Scattering Amplitudes from Geometries

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Motivations

Search for "holographic" S-matrix theory: fascinating geometric structures underlying scattering amplitudes, in some auxiliary space

- $\mathcal{M}_n^{(g)}$: **perturbative string** amps = correlators of worldsheet CFT $\rightarrow$ **twistor strings & scattering equations**, same worldsheet but without stringy excitations [Witten; w. Cachazo, Yuan; Mason, Skinner...]
- $G_+(k,n)$: **amplituhedron for $\mathcal{N}=4$ SYM** [Arkani-Hamed et al.]

Both geometries have "factorizing" boundary structures: locality and unitarity naturally emerge (without referring to the bulk)

What questions to ask, directly in the "kinematic space", to generate local, unitary dynamics? Avatar of these geometries?

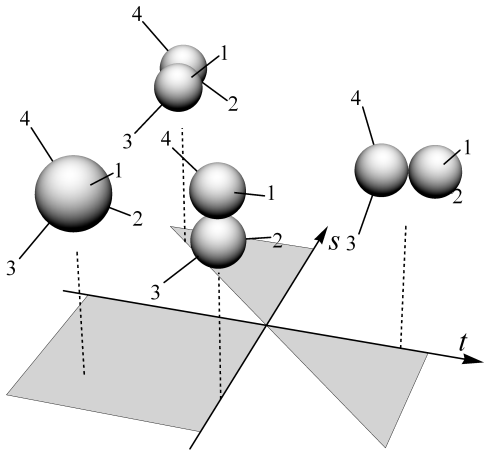
Example: scattering eqs & worldsheet

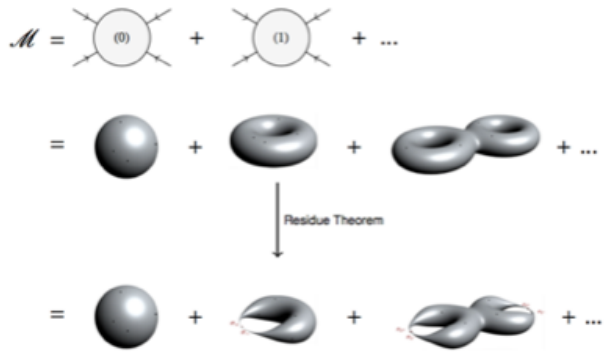
Connected formulas for a large class of massless tree amplitudes [CHY]:

$$M_n^{\text{tree}} = \int \frac{d^n\sigma}{\text{vol}\, SL(2)} \prod_a{}' \delta(E_a)\, \mathcal{I}_n(\{\sigma\}, \cdots)\,, \quad E_a := \sum_{b\neq a} \frac{s_{a,b}}{\sigma_a - \sigma_b} = 0\,.$$

"CHY integrand", $\mathcal{I}_n$, depends on theory: $\text{PT}(\alpha)\,\text{PT}(\beta)$ for bi-adjoint ϕ^3, $\text{PT}\,\text{Pf}'\Psi_n(\epsilon)$ for Yang-Mills and $\text{Pf}'\Psi_n(\epsilon)^2$ for gravity, *etc.*

Scattering Equations are universal: connect boundaries of $\mathcal{M}_n^{(0)}$ to physical singularities, $(\sum_{i\in L} p_i)^2 \to 0 \iff \mathcal{M}_n^{(0)} \to \mathcal{M}_L^{(0)} \times \mathcal{M}_R^{(0)}$





Amplitudes as Forms

Scattering amps as differential forms on kinematic space $\to$ a new picture for amplituhedron [Arkani-Hamed, Thomas, Trnka] & more!

Forms on momentum-twistor space = (bosonized) superamplitude in $\mathcal{N}=4$ SYM: $\eta_i \to dZ_i \implies \Omega_n^{(4k)}$ for N^{k}MHV tree

(tree) amplituhedron geometry="positive" region $\cap$ $4k$-dim subspaces $\Omega_n^{(4k)}|_{\text{subspace}}$ = **canonical form** of **positive geometry** [Arkani-Hamed, Bai, Lam]

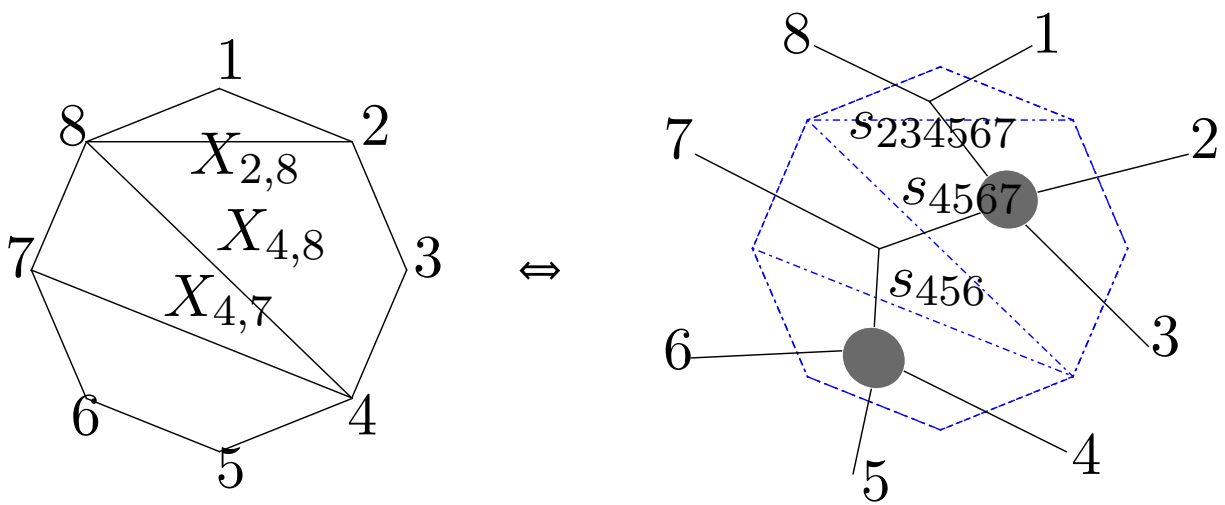
This talk: identical structure for wide variety of theories in any dim:

