Bethe/Gauge correspondence revisited:

the Ruijsenaars-Schneider case

Antonio Sciarappa

KIAS

Based on arXiv:1809.10294 with Y. Hatsuda (Rikkyo Univ.) and S. Zakany (Geneva Univ.)

November 2018 - East Asia Joint Workshop on Fields and Strings 2018

Overview

5d $\mathcal{N}=1^*$ $SU(N)$ SYM conjecturally related to elliptic integrable system:

5d $\mathcal{N}=1^*$ $SU(N)$ on $\mathbb{R}^4_{\epsilon_{1,2}} \times S^1_R$ $\Longleftrightarrow$ N - particle Ruijsenaars-Schneider

The precise relation depends on the values of Omega background parameters $\epsilon_{1,2}$:

$\epsilon_1 = 0, \epsilon_2 = 0$: classical, time-independent (Seiberg-Witten theory)

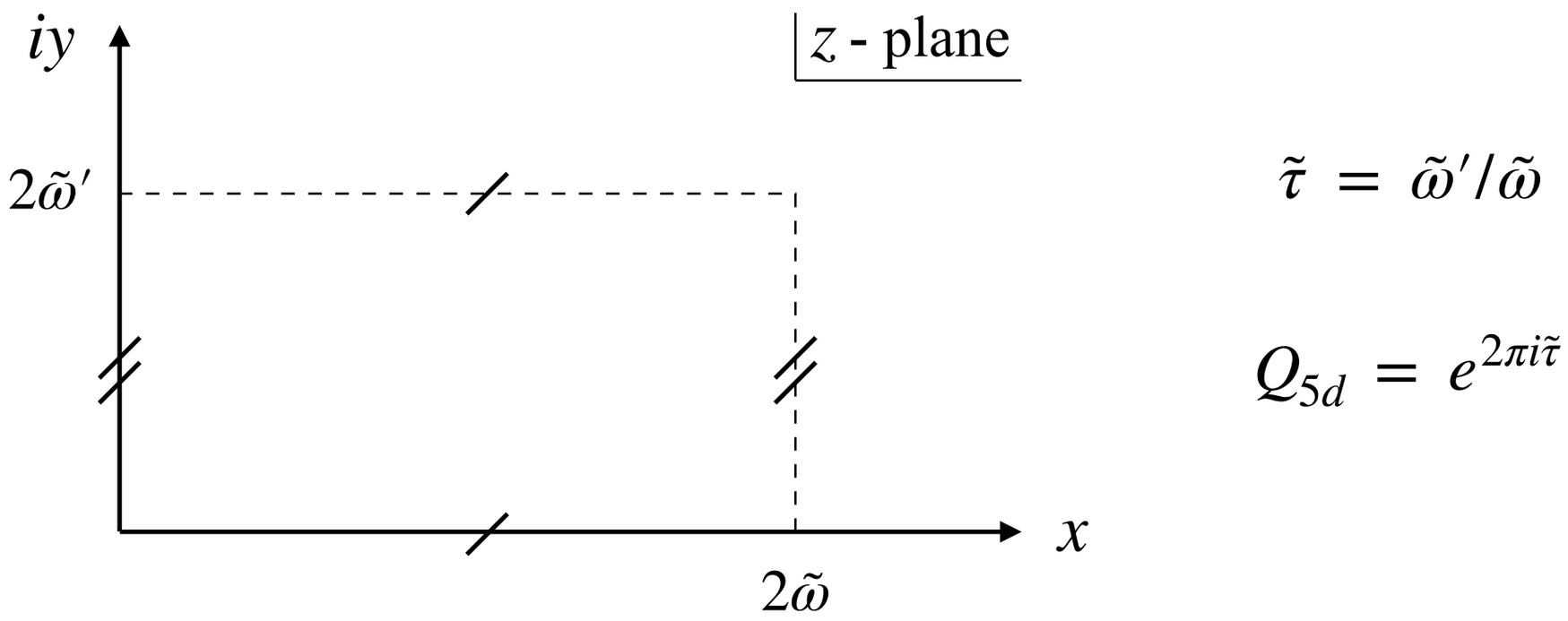
Today: $\epsilon_1 = \hbar, \epsilon_2 = 0$: quantum, time-independent (Nekrasov-Shatashvili)

$\epsilon_1 \neq 0, \epsilon_2 \neq 0$: quantum, time-dependent (general case)

Goal: test this conjecture (in NS limit) and use it to solve Ruijsenaars-Schneider

