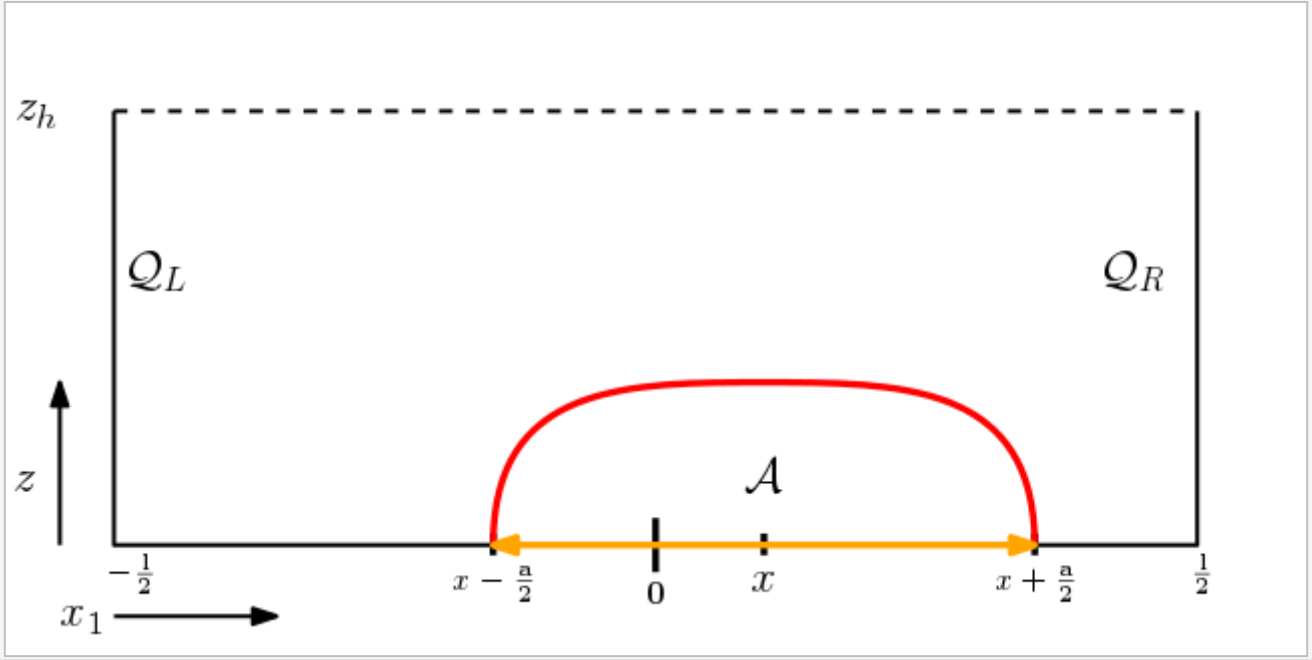
Holographic Entanglement Entropy

- $a = 2z_0 \int_0^1 \frac{v^d dv}{\sqrt{\left(1-(bv)^{d+1}\right)(1-v^{2d})}}, \qquad v = z/z_0, \qquad b = z_0/z_h;$
- $S_{EE}^{\mathcal{A}} = \frac{l_{AdS}^d}{2(d-1)G_N^{(d+2)}} \left(\frac{L}{\epsilon}\right)^{d-1} - \frac{S_{BH}}{L \cdot G_N^{(d+2)} \cdot (d-1)} \cdot \left[\frac{a}{b^d} \right.$

  $\left. -bz_0 \int_0^1 \frac{(d-3)vdv}{\sqrt{(1-(bv)^{d+1})(1-v^{2d})}} - bz_0 \int_0^1 \frac{(d+3)v^{2d+1}dv}{\sqrt{(1-(bv)^{d+1})(1-v^{2d})}} \right].$
- $S_{BH} = \frac{l_{AdS}^d L^d}{4z_H^d}, \qquad \int_{\mathbb{R}^{d-1}} d^{d-1}\mathbf{x} = L^{d-1},$
- We will set $\ell_{AdS} = 1$, $z_h = 1.5$. $\ell = 1/2$ and $L = 1$

