Universal Holographic Two-point Functions in Massless Higher-order Gravities

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Introduction

Correlation functions of physical operators encode all the physical information in a quantum field theory.

There is a universal operator in relativistic QFTs, namely the energy-momentum tensor T_{ij}.

In a conformal field theory (CFT), the two-point function is uniquely determined by the conformal symmetry, up to an overall constant

$$\langle T_{ij}(x)T_{kl}(0)\rangle = \frac{C_T\,\mathcal{I}_{ijkl}(x)}{x^{2d}}\,,$$

where $\mathcal{I}_{ijkl}(x)$ is the spacetime tensor

$$\begin{aligned}\mathcal{I}_{ijkl}(x) &= \frac{1}{2}\Big(I_{ik}(x)I_{jl}(x)+I_{il}(x)I_{jk}(x)\Big)-\frac{1}{d}\eta_{ij}\eta_{kl}\,,\\ I_{ij}(x) &= \eta_{ij}-\frac{2x_ix_j}{x^2}\,.\end{aligned}$$

The constant C_T depends on the specificity of a CFT.

What can the AdS/CFT tell us about C_T?

AdS/CFT

The AdS/CFT correspondence provides a powerful tool for calculating the two-point functions. (Maldacena; Witten; Gubser, Klebanov, Polyakov; ...)

The field/operator duality states that any field ϕ in the bulk has the corresponding gauge invariant operator $\mathcal{O}_\phi$ defined in the boundary. The boundary value ϕ_0 of ϕ is identified as the source coupled to the operator, and furthermore the partition function of the boundary CFT_d is identified with the on-shell action of AdS_{d+1} gravities

$$S_{\text{gr}}\Big|_{\phi_0} = \langle \exp(-\int d^d x\, \phi_0 \mathcal{O}_\phi) \rangle \,.$$

With this identification, the n-point functions of the operator $\mathcal{O}_\phi$ of the CFT_d can be computed by evaluating the on-shell action of the gravity theories:

$$\langle \mathcal{O}_\phi(x_1) \cdots \mathcal{O}_\phi(x_n) \rangle \sim \frac{\delta^n S}{\delta\phi_0(x_1) \cdots \delta\phi_0(x_n)} \sim \frac{\delta^{n-1} \langle \mathcal{O}_\phi(x_1) \rangle}{\delta\phi_0(x_2) \cdots \delta\phi_0(x_n)} \,.$$

Metric and energy-momentum tensor

The holographic dictionary states that the bulk metric corresponds to the boundary energy-momentum tensor.

The linear perturbation h_{ij} of the AdS background

$$ds^2 = \frac{\ell^2 dr^2}{r^2} + r^2 \eta_{ij} dx^i dx^j + h_{ij} dx^i dx^j$$

decouples from the matter field in Einstein gravity with minimally-coupled matter. The only relevant bulk Lagrangian then must be

$$\mathcal{L}_{d+1} = \sqrt{-g}(R - 2\Lambda_0)\,, \qquad \Lambda_0 = \frac{d(d-1)}{2\ell^2}\,,$$

where ℓ is the AdS radius.

Thus without any calculation, one can deduce that

$$C_T = C_T(\Lambda_0)\,.$$

Holographic 2-point function

Indeed, the holographic two-point functions are given by

$$C_T = \frac{\Gamma(d+2)}{16(-\pi)^{\frac{d}{2}+1}(d-1)\Gamma(\frac{d}{2})}\mathcal{C}_T\,,$$

with (numerical stripped the quantity)

$$\mathcal{C}_T = \ell^{d-1}\,.$$

(Liu, Tseytlin hep-th/9804083)

It turns out that in $d = 4$, the holographic conformal anomalies are also (Freedman,Gubser,Pilch,Wanter hep-th/9904017)

$$a = c = \ell^3\,.$$

(Here we stripped off the inessential numerical from expressions.) This seems to suggest that

$$\mathcal{C}_T = a\,.$$

Is this universal? It is dangerous to conclude so much from a system with just 1 non-trivial parameter.