Conventions

We will consider systems of N particles living on a rectangular torus $T^2_{\tilde{\tau}}$



at positions labelled by complex coordinate variables

$$z_j = x_j + iy_j\,, \qquad j = 1, \ldots, N$$

Elliptic Ruijsenaars-Schneider

Classical N - particle elliptic Ruijsenaars-Schneider Hamiltonian:

$$H_1 = \sum_{j=1}^{N} W_j^* \, W_j \, e^{p_j} \qquad \text{with} \qquad W_j = \prod_{k \neq j}^{N} \sqrt{\frac{\sigma(z_j - z_k + ig \,|\, \tilde{\omega}, \tilde{\omega}')}{\sigma(z_j - z_k \,|\, \tilde{\omega}, \tilde{\omega}')}}$$

H_1 is part of a set of N conserved quantities H_j in involution ($\{H_j, H_k\} = 0$)

$\Longrightarrow$ define a classical "*relativistic*" integrable system

Remark 1: quantities H_j depend *exponentially* on momenta ("*relativistic*")

Remark 2: classical integrable system is complex, i.e. $z_j, p_j, g \in \mathbb{C}$

Quantum N - particle elliptic Ruijsenaars-Schneider:

$$\hat{H}_1 = \sum_{j=1}^{N} W_j^* \, e^{-i\hbar\partial_{z_j}} W_j \quad \text{with} \quad W_j = \prod_{k\neq j}^{N} \sqrt{\frac{\sigma(z_j - z_k + ig \,|\, \tilde{\omega}, \tilde{\omega}')}{\sigma(z_j - z_k \,|\, \tilde{\omega}, \tilde{\omega}')}}$$

H_1 promoted to finite-difference operator $\hat{H}_1$ after canonical quantization

$$[\hat{z}_j \,, \hat{p}_k] = i\hbar\delta_{j,k}$$

and also H_j promoted to commuting finite-difference operators $\hat{H}_j$

$$\{H_j \,, H_k\} = 0 \quad \Longrightarrow \quad [\hat{H}_j \,, \hat{H}_k] = 0$$

"Quantum" elliptic Ruijsenaars-Schneider problem: find $\psi(\vec{z})$, E_j such that

$$\hat{H}_j \psi(\vec{z}) = E_j \psi(\vec{z})$$

However, not yet a true Quantum Mechanical problem since $z_j\,,\, g\,,\, \hbar\,,\, E_j \in \mathbb{C}$;

must specify Hilbert space / reality conditions on parameters for self-adjointness

Among other possibilities, there are two natural self-adjointness domains:

$$A): \quad (z_j\,,\, g\,,\, \hbar) \in \mathbb{C} \;\longrightarrow\; (x_j\,,\, g_x\,,\, \hbar_x) \in \mathbb{R}_+\,, \quad E_j \in \mathbb{R}_+\,, \quad \mathscr{H} = L^2([0\,,\, 2\tilde{\omega}]^{N-1})$$

$$B): \quad (z_j\,,\, g\,,\, \hbar) \in \mathbb{C} \;\longrightarrow\; (iy_j\,,\, ig_y\,,\, i\hbar_y) \in i\mathbb{R}_+\,, \quad E_j \in \mathbb{R}_+\,, \quad \mathscr{H} = L^2([0\,,\, 2\tilde{\omega}']^{N-1})$$

associated to Quantum Mechanical problems with discrete energy levels for $g > 0$

The A, B problems are related by S-duality, i.e. exchange real / imaginary period

$$\Longrightarrow \quad \boxed{E^{(A)}_{j,n}(\tilde{\omega}, \tilde{\omega}') \;=\; E^{(B)}_{j,n}(-i\tilde{\omega}', i\tilde{\omega})}$$

Ruijsenaars-Schneider and gauge theories

How can we solve the quantum elliptic Ruijsenaars-Schneider problem (A/B)?