- Bi-adjoint ϕ^3 from kinematic and worldsheet associahedra
- YM/NLSM: "geometrizing" color & its duality to kinematics
- Real and complex integrals of forms, double-copy & strings?

Kinematic Space

The kinematic space, $\mathcal{K}_n$, for n massless momenta p_i is spanned by Mandelstam variables $s_{i\,j}$ with $\sum_{j\neq i} s_{i\,j} = 0\,;\quad \dim \mathcal{K}_n = \frac{n(n-3)}{2}$.

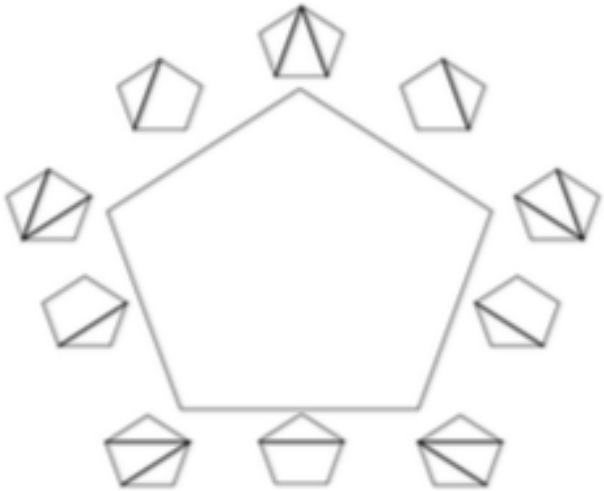
Given an ordering $(12\cdots n)$, planar variables $X_{i,j} := s_{i,i+1,\cdots,j-1}$ dual to $\frac{n(n-3)}{2}$ diagonals of a n-gon form a basis of $\mathcal{K}_n$

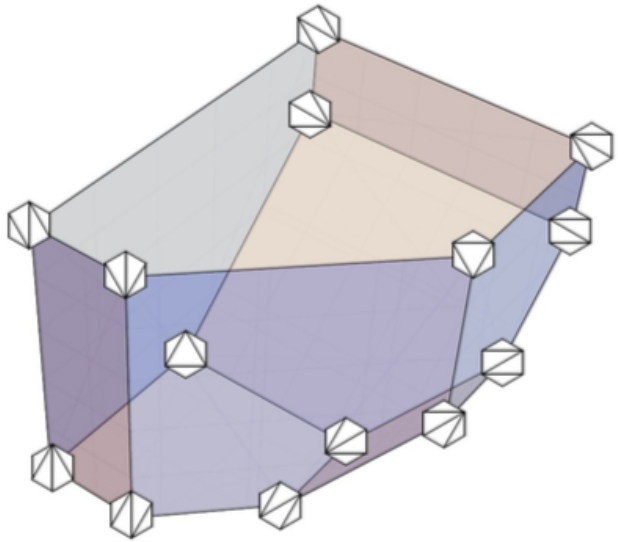


A planar cubic tree graph consists of $n-3$ compatible planar variables as poles, and it is dual to a full triangulation of the n-gon.

The Associahedron

The **associahedron** polytope encodes combinatorial "factorization": each co-dim d face represent a triangulation with d diagonals or planar tree with d propagators (vertices $\leftrightarrow$ planar cubic trees)

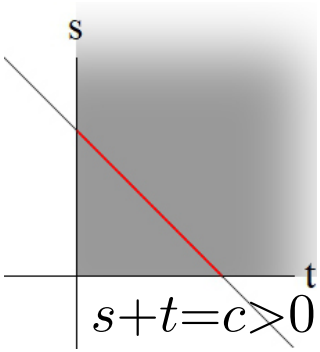




Universal factorization structures of any massless tree amps (in particular ϕ^3), but how to realize it directly in kinematic space?

