Triangulation of 2-loop MHV Amplituhedron from sign flips

Ryota Kojima
Sokendai, KEK theory centor

Based on

RK “Triangulation of 2-loop MHV Amplituhedron from sign flips” to appear

RK and Jaroslav Trnka(UC Davis, QMAP), in progress

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Introduction

- Scattering amplitude: Basic objects in QFT.

- Recently there are many progress about the structure of scattering amplitude in Planar N=4 SYM.

- One of the important progress: Geometric structure

The scattering amplitudes can be obtained from some geometrical object : Amplituhedron $\mathcal{A}_{n,m,k}(Z)$

  - defined as a generalization of the interior of polytope

  - no reference to usual field theory notions: feynman diagrams, Lagrangian…

Introduction

- The amplituhedron $\mathcal{A}_{n,m,k}(Z)$: k plane Y in k+m dimensional space

(N. Arkani-Hamed, J. Trnka 2013)

$$Y_a^I = C_{\alpha a} Z_\alpha^I$$

$$C = (\vec{c}_1, \cdots, \vec{c}_n) \in G_+(k,n)$$
$$Z = (Z_1^I, \cdots, Z_n^I) \in M_+(k+m,n)$$

$G_+(k,n)$:positive Grassmannian
M_+ :positive matrix

$$a = 1, \cdots, k$$
$$I = 1, \cdots, k+m$$
$$\alpha = 1, \cdots, n$$

- Positive Grassmannian $G_+(k,n)$: k-plane in n-dimensional space

$$C = (\vec{c_1}, \cdots, \vec{c_n})/GL(k) \qquad \langle c_{a_1} \cdots c_{a_k} \rangle > 0 \qquad a_1 < \cdots < a_k$$

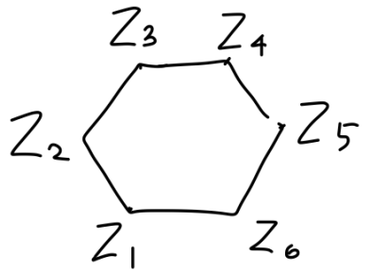
- Positive matrix M_+

$$Z = (Z_1^I, \cdots, Z_n^I) \qquad \langle Z_{a_1}^I, \cdots, Z_{a_{k+m}}^I \rangle > 0 \qquad a_1 < \cdots < a_{k+m}$$

$$I = 1, \cdots, k+m$$

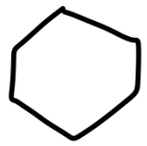
Introduction

- Let consider a point inside a convex polygon in projective plane.



$$Y = c_1 Z_1 + c_2 Z_2 + \cdots + c_n Z_n$$

$$c_j > 0$$



convex polygon



not convex polygon

- Convexity for the polygon is that matrix made by vertices Z_i has all positive minors:

$$\langle Z_{a_1} Z_{a_2} Z_{a_3} \rangle > 0 \qquad a_1 < a_2 < a_3$$

- The interior of the polygon is

$$Y = C \cdot Z$$

$$C = (c_1 c_2 \cdots c_n) \in G_+(1, n)$$
$$Z = (Z_1 Z_2 \cdots Z_n) \in M_+(3, n)$$

M_+ :positive matrix

- We can generalize this to the Grassmannian and higher dimension.

Introduction

$$Y = C \cdot Z$$

$$C = (c_1 c_2 \cdots c_n) \in G_+(1, n)$$

$$Z = (Z_1 Z_2 \cdots Z_n) \in M_+(3, n)$$

$$\downarrow$$

$$Y_a^I = C_{\alpha a} Z_a^I$$

$$C = (\vec{c}_1, \cdots, \vec{c}_n) \in G_+(k, n)$$

$$Z = (Z_1, \cdots, Z_n) \in M_+(k+m, n)$$

This is the definition of the amplituhedron! $\mathcal{A}_{n,m,k}(Z)$

- The amplituhedron is defined as the generalization of the interior of convex polygon.