- Numerically: always possible, by numerically diagonalizing the operators $\hat{H}_j$

  (however, usually quite time consuming)

- Analytically: as before, no known general procedure…

Proposal: try to use the conjectural Bethe/Gauge correspondence

"relativistic" Quantum Integrable Systems		5d Supersymmetric Gauge Theory (NS)
(N-particle elliptic Ruijsenaars-Schneider)	$\Longleftrightarrow$	($\mathcal{N}=1^*$ $SU(N)$ on $\mathbb{R}^4_{\epsilon_{1,2}} \times S^1$ at $\epsilon_1 = \hbar, \epsilon_2 = 0$)

to construct the desired analytic solution to the Ruijsenaars-Schneider problem

The conjectural Bethe/Gauge dictionary (naive 5d uplift of 4d NS proposal):

Quantum Integrable System		**Supersymmetric Gauge Theory**
N particles system		$SU(N)$ gauge theory
Planck constant $\hbar$		"NS limit" $\epsilon_1 \sim \hbar$, $\epsilon_2 = 0$
parameter $Q_{5d} = e^{2\pi i\tilde{\tau}}$		instanton fugacity Q_{5d}
coupling g		adjoint mass m
energies $E_j(\vec{a})$	$\Longleftrightarrow$	antisym. Wilson loops $\langle W_{\wedge^j} \rangle_{NS}$
eigenfunction $\psi(\vec{z})$		codim. 2 observable $\langle Z_{3d/5d}(\vec{z}) \rangle_{NS}$
A / B quantization conditions		A / B SUSY vacua equations
quantum period a_j		vector multiplet scalar vev a_j
quantum period $a_{D,j} = \partial_{a_j} F$		superpotential $F_{NS} = -\ln\langle Z_{5d} \rangle_{NS}$

Let us test the 5d Bethe/Gauge proposal, for the 2-particle system case

- Let us focus on computing the discrete energy levels $E_n^{(B)}$, $E_n^{(A)}$

- According to Bethe/Gauge conjecture, energy computable via localization as

$$E(a) \sim \langle W_{\square}(a) \rangle_{NS}$$

- Get B / A energies $E_n^{(B)}$, $E_n^{(A)}$ by imposing reality conditions on parameters

  (choice Hilbert space) + appropriate quantization conditions on a :

  $a \sim 2\pi n\hbar_x$ (problem B), $\qquad a_D(a) = \partial_a F_{NS}(a) \sim 2\pi n\hbar_y$ (problem A)

- Results should match numerical spectrum and should satisfy S-duality

$$E_n^{(A)}(\tilde{\omega}, \tilde{\omega}') = E_n^{(B)}(-i\tilde{\omega}', i\tilde{\omega})$$

However, Bethe/Gauge prescription inconsistent for A-model, S-duality broken:

- B-model:

$\tilde{\omega}' = i\pi,\ \tilde{\omega} = \frac{\pi^2}{3},\ g_x = \sqrt{7}, \hbar_x = \sqrt{2}$	$E_0^{(\mathrm{B})}$	$E_1^{(\mathrm{B})}$
Gauge theory - $O(Q_{5\mathrm{d}}^2)$	3.41632604117...	6.31880651657...
Gauge theory - $O(Q_{5\mathrm{d}}^3)$	3.41635025554...	6.31891130483...
Gauge theory - $O(Q_{5\mathrm{d}}^4)$	3.41635065566...	6.31891099761...
Numerical	3.41635065432...	6.31891099879...

- A-model:

$\tilde{\omega}' = \frac{i\pi^2}{3},\ \tilde{\omega} = \pi,\ g_y = \sqrt{7},\ \hbar_y = \sqrt{2}$	$E_0^{(\mathrm{A})}$	$E_1^{(\mathrm{A})}$
Gauge theory - $O(Q_{5\mathrm{d}}^2)$	3.41642587059...	6.31890324086...
Gauge theory - $O(Q_{5\mathrm{d}}^3)$	3.41644610533...	6.31891130715...
Gauge theory - $O(Q_{5\mathrm{d}}^4)$	3.41644619284...	6.31891133951...
Numerical	3.41635065432...	6.31891099879...

Why mismatch for the A-model?

- B-model energy ok $\Longrightarrow$ problem must be in A-type quantization conditions

- Conjectural quantization conditions written in terms of $F_{NS} = -\ln\langle Z_{5d}\rangle_{NS}$:

$$F_{NS}^{inst} = -Q_{5d}\frac{(1-\mu)(1-q\mu^{-1})}{(1-q)(1-q\alpha)(1-q\alpha^{-1})}\left[(1+\mu)(1+q^2\mu^{-1})+q(\mu+\mu^{-1})-2q(\alpha+\alpha^{-1})-2q\right]+O(Q_{5d}^2)$$

with $q = e^{\frac{2\pi\epsilon_1}{2\tilde{\omega}}}$, $\mu = e^{\frac{2\pi m}{2\tilde{\omega}}}$, $\alpha = e^{\frac{2\pi a}{2\tilde{\omega}}}$