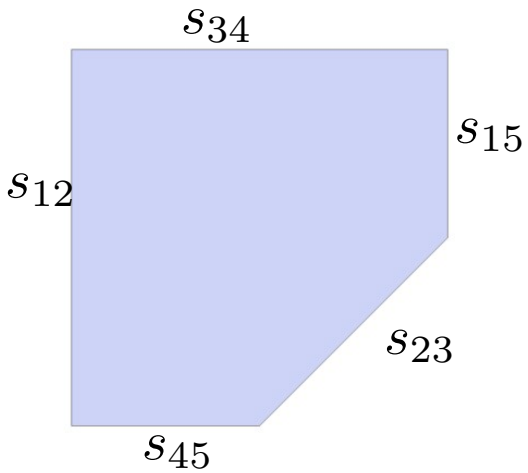
Kinematic Associahedron

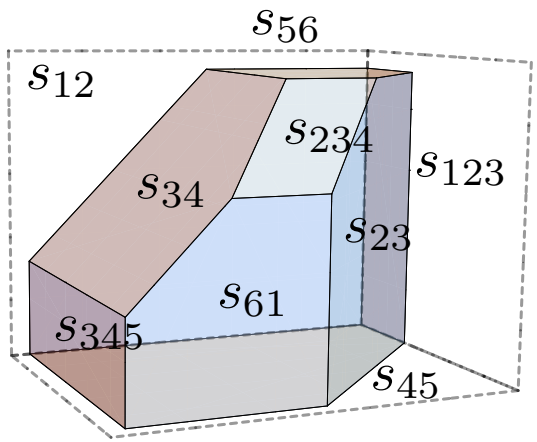
Δ_n: all $X_{i,j} \geq 0$ (top-dim cone); H_n: $(n-3)$-dim subspace defined by $X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} (= -s_{ij}) = c_{i,j}$ as *positive constants*, for all non-adjacent $1 \leq i, j < n$. Then $\mathcal{A}_n := \Delta_n \cap H_n$ [ABHY]



$$e.g.\ \mathcal{A}_4 = \{s > 0, t > 0\} \cap \{-u = \text{const} > 0\}$$

$$\mathcal{A}_5 = \{s_{12}, \cdots, s_{51} > 0\} \cap \{-s_{13}, -s_{14}, -s_{24} = \text{const} > 0\}$$



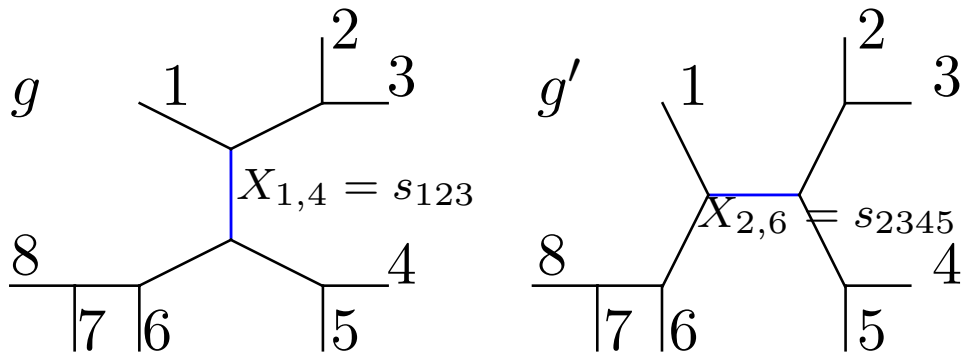


Planar Scattering Forms

The *planar scattering form* for ordering $(12\cdots n)$ is a sum of rank-$(n-3)$ $d\log$ forms for $\mathbf{Cat}_{n-2}$ planar cubic graphs with $\text{sign}(g)=\pm 1$:

$$\Omega_n^{(n-3)} := \sum_{\text{planar } g} \text{sign}(g) \bigwedge_{a=1}^{n-3} d\log X_{i_a,j_a}$$

with $\text{sign}(g) = -\text{sign}(g')$ for any g, g' related by a *mutation*



Sign-flip rule fixed by projectivity: invariant under *local* $\text{GL}(1)$ transf. $X_{i,j} \to \Lambda(X) X_{i,j}$ (well-defined in a projectivized space)

Projectivity is equivalent to requiring that the form only depends on *ratios* of variables, *e.g.* $\Omega_4^{(1)} = \frac{ds}{s} - \frac{dt}{t} = d\log\frac{s}{t}$ and

$$\begin{aligned}
\Omega_5^{(2)} = &\quad \tfrac{ds_{12}}{s_{12}} \wedge \tfrac{ds_{34}}{s_{34}} + \tfrac{ds_{23}}{s_{23}} \wedge \tfrac{ds_{45}}{s_{45}} + \cdots + \tfrac{ds_{51}}{s_{51}} \wedge \tfrac{ds_{23}}{s_{23}} \\
= &\quad d\log\tfrac{X_{1,3}}{X_{2,4}} \wedge d\log\tfrac{X_{1,3}}{X_{1\,4}} + d\log\tfrac{X_{1,3}}{X_{2,5}} \wedge d\log\tfrac{X_{3,5}}{X_{2,4}} \\
\Omega_6^{(2)} = &\quad \textstyle\sum_{g=1}^{14} \pm \wedge (d\log X)^3 = \sum \pm\, d\log \text{ratio}'s
\end{aligned}$$

It follows immediately that $\Omega^{(n-3)}$ is cyclically invariant up to a sign $i \to i{+}1$: $\Omega_n^{(n-3)} \to (-1)^{n-3}\,\Omega_n^{(n-3)}$, and it factorizes correctly *e.g.*

$$X_{1,m} = s_{1,\cdots,m-1} \to 0: \qquad \Omega_n \to \Omega_m \wedge d\log X_{1,m} \wedge \Omega_{n-m+2}$$

Projectivity is a remarkable property of $\Omega_n^{(n-3)}$, not true for each diagram or any proper subset of planar Feynman diagrams.