- m=4 is the physical amplituhedron.

Introduction

- The amplituhedron $\mathcal{A}_{n,m,k}(Z)$: k plane Y in k+m dimensional space

$$Y_a^I = C_{\alpha a} Z_\alpha^I$$

$$C = (\vec{c}_1, \cdots, \vec{c}_n) \in G_+(k,n)$$
$$Z = (Z_1^I, \cdots, Z_n^I) \in M_+(k+m,n)$$

$G_+(k,n)$:positive Grassmannian
M_+ :positive matrix

$$a = 1, \cdots, k$$
$$I = 1, \cdots, k+m$$
$$\alpha = 1, \cdots, n$$

Conjecture

Canonical form of $\mathcal{A}_{n,m=4,k}(Z)$ $\leftrightarrow$ n-point N^{k}MHV amplitude

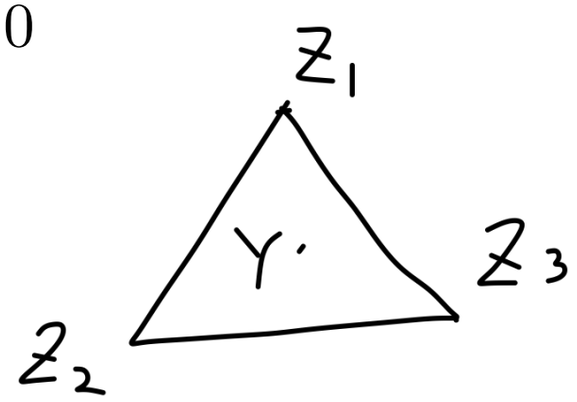
The definition of canonical form Ω :

Ω has logarithmic singularities
on all the boundaries of amplituhedron.

Introduction

- Let consider the simplest case: triangle. $\mathcal{A}_{m=2,k=1,n=3}$

$$Y = Z_1 + c_2 Z_2 + c_3 Z_3 \qquad c_2, c_3 > 0$$



- Boundary of the triangle is

$$c_2 \to 0, \quad c_3 \to 0$$

- The canonical form is

$$\Omega = \frac{dc_2}{c_2}\frac{dc_3}{c_3} = \frac{\langle Y d^2 Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} \qquad \langle 123\rangle = \epsilon_{abc} Z_1^a Z_2^b Z_3^c$$

$$d^2 Y = dc_2 dc_3 Z_2 Z_3$$

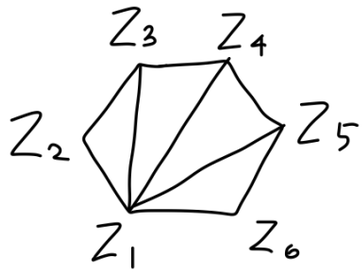
- Consider simplex in $\mathbb{P}^4$, $\mathcal{A}_{m=4,k=1,n=5}$

$$\Omega_P = \frac{\langle Y d^4 Y\rangle \langle 12345\rangle^4}{\langle Y1234\rangle\langle Y2345\rangle\langle Y3451\rangle\langle Y4512\rangle\langle Y5123\rangle}$$

5-pt NMHV amplitude

Introduction

- How to obtain canonical form for general amplituhedron?



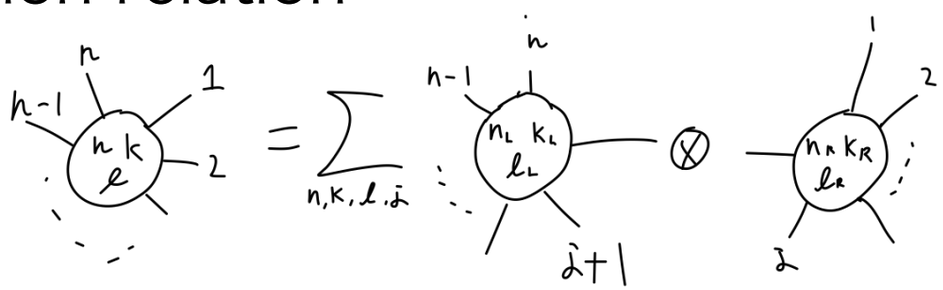
⟶ Triangulation

$$\Omega = \frac{\langle Y d^2 Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y d^2 Y\rangle \langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y d^2 Y\rangle \langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y d^2 Y\rangle \langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

- More general case, it is difficult to triangulation.