Holographic 2-pt function in Einstein-Gauss-Bonnet

The holographic two-point functions in Einstein-Gauss-Bonnet gravity were later obtained. (BEMPSS, 0911.4257)

In $d = 4$, it turns out to be

$$\mathcal{C}_T = c\,.$$

In $d = 4$, conformal anomaly in general has two charges:

$$\langle T^{\mu}_{\mu} \rangle \sim aE + cW^2\,.$$

where E is the Euler density and W is the Weyl tensor. (Duff 1977,hep-th/9308075;...)

The holographic a and c charges are different for Einstein-Gauss-Bonnet gravity.

So the universal formula is $\mathcal{C}_T = c$ instead of a?

But $\mathcal{C}_T = c$ can be universal beyond $d = 4$.

The a-charge measures the massless degrees of freedom, whilst the physical interpretation of c is not clear.

Furthermore, conformal anomaly in higher dimensions takes the form

$$\langle T^{\mu}_{\mu} \rangle \sim aE + \sum_i c_i I_i \,.$$

The number of Weyl invariants increases as d becomes larger.

Thus the generalization of $\mathcal{C}_T = c$ to higher dimensions are not at all clear.

These seem to suggest that there is no universal expression for

$$\mathcal{C}_T$$

We need more examples!

Massless higher-order gravities

Einstein gravity extended with higher-order curvature invariants has a long history. (See, e.g. Bueno, Cano, Min, Visser, 1610.08519 for a recent review.)

The vacua can be Minkowski, (A)dS spacetimes.

Linear spectrum can have massive modes, in addition to the usual (massless) graviton.

For appropriate curvature invariant combinations, the massive modes can all decouple, and then the spectrum contains only the graviton.

We refer these to massless gravities

The mentioned Einstein and Einstein-Gauss-Bonnet gravities are clear examples.

Quadratic invariants

In the study of higher-order gravities, one breakthrough was the demonstration that Einstein gravity extended with quadratic invariants

$$\mathcal{L} = \sqrt{-g}\Big(R + \alpha R^2 + \beta R^{\mu\nu} R_{\mu\nu}\Big)\,,$$

is renormalizable (Stelle, 1976).

The vacuum is Minkowski, (quadratic curvature terms do not alter the vacuum in $D = 4$.) The linearized spectrum contains the usual graviton, plus massive scalar and ghosty massive spin-2 modes (Stelle)

$$\mu_0^2 = \frac{1}{2(3\alpha + \beta)}\,, \qquad \mu_2^2 = -\frac{1}{\beta}\,.$$

It turns out that the theory admits new static black holes over and above the usual Schwarzschild black hole. (Lü, Perkins, Pope, Stelle, 1502.01028,1508.00010)

General properties

Consider Einstein gravity coupled to a bare cosmological constant Λ_0 extended with polynomial invariants of Riemann tensors

$$\mathcal{L} = \sqrt{-g}\Big(R - 2\Lambda_0 + P_n(R_{\mu\nu\rho\sigma})\Big)\,,$$

Assuming that the theory admits AdS with radius ℓ, and denoting α_i as the bare coupling constant of each polynomial term, then the equations of motion implies that

$$\Lambda_0 = \Lambda_0(\ell, \alpha_i)\,.$$

There are only two massive modes, one spin-0 and one spin-2 of masses (μ_0, μ_2):

$$\frac{1}{\mu_0^2} \sim f(\alpha_i)\,, \qquad \frac{1}{\mu_2^2} \sim g(\alpha_i)\,,$$

where f and g are linear functions of α_i.

Ghost-free and a-theorem constraints

The massive spin-2 modes are ghosts and it must be decoupled. The massive scalar mode is harmless at the first sight, but holographic a-theorem requires the decoupling of the massive scalar modes also. (Li,Lü,Wu, 1711.03650)

Thus we have

$$\begin{aligned} \text{equations of motion}: &\quad \Lambda_0 = \Lambda_0(\ell, \alpha_i)\,, \\ \text{ghost free}: &\quad g(\alpha_i) = 0\,, \\ a\text{-theorem}: &\quad f(\alpha_i) = 0\,. \end{aligned}$$

Linearized equation for the remaining graviton becomes (after applying appropriate gauge)

$$\kappa_{\rm eff}\Big(\Box + \frac{2}{\ell^2}\Big)h_{ij} = 0\,, \qquad \kappa_{\rm eff}(\ell, \alpha_i) = \frac{1}{16\pi}(1 + \cdots)\,,$$

where we set the convention such that $\kappa_{\rm eff} = 1/(16\pi)$ for Einstein gravity and the ellipses denote the higher-order contributions.