Canonical Form of $\mathcal{A}_n$

Unique form of any positive geometry= "volume" of the dual: $\Omega(A)$ has $d\log$ singularities on all boundaries ∂A with $\text{Res} = \Omega(\partial A)$

For simple polytopes: $\sum_v \pm \wedge d\log F$ for faces $F = 0$ adjacent to v

Canonical form of $\mathcal{A}_n$ = Pullback of Ω_n to $H_n \propto$ planar ϕ^3 amplitude!

$$e.g. \qquad \Omega(\mathcal{A}_4) = \Omega_4^{(1)}|_{H_4} = (\frac{ds}{s} - \frac{dt}{t})|_{-u=c>0} = (\frac{1}{s} + \frac{1}{t})\, ds$$

$$\Omega(\mathcal{A}_5) = \Omega_5^{(2)}|_{H_5} = \left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right) ds_{12} \wedge ds_{34}$$

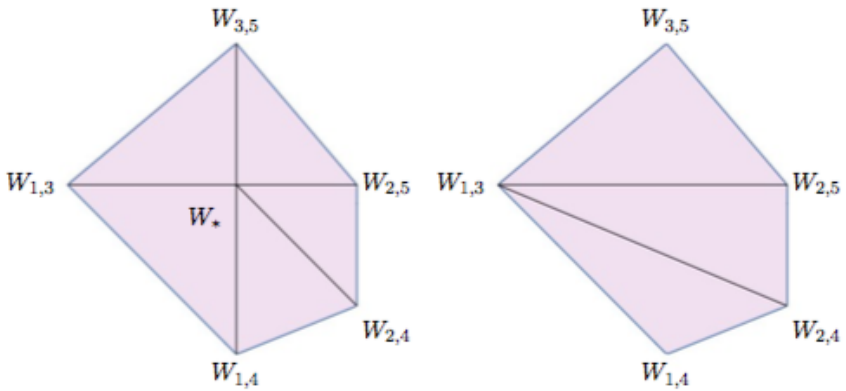
$$\Omega(\mathcal{A}_n) = \sum \text{sgn}(g) \bigwedge_{a=1}^{n-3} d\log X_{i_a,j_a} = d^{n-3}X\ m(12\cdots n|12\cdots n)$$

Similarly for $m(\alpha|\beta)$: "volume" of degenerate $\mathcal{A}_n$ (faces at infinity)

Triangulations & Recursion Relations for ϕ^3 Amps

Geometric picture: Feynman-diagram expansion = triangulation of the dual into $\mathbf{Cat}_{n-2}$ simplices by introducing the point at "infinity"

Triangulate the dual or itself in other ways $\rightarrow$ new rep. of ϕ^3 amps



$$\Omega(\mathcal{A}_5) = d^2\mathbf{s}\,\left(\frac{1}{s_{12}s_{34}} + \cdots + \frac{1}{s_{51}s_{23}}\right)$$

$$= d^2\mathbf{s}\,\left(\frac{s_{12}+s_{51}}{s_{12}s_{34}s_{51}} + \frac{s_{12}+s_{51}}{s_{12}s_{51}s_{23}} + \frac{s_{12}-s_{45}+s_{23}}{s_{12}s_{23}s_{45}}\right)$$

$$= \text{sum of 3 triangles of } \mathcal{A}_5 \text{ itself}$$

$$(\text{each with spurious poles} \sim s^2)$$

Similar to "local" or "BCFW" triangulations of the amplituhedron: BCFW-like recursion & hidden properties of ϕ^3 amplitudes [w. Yang]

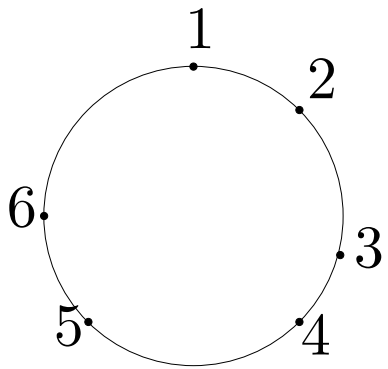
Worldsheet Associahedron & Scattering Equations

A well-known associahedron: minimal blow-up of the open-string worldsheet $\mathcal{M}_{0,n}^{+} := \{\sigma_1 < \sigma_2 < \cdots < \sigma_n\}/\mathrm{SL}(2,\mathbb{R})$ [Deligne, Mumford]