- Triangulation for the amplituhedron ↔ Recursion relation for the amplitude

- Example: BCFW recursion relation (Britto, Cachazo, Feng, Witten 2005)



- We can obtain the triangulation of the general amplituhedron from the BCFW.

- However the origin of this triangulation is QFT, not the geometry.

Introduction

- Motivation:

  Is it possible to triangulate the general amplituhedron

  from only geometry?

- Some simple case ⋯ straightforward.

- In this work, we consider 2-loop MHV amplituhedron from the sign flip definition.
- We found that
  - triangulation of n-point 2-loop MHV amplituhedron.
  - formula for n-point 2-loop MHV amplitude.

Amplituhedron from sign flip

$$\mathcal{A}_{n,m,k}(Z)$$

$$Y_a^I = C_{\alpha a} Z_a^I \qquad C = (\vec{c}_1, \cdots, \vec{c}_n) \in G_+(k,n)$$

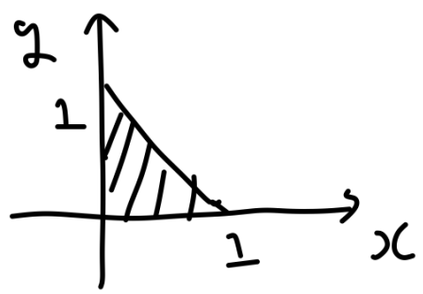
$$Z = (Z_1, \cdots, Z_n) \in M_+(k+m,n)$$

- The amplituhedron is defined as the generalization of the interior of convex polygon.

- There are two definition of the interior of convex polytope:

- vertex-centered description: $Y = c_1 Z_1 + c_2 Z_2 + \cdots + c_n Z_n \qquad c_a > 0$

- face-centered: Cut out the polytope by a collection of inequalities associated with the boundaries



$x > 0$
$y > 0$
$1 - x - y > 0$

Amplituhedron from sign flip

- Is it possible to define the amplituhedron for the face centered description?

⟶ New picture of the amplituhedron: sign flip

(N. Arkani-Hamed, H.Thomas, J. Trnka 2017)

Y is in the m=2 k amplituhedron only if

Y is k plane in k+2 dimensional space which satisfies

$[Yii+1] > 0$ and $\{[Y12], \cdots, [Y1n]\}$ has precisely k sign flips.

Y is in the m=4 k amplituhedron only if

Y is k plane in k+4 dimensional space which satisfies

$[Yii+1jj+1] > 0$ and $\{[Y1234], \cdots, [Y123n]\}$ has precisely k sign flips.

$$[Yij] \equiv \langle (Y_1 Y_2) Z_i Z_j \rangle$$

Amplituhedron from sign flip

- m=2 k=1 n=6 tree amplituhedron: 4 patterns of sign flip

[Y12]	[Y13]	[Y14]	[Y15]	[Y16]
+	−	−	−	−
+	+	−	−	−
+	+	+	−	−
+	+	+	+	−

- From each pattern, we can construct the canonical form.