Freedom from ghost requires $\kappa_{\rm eff} \geq 0$.

Examples of massless gravities

(1) Einstein-Gauss-Bonnet

$$L_{d+1} = R + \frac{d(d-1)}{\ell_0^2} + \alpha(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\,.$$

We have

$$\frac{1}{\ell_0^2} - \frac{1}{\ell^2} + \frac{(d-2)(d-3)\alpha}{\ell^4} = 0\,,$$

$$\kappa_{\rm eff} = \frac{1}{16\pi}\Big(1 - \frac{2\alpha(d-2)(d-3)}{\ell^2}\Big)\,.$$

In $d = 4$, we have

$$16\pi\kappa_{\rm eff} = 1 - \frac{4\alpha}{\ell^2}\,.$$

(Guass-Bonnet term is the only quadratic combination that yields massless gravity.)

Examples of massless gravities

(2) Einstein-Lovelock

$$L = \sum \frac{(2k)!}{2^k}\, a_k\, \delta^{\mu_1\mu_2\cdots\mu_{2k}}_{\nu_1\nu_2\cdots\nu_{2k}} R^{\nu_1\nu_2}_{\mu_1\mu_2} R^{\nu_3\nu_4}_{\mu_3\mu_4} \cdots R^{\nu_{2k-1}\nu_{2k}}_{\mu_{2k-1}\mu_{2k}}\,,$$

with

$$a_0 = \frac{d(d-1)}{\ell_0^2}\,, \qquad a_1 = 1\,.$$

We have a constraint

$$\text{eom}: \qquad \sum_{k\geq 0} \frac{d!}{(d-2k)!}(-\frac{1}{\ell^2})^k a_k = 0\,.$$

and

$$\kappa_{\text{eff}} = \frac{1}{16\pi} \sum_{k\geq 1} \frac{(-1)^{k+1} k(d-2)! a_k}{\ell^{2(k-1)}(d-2k)!}\,.$$

Examples of massless gravities

(3) Einstein-Riemann-cubics $L = R + \frac{d(d-1)}{\ell_0^2} + H^{(3)}$

$$
\begin{aligned}
H^{(3)} \;=\; & e_1 R^3 + e_2 R\, R_{\mu\nu} R^{\mu\nu} + e_3 R^\mu_\nu R^\nu_\rho R^\rho_\mu + e_4 R^{\mu\nu} R^{\rho\sigma} R_{\mu\rho\nu\sigma} \\
& + e_5 R R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} + e_6 R^{\mu\nu} R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} \\
& + e_7 R^{\mu\nu}{}_{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} R^{\alpha\beta}{}_{\mu\nu} + e_8 R^\mu{}_\nu{}^\alpha{}_\beta R^\nu{}_\rho{}^\beta{}_\gamma R^\rho{}_\mu{}^\gamma{}_\alpha \,.
\end{aligned}
$$

Constraints

$$
\begin{aligned}
& \frac{1}{\ell_0^2} - \frac{1}{\ell^2} - \frac{(d-5)}{(d-1)\ell^6}\Big(d^2(d+1)^2 e_1 + d^2(d+1) e_2 + d^2 e_3 \\
& + d^2 e_4 + 2d(d+1) e_5 + 2d e_6 + 4 e_7 + (d-1) e_8 \Big) = 0 \,, \\
& (d+1) d e_2 + 3 d e_3 + (2d-1) e_4 + 4(d+1) d e_5 + 4(d+1) e_6 \\
& + 24 e_7 - 3 e_8 = 0 \,, \\
& 12(d+1) d^2 e_1 + \left(d^2 + 10d + 1\right) d e_2 + 3(d+1) d e_3 \\
& + \left(2d^2 + 5d - 1\right) e_4 + 4(d+5) d e_5 \\
& + 4(2d+1) e_6 + 3(d-1) e_8 + 24 e_7 = 0 \,.
\end{aligned}
$$

Thus (one combination is Lovelock of 3'rd order)

$$
16\pi \kappa_{\rm eff} = 1 + \frac{1}{\ell^4}(d-5)(d-2)(3(d+1) d e_1 + 2 d e_2 + e_4 + 4 e_5) \,.
$$

Examples of massless gravities

(4) Quasi-topological Ricci Polynomials (Li,Liu,Lü, 1708.07198)

$$L = R + \frac{d(d-1)}{\ell_0^2} + P_n(R_{\mu\nu})\,.$$

Massless condition implies that

$$\kappa_{\text{eff}} = \frac{1}{16\pi}\,.$$

Higher-order terms give no contribution to equations of motion at the linear level—quasi-topological.