This is non-trivial in σ's but becomes manifest *e.g.* using cross ratios

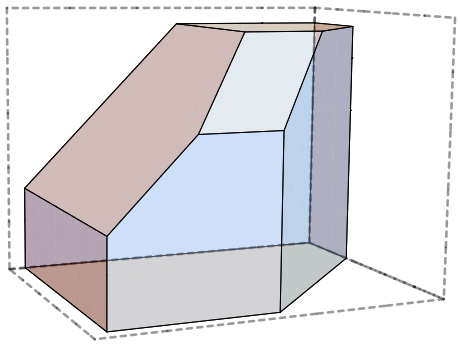
The *canonical form* of $\overline{\mathcal{M}}_{0,n}^{+}$ is the "Parke-Taylor" form

$$\omega_n^{\mathrm{WS}} := \frac{1}{\mathrm{vol}\,[\mathrm{SL}(2)]} \prod_{a=1}^{n} \frac{d\sigma_a}{\sigma_a - \sigma_{a+1}} := \mathrm{PT}(1,2,\cdots,n)\, d\mu_n$$



scattering equations

as a map from $\overline{\mathcal{M}}_{0,n}^{+}$ to $\mathcal{A}_n$



Pushforward from the wolrdsheet

On H_n, scattering eqs provide a diffeomorphism from $\overline{\mathcal{M}}_{0,n}^{+}$ to $\mathcal{A}_n$:

$$s_{a,a+1} = \sigma_{a,a+1} \sum_{1<i+1\leq a\leq j<n} \frac{c_{i,j}}{\sigma_{i,j}} \qquad \text{for } a = 1,\ldots,n-3 \ (\sigma_n \to \infty)$$

Diff $A \to B \implies$ pushforward $\Omega(A) \to \Omega(B)$ [Arkani-Hamed, Bai, Lam]

$$y = f(x) \qquad \Longrightarrow \qquad \Omega(B)_y = \sum_{x=f^{-1}(y)} \Omega(A)_x$$

Canonical form of $\mathcal{A}_n$, $\Omega_n^{\phi^3}$ is the pushforward of $\omega_n^{\rm WS}$ by summing over $(n-3)!$ sol. of scattering eqs (geometric origin of CHY)

$$\sum_{\rm sol.} d\mu_n \ {\rm PT}(\alpha)|_{H(\alpha)} = m(\alpha|\alpha) \ d^{n-3}\mathbf{s}$$

Projective Scattering Forms

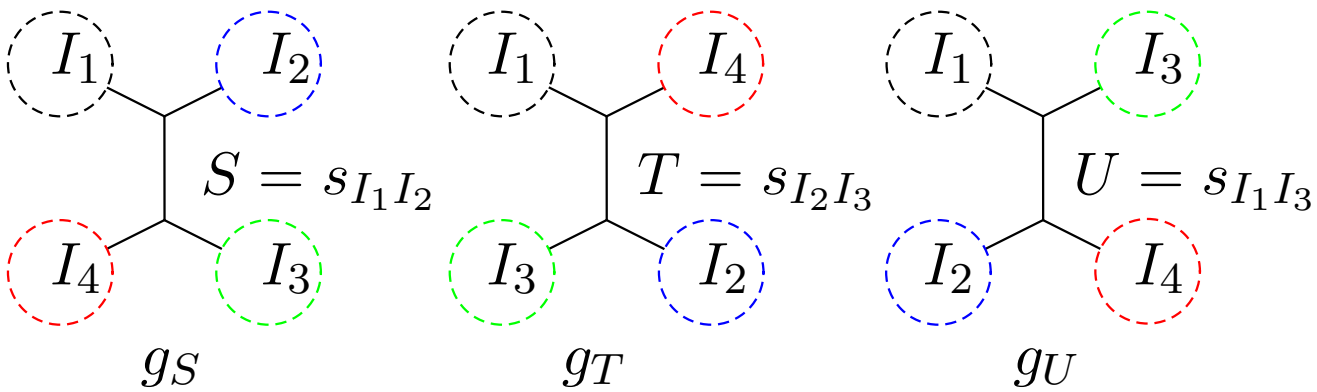
General scattering forms: sum over all cubic graphs with numerators

$$\Omega[N] = \sum_g N(g) \bigwedge_{I=1}^{n-3} d\log s_I \,, \quad e.g.\ N_s\, d\log s + N_t\, d\log t + N_u\, d\log u$$

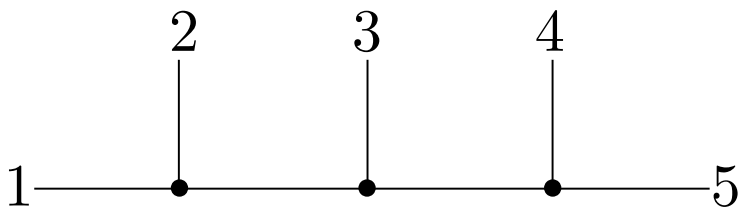
Projectivity: require $\Omega[N]$ to be *i.e.* covariant under $s_I \to \Lambda(s) s_I$

$\Longrightarrow$ kinematic numerators can be chosen to satisfy Jacobi identities

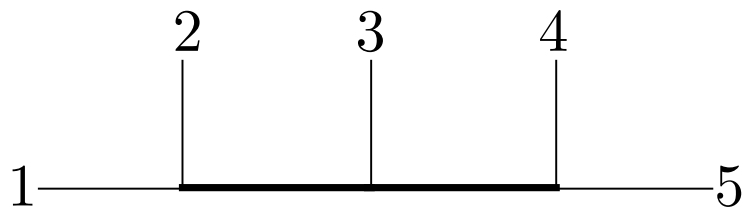
$$N(g_S) + N(g_T) + N(g_U) = 0 \,, \quad e.g.\ N_s + N_t + N_u = 0$$