- Sign flip takes place in i_1, i_2 , we can expand Y as

$$Y_1 = Z_1 + x_1 Z_{i_1} + y_1 Z_{i_1+1}$$
$$Y_2 = -Z_1 + x_2 Z_{i_2} + y_2 Z_{i_2+1}$$

- Then from sign flip, $x_1, y_1, x_2, y_2 > 0$ and the canonical form for this flip pattern is

$$\Omega_{i_1 i_2} = \frac{dx_1}{x_1}\frac{dy_1}{y_1}\frac{dx_2}{x_2}\frac{dy_2}{y_2}$$

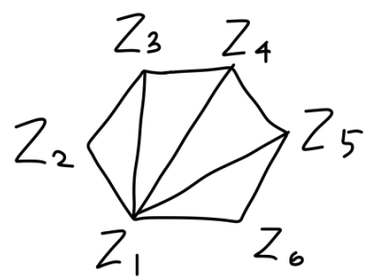
- The canonical form for this amplituhedron is $\Omega = \sum_{\text{all flips}} \Omega_{ij}$

Amplituhedron from sign flip

- The sum of these forms are the full form of this amplituhedron

$$\Omega = \frac{\langle Y d^2 Y\rangle \langle 123\rangle^2}{\langle Y12\rangle\langle Y23\rangle\langle Y31\rangle} + \frac{\langle Y d^2 Y\rangle \langle 134\rangle^2}{\langle Y13\rangle\langle Y34\rangle\langle Y41\rangle} + \frac{\langle Y d^2 Y\rangle \langle 145\rangle^2}{\langle Y14\rangle\langle Y45\rangle\langle Y51\rangle} + \frac{\langle Y d^2 Y\rangle \langle 156\rangle^2}{\langle Y15\rangle\langle Y56\rangle\langle Y61\rangle}$$

- This is the direct triangulation of the amplituhedron.



- Each sign flip pattern ⟷ Cell of the triangulated amplituhedron.
- Only m=1,2 case, we can triangulate from sign flip.
- m=4 amplituhedron, there isn't simple relation between a particular cell and one of the flip pattern.

Triangulation from sign flip

- 1-loop MHV (k=0) physical amplituhedron is isomorphic to the m=2 k=2 unphysical amplituhedron

$$\mathcal{A}(m=4,k=0,n;l=1) \simeq \mathcal{A}(m=2,k=2,n;l=0)$$

- m=2, k=2 tree amplituhedron from sign flip

$$\Omega(\mathcal{A}(m=2,k=2,n)) = \langle Y d^2 Y_1\rangle\langle Y d^2 Y_2\rangle \times \sum_{1<i<j<n} \frac{\langle Y_1 Y_2 (1ii+1)\cap(1jj+1)\rangle^2}{\langle Y_1 Y_2 1i\rangle\langle Y_1 Y_2 1i+1\rangle\langle Y_1 Y_2 ii+1\rangle\langle Y_1 Y_2 1j\rangle\langle Y_1 Y_2 1j+1\rangle\langle Y_1 Y_2 jj+1\rangle}$$

- $(Y_1, Y_2) \to (A, B)$ it correspond to 1-loop k=0 amplitude.

- This triangulation ⋯ direct triangulation of the amplituhedron

Triangulation from sign flip

- How about more complicated case?

$\longrightarrow$ 2-loop MHV amplituhedron

- In the sign flip definition, the 2-loop MHV amplituhedron is defined

> Two 2-plane in 4 dimensional space $(AB)_1, (AB)_2$ that
>
> $(AB)_1, (AB)_2$ are 1-loop MHV amplituhedron
>
> And there are further constraint, $\langle (AB)_1 (AB)_2 \rangle > 0$

- $(AB)_1, (AB)_2$ are 1-loop MHV amplituhedron

$$x_1, w_1, y_1, z_1 > 0, \ x_2, w_2, y_2, z_2 > 0$$

- The region of these parameters are further bounded.