Holographic Two-Point Functions

Constructing two-point functions require solutions of the linear perturbation

$$\kappa_{\text{eff}}\Big(\Box + \frac{2}{\ell^2}\Big)h_{ij} = 0\,.$$

For the AdS spactimes in planar coordinates

$$d\bar{s}^2_{d+1} = \frac{\ell^2 dr^2}{dr^2} + r^2 dx^i dx^j \eta_{ij}\,,$$

the solutions take the form

$$h_{ij} = c_{ij} r^{-\frac{d-4}{2}} c_{ij} H_{\frac{d}{2}}(\frac{\ell p}{r})\, e^{\mathrm{i}p\cdot x}\,,$$

where c_{ij}'s satisfies appropriate gauge conditions and H is an Hankel function.

Energy-momentum tensor

The boundary energy-momentum tensor is given by

$$\langle T_{ij} \rangle = T_{ij}(\hat{h}) r^{d-2} \Big|_{r\to\infty} ,$$

where $T_{ij}(\hat{h})$ is the Brown-York energy-momentum tensor associated with the embedding boundary metric $\hat{h}_{ij}$

$$ds^2_{d+1} = \frac{\ell^2 dr^2}{r^2} + \hat{h}_{ij} dx^i dx^j \, .$$

To compute the Brown-York tensor, we need to know the full action

$$S = S_{\text{bulk}} + S_{\text{surf}} + S_{\text{ct}} \, .$$

Refs for S_{surf}:

- Einstein-Gauss-Bonnet: (Liu, Sabra 0807.1256)
- General Lovelock: (Liu, Lü, Pope, 1708.02329)
- Riemann poly.: (Deruelle, Sasaki, Sendouda, Yamauchi, 0908.0679)

The upshot is that the two-point functions take the form

$$\langle T_{ij} T_{kl} \rangle \sim \kappa_{\text{eff}} \frac{\delta h_{ij}^{(d)}}{\delta h_{kl}^{(0)}} \,,$$

where

$$h_{ij} = r^2 (h_{ij}^{(0)} + \cdots + \frac{h_{ij}^{(d)}}{r^d} + \cdots) \,.$$

The result

After some tedious calculations, we find

$$\mathcal{C}_T = 16\pi\,\kappa_{\text{eff}}\,\ell^{d-1}\,.$$

This is a nice "universal" result, but does not seem to tell us much about the CFT itself.

What does this quantity mean in terms of quantities of the CFT?

Can this be expressed in terms of central charges, as in the examples mentioned in the beginning of this talk?

Central charges in $d = 4$

Consider the bulk quadratic theory in $D = 5$ $(d = 4)$

$$L = R - 2\Lambda_0 + e_1 R^2 + e_2 R_{\mu\nu} R^{\mu\nu} + e_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \,.$$

The a and c charges are

$$a = \ell^3 - 4(10e_1 + 2e_2 + e_3)\ell \,, \quad c = \ell^3 - 4(5e_1 + 2e_2 - e_3)\ell \,.$$

If we take the massless combination (Gauss-Bonnet)

$$\{e_1, e_2, e_3\} = \{1, -4, 1\}\alpha \,,$$

we have (Myers, Sinha, 1006.1263 in different parametrization)

$$a = \ell^3 - 12\ell\alpha \,, \quad c = \ell^3 - 4\ell\alpha \,.$$

On the other hand, we have $16\pi\kappa_{\rm eff} = 1 - 4\alpha/\ell^2$, and hence

$$\mathcal{C}_T = \ell^3 - 4\ell\alpha \,.$$

Thus indeed we have $\mathcal{C}_T = c$. But we also have an identity

$$c = \frac{1}{3}\ell\frac{\partial a}{\partial \ell} \,, \qquad \to \qquad \mathcal{C}_T = \frac{1}{3}\ell\frac{\partial a}{\partial \ell} \,.$$