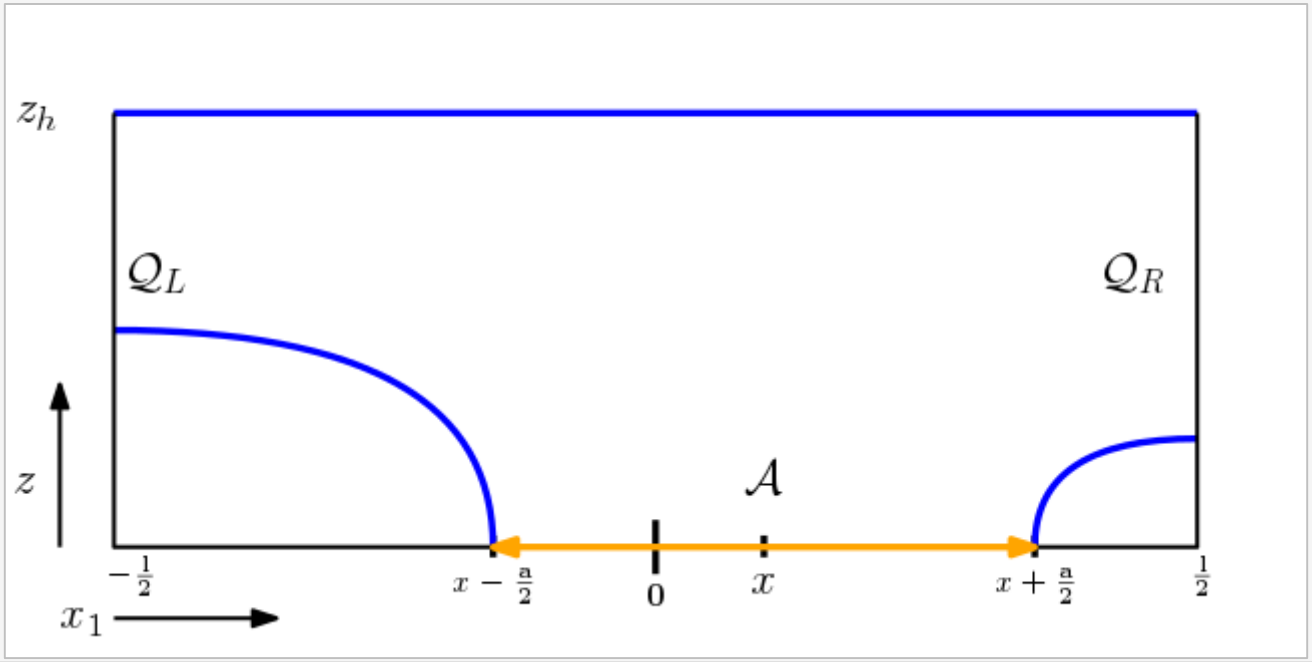
The Sunset Shape

- $S^{\mathcal{A}}_{sunset} = S^{\mathcal{A}}_{EE}(a),$
- $S^{\mathcal{A}^c}_{R-sunset} = S^{\mathcal{A}}_{EE}(a) + S_{BH}.$