Color is Kinematics



$$f^{a_1a_2b}f^{ba_3c}f^{ca_4a_5} \quad \leftrightarrow$$



$$\mathrm{d}s_{12}\wedge\mathrm{d}s_{45}$$

Duality between *color factors* and *differential forms on* $\mathcal{K}_n$ for cubic graphs: $C(g)$ and $W(g)$ satisfy the same algebra.

$$\textbf{Claim}: \ W(g) := \pm \bigwedge_{I=1}^{n-3} ds_I \implies W(g_S)+W(g_T)+W(g_U) = 0$$

Scattering forms are color-dressed amps without color factors.
For U(N), partial amps are pullbacks to subspaces (as bi-adjoint ϕ^3).

Uniqueness of YM and NLSM Forms

Remarkably rigid objects encoding full amps in YM & NLSM

Gauge invariance: $\Omega^{\rm YM}$ invariant under every shift $\epsilon_i^\mu \to \epsilon_i^\mu + \alpha p_i^\mu$

Adler zero: $\Omega^{\rm NLSM}$ vanishes under every soft limit $p_i^\mu \to 0$

Key: forms are projective $\Longrightarrow$ unique $\Omega^{\rm YM}$ and $\Omega^{\rm NLSM}$!

Implies the amp "uniqueness theorem" [Arkani-Hamed, Rodina, Trnka]:
$(n-1)!$ parameters for amp vs. unique form up to an overall const.

Alternatively they are pushforward of rigid worldsheet objects

$$\Omega_n^{\rm YM} = \sum_{\rm sol.} d\mu_n {\rm Pf}'\Psi_n\,, \quad \Omega_n^{\rm NLSM} = \sum_{\rm sol.} d\mu_n \det{}' A_n$$

Integrals of Canonical Forms

Natural to integrate Ω_P in P provided regulators for all facets W_a's

$$\mathcal{I}_P \equiv \epsilon^d \int_P \Omega_P(Y) \prod_{\text{facets}} (Y \cdot W_a)^{\epsilon X_a}, \ e.g. \ \epsilon \int_0^1 \frac{dx}{x(1-x)} x^{\epsilon A}(1-x)^{\epsilon B},$$

The regulators ensure no log divergences, and the leading term in ϵ comes from vertices, *e.g.* for simple polytopes (d facets for each):

$$\lim_{\epsilon\to 0} \mathcal{I}_P(\{X\}) = \sum_{\text{vertex } i} \prod_a^d \frac{1}{X_a}, \quad e.g. \ \lim_{\epsilon\to 0} \mathcal{I}_{[0,1]} = \frac{1}{A} + \frac{1}{B}$$

$$\lim_{\epsilon\to 0} \epsilon^2 \int \frac{dxdy}{x(y-x)(1-y)} x^{\epsilon A}(y-x)^{\epsilon B}(1-y)^{\epsilon C} = \frac{1}{AB} + \frac{1}{BC} + \frac{1}{CA}.$$

Looks like *canonical function* of some polytope in X space.

Integral vs. Pushforward

The geometry Q can be obtained via a map from P to X space:
"scattering equations" =saddle-point of "Koba-Nielson" factors

$$\sum_a X_a \, d\log(Y \cdot W_a) = 0 \,, \quad \text{or} \sum_a \frac{X_a}{Y \cdot W_a} W_a^I = 0 \; (I = 1, \cdots, d)$$

Conjecture: Leading term of the integral $\mathcal{I}_P$ equals canonical function of Q, $F_Q(X)$, which is given by pushforward of $\Omega(P)$:

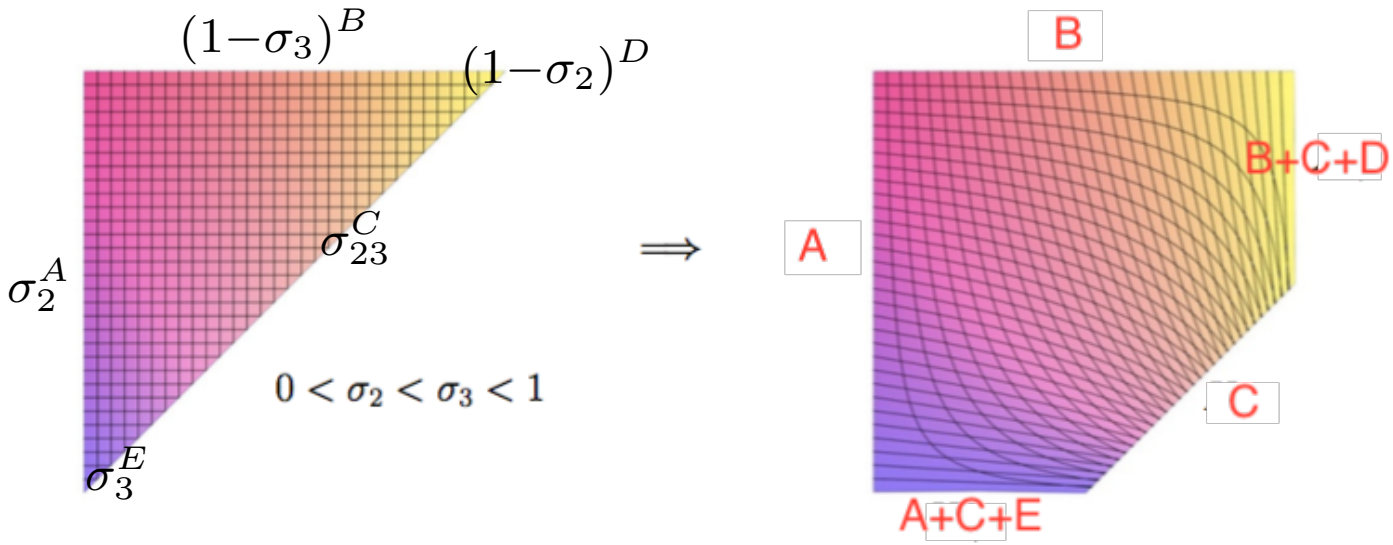
$$\lim_{\epsilon \to 0} \int_P \Omega(P) \prod_a (Y \cdot W_a)^{\epsilon X_a} = F_Q(X) \,, \text{ where}$$

$$d^d X \, F_Q(X) = \sum_{\text{sol.}} \Omega(P)|_H \,, e.g. \; \Omega(Q) = \sum_{\text{vertex } i} \pm \prod_a d\log X_a \,,$$

We believe this to be the general mechanism behind CHY:
field-theory limit of string integral = pushforward via scattering eqs

Same for integrals with extra walls passing through vertices of P: Q can be thought of as "blowup" of P at those boundaries, *e.g.*

$$\lim_{\epsilon\to 0}\int \frac{d\sigma_2 d\sigma_3}{\sigma_2\sigma_{23}(1-\sigma_3)}\text{“KN”} = \frac{1}{AB}+\left(\frac{1}{A}+\frac{1}{C}\right)\frac{1}{(A+C+E)}+\left(\frac{1}{B}+\frac{1}{C}\right)\frac{1}{(B+C+D)}$$



The leading term, and remarkably the pushforward, only depends on the combinatorics, not any details of the extra walls.

Complex Integrals vs. CHY formula

Another natural integral: $|\Omega_P(Y)|^2$ on $\mathbb{C}^d$. Same limit as the real one:

$$\mathcal{I}_P^{\mathbb{C}} := (\frac{\epsilon}{2\pi i})^d \int_{\mathbb{C}^d} |\Omega_P(Y)|^2 \prod_{\text{facets}} |Y \cdot W_a|^{\epsilon X_a}, \quad \lim_{\epsilon\to 0} \mathcal{I}_P^{\mathbb{C}} = \lim_{\epsilon\to 0} \mathcal{I}_P \,,$$

Key: the limit again comes from every vertex, which gives the same residue as the real case: $\lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{dzd\bar{z}}{|z|^2} |z|^{\epsilon A} = \lim_{\epsilon\to 0} \int \frac{rdr}{r^2} r^{\epsilon A} = \frac{1}{A}$.

The same holds for complex integrals with extra walls, where the limit equals to "CHY formula" (self-intersecting number [Mizera])

$$\lim_{\epsilon\to 0} \mathcal{I}_P^{\mathbb{C}} = \frac{1}{(2\pi i)^d} \oint_{|E^I|=\varepsilon} \frac{\Omega_P(Y)\,\hat{\Omega}_P(Y)}{\prod_{I=1}^d E^I}\,, \quad E^I := \sum_a \frac{X_a W_a^I}{Y \cdot W_a}\,.$$

Integrals of General Forms

These integrals extract rational functions in X space from forms, and we can apply them to general (non-$d\log$) forms like $\Omega^{\rm YM/NLSM}$.

Consider pullback of $\Omega^{\rm YM}(x) = \sum_g N_g^{\rm YM}(\epsilon, k) \wedge_{i,j} d\log x_{ij}$ to H_n: the regulated integral in $\mathcal{A}_n$ (all $x_{ij} > 0$) gives the partial amplitude!

$$\lim_{\epsilon\to 0} \epsilon^{n-3} \int_{x_{ij}>0} \Omega^{\rm YM}(x)|_H \prod_{i,j} x_{ij}^{\epsilon X_{ij}} = \sum_g \frac{{\rm Res}_g \Omega^{\rm YM}}{\prod X_{ij}^{(g)}} = M_n^{\rm YM} \,,$$

$$e.g. \qquad \lim_{\epsilon\to 0} \int_{0<x<c} (N_s d\log x + N_t d\log(c-x)) x^{\epsilon s}(c-x)^{\epsilon t} = \frac{N_s}{s} - \frac{N_t}{t} \,.$$

Each vertex gives $n-3$ planar poles $X_{i\,j}$, times the residue. Here X's are given by the kinematic data defining residues, $N_g^{\rm YM}(\epsilon, p)$.