Central charges in $d = 4$

Consider now Einstein-Riemann cubic theory. The a and c charges for the massless theories are (Li,Lü,Wu, 1711.03650)

$$\begin{aligned} a &= \ell^3 + 6(60e_1 + 8e_2 + e_4 + 4e_5)\ell^{-1}\,, \\ c &= \ell^3 - 2(60e_1 + 8e_2 + e_4 + 4e_5)\ell^{-1}\,. \end{aligned}$$

And we have

$$\mathcal{C}_T = 16\pi\kappa_{\rm eff}\ell^3 = \ell^3 - 2(60e_1 + 8e_2 + e_4 + 4e_5)\ell^{-1}\,.$$

Thus we have again

$$\mathcal{C}_T = c = \frac{1}{3}\ell\frac{\partial a}{\partial \ell}\,.$$

In the above equations, as was discussed, the first equality is hard to generalize, but the second equality can be generalized straightforwardly.

$\mathcal{C}_T$ and a-charges in massless higher-order gravities

- Quasi-topological Ricci polynomials (including Einstein)

$$a = \ell^{d-1}\,, \qquad \mathcal{C}_T = \ell^{d-1}\,.$$

- Gauss-Bonnet

$$\begin{aligned} a &= \ell^{d-1} - 2\alpha(d-1)(d-2)\ell^{d-3}\,, \\ \mathcal{C}_T &= \ell^{d-1} - 2\alpha(d-2)(d-3)\ell^{d-3}\,. \end{aligned}$$

- Lovelock

$$\begin{aligned} a &= \ell^{d-1}\sum_{k\geq 1}\frac{(-1)^{k+1}k(d-2)!}{\ell^{2k}(d-2k+1)!}a_k\,, \\ \mathcal{C}_T &= \ell^{d-1}\sum_{k\geq 1}\frac{(-1)^{k+1}k(d-2)!}{\ell^{2(k-1)}(d-2k)!}a_k\,. \end{aligned}$$

- Riemann-cubics

$$\begin{aligned} a &= \ell^{d-1} + \ell^{d-5}(d-1)(d-2)(3(d+1)de_1 + 2de_2 + e_4 + 4e_5)\,, \\ \mathcal{C}_T &= \ell^{d-1} + \ell^{d-5}(d-5)(d-2)(3(d+1)de_1 + 2de_2 + e_4 + 4e_5)\,. \end{aligned}$$

Universal holographic two-point functions

$$\langle T_{ij}(x)T_{kl}(0)\rangle = \frac{C_T\,\mathcal{I}_{ijkl}(x)}{x^{2d}}\,, \qquad C_T = \frac{\Gamma(d+2)}{16(-\pi)^{\frac{d}{2}+1}(d-1)\Gamma(\frac{d}{2})}\mathcal{C}_T\,,$$

with

$$\mathcal{C}_T = \frac{1}{d-1}\ell\frac{\partial a}{\partial \ell}\,.$$

An important point: We express the bare cosmological constant $\Lambda_0 = \Lambda(\ell, \alpha_i)$, where ℓ is the AdS radius and α_i are coupling constants of higher-order invariants. The $\mathcal{C}_T$ expression holds α_i fixed.

In literature, $\mathcal{C}_T$ is commonly expressed in terms of Λ_0 (or its bare radius ℓ_0) and α_i, in which case, the relation was not manifest and overlooked.

Conclusions

Conclusions:

- An universal expression relating two-point functions and the a-charges:

$$\mathcal{C}_T = \frac{1}{d-1}\ell\frac{\partial a}{\partial \ell}\,.$$

- In $d = 4$, we find

$$c = \frac{1}{3}\ell\frac{\partial a}{\partial \ell}\,.$$

Open Questions

- Prove the general case for the $\mathcal{C}_T$ expression.
- Prove and generalize the $d = 4$ result $c = \frac{1}{3}\ell\frac{\partial a}{\partial \ell}$ to higher d.
- Can these holographic identities find their avatar in a concrete example of CFTs?