- For A-model $q \in$ unit circle $\Longrightarrow$ F_{NS} not well-defined function, poles at

$$\hbar_y / 2\tilde{\omega} \in \mathbb{Q}$$

which however should be perfectly admissible values in Quantum Mechanics

Indeed, we were not careful enough in defining the quantum problem:

- The quantum Hamiltonians are not differential but finite-difference operators

  $\Longrightarrow$ eigenfunction $\psi(\vec{z})$ ambiguous, defined only up to $i\hbar$-periodic function

How to fix this ambiguity? Faddeev's observation:

- Consider a second set of N commuting finite-difference operators $\hat{\tilde{H}}_j$,

  obtained from the first set by the exchange $2\tilde{\omega} \leftrightarrow \hbar$; by construction

$$[\hat{H}_j, \hat{H}_k] = 0, \qquad [\hat{\tilde{H}}_j, \hat{\tilde{H}}_k] = 0, \qquad [\hat{H}_j, \hat{\tilde{H}}_k] = 0$$

- We then study the unambiguous modular double problem with $2N$ operators:

$$\hat{H}_j \psi(\vec{z}) = E_j \psi(\vec{z}) \qquad \hat{\tilde{H}}_j \psi(\vec{z}) = \tilde{E}_j \psi(\vec{z})$$

What are the implications for the naive 5d Bethe/Gauge prescription?

From modular duality structure expect full solution to be $2\tilde{\omega} \leftrightarrow \hbar$ symmetric,

but Bethe/Gauge formulae are not symmetric $\Longrightarrow$ proposal: restore symmetry

full solution = "symmetrized" Bethe/Gauge

The full A-type quantization conditions would then be

$$a_D = \partial_a F_{full}(a) \sim 2\pi n \hbar_y \,, \qquad F_{full}(a) = \underbrace{F_{NS}(a;\hbar_y,2\tilde{\omega})}_{\text{Bethe/Gauge}} + \underbrace{F_{NS}(a;2\tilde{\omega},\hbar_y)}_{\text{non-perturbative}}$$

The second piece naturally involves non-perturbative corrections in $\hbar \sim \epsilon_1$,

which are such that $F_{full}(a)$ is free of poles at $\hbar_y / 2\tilde{\omega} \in \mathbb{Q}$

Correct energies and S-duality restored after adding non-perturbative effects:

- B-model:

$\tilde{\omega}' = i\pi,\ \tilde{\omega} = \frac{\pi^2}{3},\ g_x = \sqrt{7}, \hbar_x = \sqrt{2}$	$E_0^{(\mathrm{B})}$	$E_1^{(\mathrm{B})}$
Gauge theory - $O(Q_{5\mathrm{d}}^2)$	3.41632604117...	6.31880651657...
Gauge theory - $O(Q_{5\mathrm{d}}^3)$	3.41635025554...	6.31891130483...
Gauge theory - $O(Q_{5\mathrm{d}}^4)$	3.41635065566...	6.31891099761...
Numerical	3.41635065432...	6.31891099879...

- A-model:

$\tilde{\omega}' = \frac{i\pi^2}{3},\ \tilde{\omega} = \pi,\ g_y = \sqrt{7},\ \hbar_y = \sqrt{2}$	$E_0^{(\mathrm{A})}$	$E_1^{(\mathrm{A})}$
Gauge theory - $O(Q_{5\mathrm{d}}^2)$	3.41633033157...	6.31890290001...
Gauge theory - $O(Q_{5\mathrm{d}}^3)$	3.41635056648...	6.31891096630...
Gauge theory - $O(Q_{5\mathrm{d}}^4)$	3.41635065398...	6.31891099866...
Numerical	3.41635065432...	6.31891099879...

What were we missing?

- Gauge theory formulae provide exact WKB solution to the A-model problem, that is the solution containing all perturbative corrections in $\hbar$

- However, full solution may involve non-perturbative contributions in $\hbar$ which gauge theory is not able to take into account

- Exact A-model quantization conditions by adding non-perturbative corrections to the gauge theory (WKB) ones, which remove the poles at $\hbar_y / 2\tilde{\omega} \in \mathbb{Q}$

- Need a "non-perturbatively" corrected version of Bethe/Gauge for 5d theories / non-perturbative corrections to topological string theory (NS on $S^5_{\omega_1,\omega_2,\omega_3}$?)

Thanks!