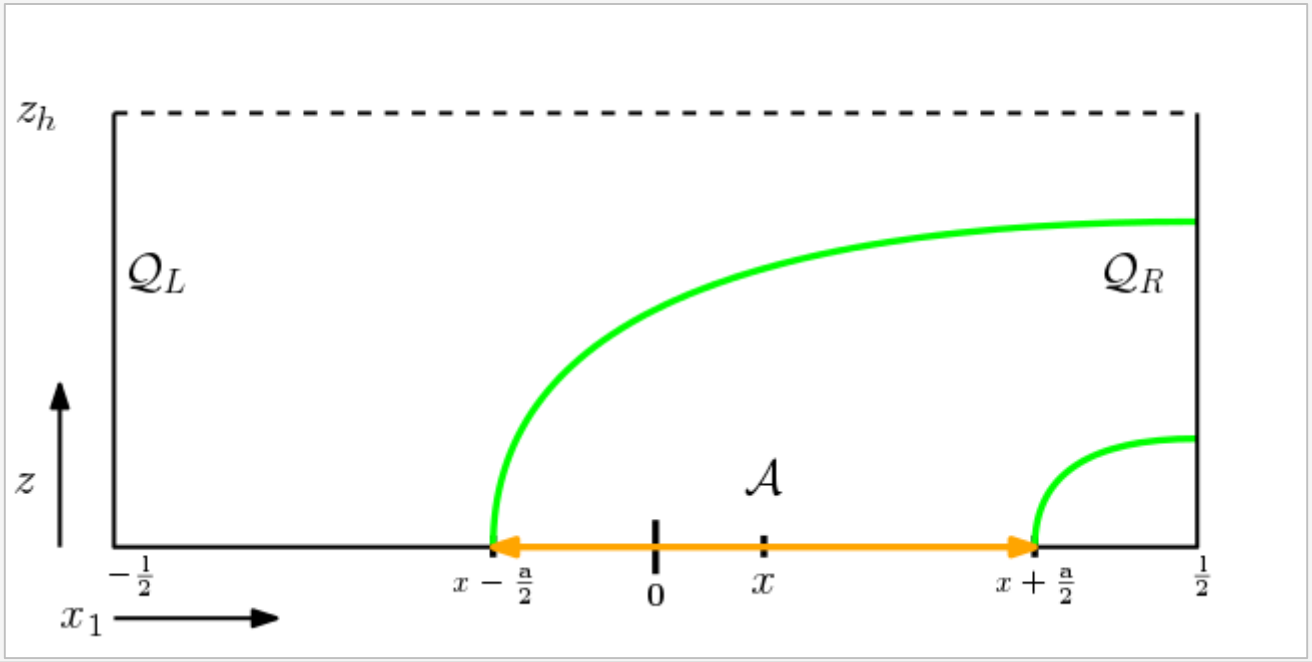
The Sky Shape

- $S^{\mathcal{A}}_{sky} = \frac{1}{2} S^{\mathcal{A}}_{EE}(l - a + 2|x|) + \frac{1}{2} S^{\mathcal{A}}_{EE}(l - a - 2|x|) + S_{BH},$
- $S^{\mathcal{A}^c}_{R-sky} = \frac{1}{2} S^{\mathcal{A}}_{EE}(l - a + 2|x|) + \frac{1}{2} S^{\mathcal{A}}_{EE}(l - a - 2|x|).$



The Rainbow Shape

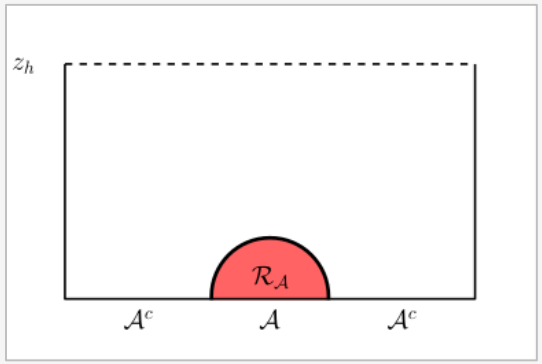
- $S^{\mathcal{A}}_{rainbow} = \frac{1}{2} S^{\mathcal{A}}_{EE}(l + a - 2|x|) + \frac{1}{2} S^{\mathcal{A}}_{EE}(l - a - 2|x|),$
- $S^{\mathcal{A}^c}_{R-rainbow} = \frac{1}{2} S^{\mathcal{A}}_{EE}(l + a - 2|x|) + \frac{1}{2} S^{\mathcal{A}}_{EE}(l - a - 2|x|) + S_{BH}.$

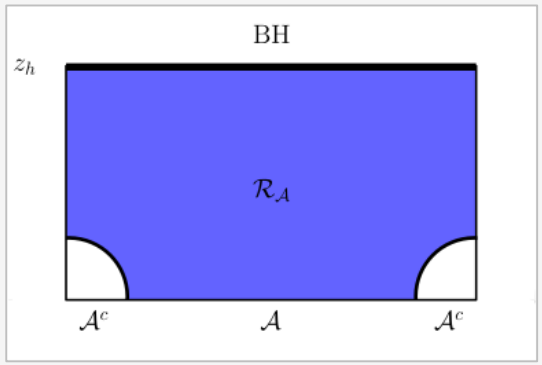


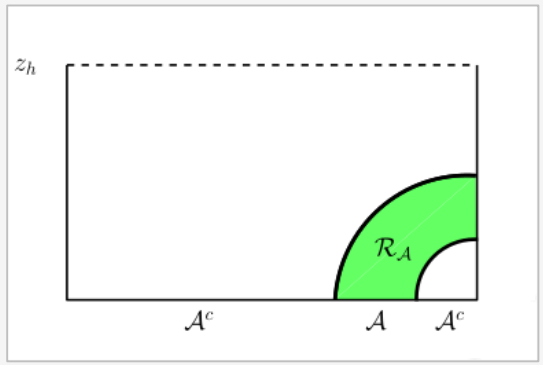
HEE of Different Shapes

- Different shapes of $S_{EE}^{\mathcal{A}}$ and $S_{EE}^{\mathcal{A}^c}$ determined by the position and size of $\mathcal{A}$.