Complex Integrals and Gravity Amplitude

Complex integral of general $|\Omega|^2$: pullback to *any* generic subspace and put walls for *all* poles of Ω, and we get residue squared

$$\lim_{\epsilon\to 0}\int_{\mathbb{C}^d}|\Omega^{(d)}(Y)|^2\prod_a|Y\cdot W_a|^{\epsilon X_a}=\sum_{\text{vertex } i}\frac{|\text{Res}_i\ \Omega^{(d)}|^2}{\prod_a^d X_a}\,,$$

If we take Ω_n^{YM} in BCJ form, it gives exactly the gravity amplitude!

$$\lim_{\epsilon\to 0}\int_{\mathbb{C}^d}|\Omega_n^{\text{YM}}(x)|^2\prod_I^{2^{n-2}-1}|x_I|^{\epsilon s_I}=\sum_g\frac{|\text{Res}_g\Omega_n^{\text{YM}}|^2}{\prod s_I^{(g)}}=M_n^{\text{GR}}\,,e.g.$$

$$\lim_{\epsilon\to 0}\int_{\mathbb{C}}|\frac{N_s dx}{x}+\frac{N_t dy}{y}+\frac{N_u dz}{z}|^2|x|^{\epsilon s}|y|^{\epsilon t}|z|^{\epsilon u}=\frac{|N_s|^2}{s}+\frac{|N_t|^2}{t}+\frac{|N_u|^2}{u}\,.$$

Projectivity Needed for Double Copy

It is crucial to start with a *projective form*, otherwise the integral also has non-vanishing residue at infinity $\sim N_s + N_t + N_u$. In general, projectivity ensures the absence of pole at infinity along any direction!

Projectivity $\leftrightarrow$ No pole at infinity $\leftrightarrow$ Double copy from $|\Omega|^2$

Different ways to represent $\Omega^{\rm YM/NLSM} \rightarrow$ the integral always gives the same limit. Also asymmetric: $\lim_{\epsilon\to 0}\int \Omega_n^{\rm YM}(\Omega_n^{\rm NLSM})^* = M_n^{\rm BI}$

Any projective form admits $\Omega = \sum_\alpha N_\alpha \Omega^{\phi^3}(\alpha)$, and integral gives

$$\lim_{\epsilon\to 0}\int_{\mathbb{C}^{n-3}} \Omega_L\, \Omega_R^* \prod_I |x_I|^{\epsilon s_I} = \sum_{\alpha,\beta} N_L(\alpha) N_R(\beta)\; m(\alpha|\beta)\,.$$

which in particular implies KLT if expanded in a $(n-3)!$ basis.

Minkowski-sum Decomposition

"Soft limit" of $\mathcal{A}_n$: as $c_{1,i} \to 0$ for $i = 3, 4, \cdots, n-1$, $\mathcal{A}_n$ collapse to lower dim, and the amplitude vanishes (manifest from geometry!)

The basic building blocks are simplices of various dim, $\Delta_{i,j}$, as degenerate limit of $\mathcal{A}_n$: all c's vanish except for one $c_{i,j} \neq 0$

$\mathcal{A}_n$ = Minkowski sum of $(n-2)(n-3)/2$ $\Delta_{i,j}$'s (only $\Delta_{1,n-1}$ top dim.)
Same for all Cayley polytopes [w. Gao, Zhang] (all $\Delta_{i,j}$ are lower dim.)



general sum of simplices → all generalized permutohedra [Postnikov]

String Worldsheet from Associahedron

For $\mathcal{A}_n$, $\Delta_{1,n-1}$ is the unique $(n-3)$-dim simplex $\rightarrow$ string worldsheet!

Solving constraints $X_{i,j}+X_{i-1,j-1}-X_{i,j-1}-X_{i-1,j}=0 \implies$
$X_{i,j}=c_{1,n-1}(z_{j-1}-z_i)=z_{j-1}-z_i$ with fixing $c_{1,n-1}=1$.

(1) Degenerate: $\int_{\mathcal{A}_n} \rightarrow \int_{\Delta_{1,n-1}}$ (2) One extra wall for each $\Delta_{i,j}$, *e.g.*

$$\int_{\Delta_{1,n-1}} \Omega(\mathcal{A}_n) \prod_a (Y\cdot W_a)^{\alpha' X_a} = \int_{z_{i-1}<z_i} \omega_n^{\rm WS}(z) \prod_{i,j} z_{i,j}^{\alpha' s_{i,j}}$$

This is exactly worldsheet integral with Koba-Nielson factor!

While (a special) triangulation of $\mathcal{A}_n^*$ gives Feynman diagrams, it seems the Minkowski sum naturally leads to string worldsheet!

Outlook

- How to α'-deform canonical forms? String amps from $\mathcal{A}_n$? Rediscover cluster algebra (associahedra) [w. Arkani-Hamed, Thomas]
- Loops: halohedra *etc.* at one loop [Salvatori] picture similar to amplituhedron? connections to ambitwistor strings?
- 4d amplituhedron in momentum space; forms combining helicity amps & pushforward from twistor-string worldsheet [w. C. Zhang]
- A unified geometric picture for amplitudes & more, *e.g.* cosmological polytopes [Arkani-Hamed, Benincasa, Postnikov], AdS?

Thank you !