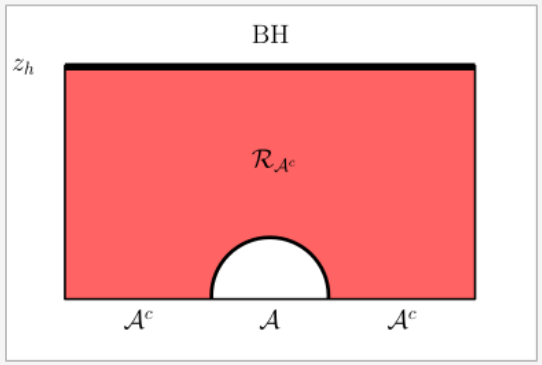
$S_{EE}^{\mathcal{A}}$:

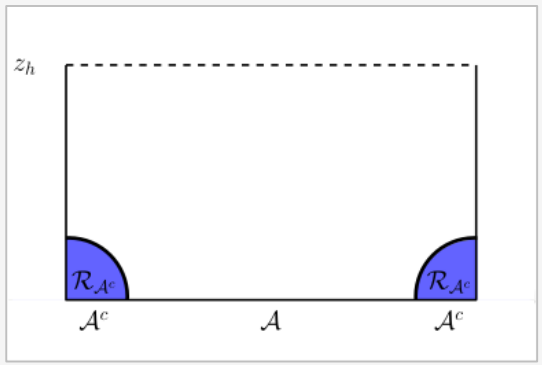


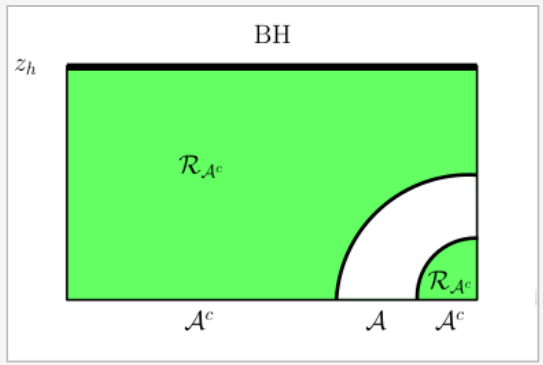




$S_{EE}^{\mathcal{A}^c}$:

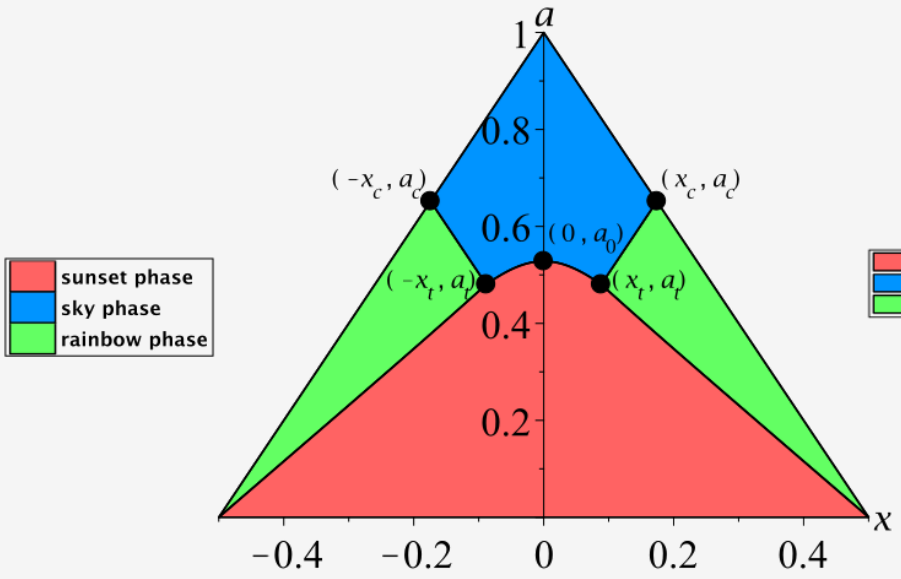




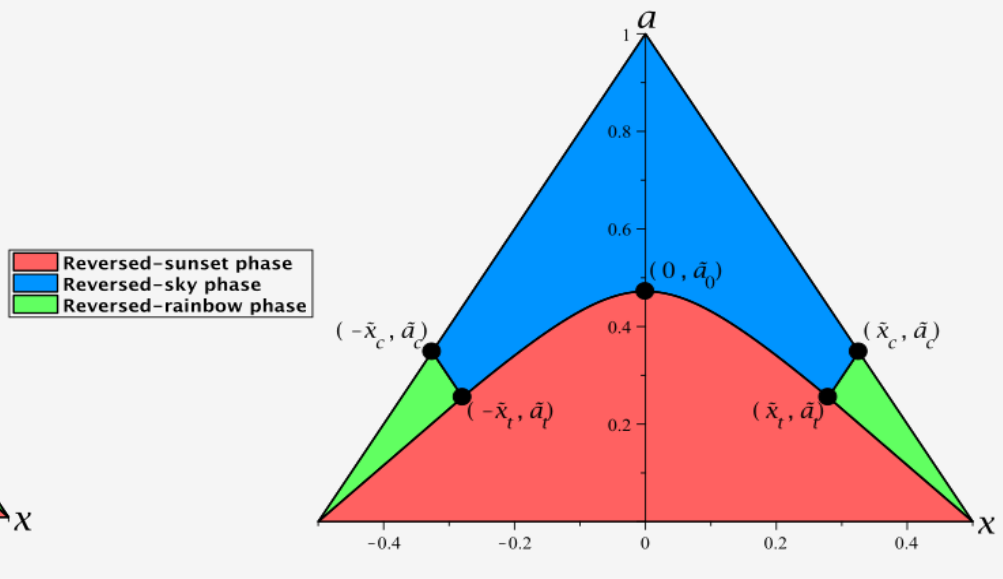


Phase Diagram of HEE

Phase Diagram of $\mathcal{S}_{EE}^{A}$



Phase Diagram of $\mathcal{S}_{EE}^{A^c}$



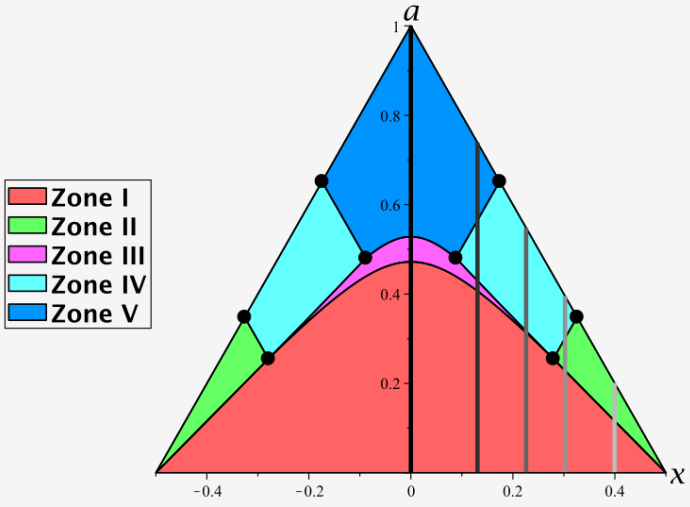
Araki-Lieb Inequality and Entanglement Plateau

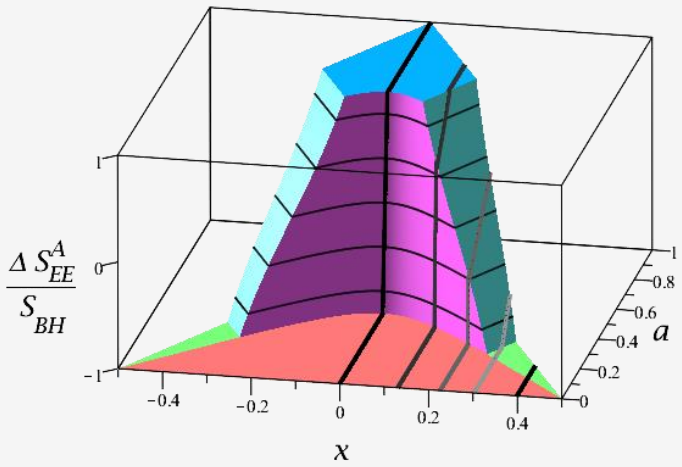
- Araki-Lieb Inequality:

$$|S_{EE}^{\mathcal{A}} - S_{EE}^{\mathcal{A}^c}| \leq S_\rho, \quad \text{where } \rho \text{ is the entire density matrix.}$$

- In HEE case [Hubeny, Maxfield, Rangamani, Tonni, 1306.4004]:

$$|\Delta S_{EE}^{A}| = |S_{EE}^{\mathcal{A}} - S_{EE}^{\mathcal{A}^c}| \leq S_{BH}.$$





Summary

- We propose the homology constraint of minimal surfaces in bulk spacetime dual to the static states of BCFT.
- We have solved traceless B.C. to determine the hypersurface Q in AdS-Schwarzschild BH bulk spacetime.
- We calculate the area of the minimal surface(HEE) of a given entanglement region using Poincaré coordinates.
- There are 3 different shapes of minimal surface because of the boundary effect.
- Araki-Lieb Inequality is satisfied.

Thank you