Triangulation from sign flip

- 4-pt case

$$Z_A = Z_1 + x_1 Z_2 + w_1 Z_3, \quad Z_B = -Z_1 + y_1 Z_3 + z_1 Z_4,$$
$$Z_C = Z_1 + x_2 Z_2 + w_2 Z_3, \quad Z_D = -Z_1 + y_2 Z_3 + z_2 Z_4,$$

$$\langle ABCD \rangle = \langle 1234 \rangle \{(x_1 - x_2)(y_1 z_2 - z_1 y_2) + (z_1 - z_2)(w_1 x_2 - x_1 w_2)\} > 0$$

- The case of

$$(y_1 z_2 - z_1 y_2), (z_1 - z_2), (w_1 x_2 - x_1 w_2) > 0$$

$$0 < x_1, \quad 0 < x_2 < x_1 + \frac{(z_1 - z_2)(w_1 x_2 - x_1 w_2)}{y_1 z_2 - z_1 y_2},$$
$$0 < w_1, \quad 0 < w_2 < \frac{w_1 x_2}{x_1}$$
$$0 < z_1, \quad 0 < z_2 < z_1,$$
$$0 < y_1, \quad 0 < y_2 < \frac{y_1 z_2}{z_1}$$

$$\Omega_1 = \frac{1}{x_1}(\frac{1}{x_2} - \frac{1}{x_2 - x_1 - a})\frac{1}{w_1}(\frac{1}{w_2} - \frac{1}{w_2 - \frac{w_1 x_2}{x_1}})\frac{1}{y_1}(\frac{1}{y_2} - \frac{1}{y_2 - \frac{y_1 z_2}{z_1}})\frac{1}{z_1}(\frac{1}{z_2} - \frac{1}{z_2 - z_1})$$

- Another case can be obtained similarly.

$$\Omega = \sum_{i=1}^{8} \Omega_i = \frac{x_1 y_1 z_2 + x_2 y_2 z_1 + x_2 w_1 z_1 + x_1 w_2 z_2}{x_1 x_2 w_1 w_2 y_1 y_2 z_1 z_2 \{(x_1 - x_2)(y_1 z_2 - y_2 z_1) + (z_1 - z_2)(w_1 x_2 - w_2 x_1)\}}$$

- This is 2-loop k=0 4pt amplitude.

Triangulation from sign flip

- Example: 5-pt case

- There are 3×3=9 cells

$[AB12]\cdots[AB15]>0$

$[AB12]$	$[AB13]$	$[AB14]$	$[AB15]$	
+	−	+	+	$\cdots 23$
+	+	−	+	$\cdots 34$
+	−	−	+	$\cdots 24$

$$\Omega = \Omega_{2323} + \Omega_{3434} + \Omega_{2424} + \Omega_{2324} + \Omega_{2334} + \Omega_{2434} + \Omega_{2423} + \Omega_{3423} + \Omega_{3424}$$

$$\begin{aligned}\Omega_{2434} = &\frac{\langle 123A_4\rangle\langle 134C_4\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\langle AB12\rangle\langle AB13\rangle\langle AB14\rangle\langle AB15\rangle\langle AB23\rangle\langle AB45\rangle\langle ABCD\rangle\langle CD13\rangle\langle CD14\rangle^2\langle CD15\rangle\langle CD34\rangle\langle CD45\rangle} \\ &\times\Big\{\langle 123A_4\rangle(\langle AB45\rangle\langle CD13\rangle\langle CD14\rangle + \langle AB15\rangle\langle CD34\rangle\langle CD14\rangle) \\ &-\langle 345A_2\rangle\langle AB14\rangle\langle CD14\rangle\langle CD15\rangle + \langle 123C_4\rangle\langle CD14\rangle\langle AB45\rangle\langle AB13\rangle\Big\}\end{aligned}$$

- The sum of these forms correspond to 5pt 2-loop MHV amplitude.

Triangulation from sign flip

- n-pt case, there are $[\frac{1}{2}(n-3)(n-2)]^2$ cells from the way to chose i,j,k,l .

- There are 13 patterns that $i,j,k,l = 2,3,\cdots,n-1, \quad i<j, k<l$

$$
\begin{aligned}
&i<k<l<j\cdots(1), \quad i<k<j<l\cdots(2), \quad i<j<k<l\cdots(3), \quad i=k<l<j\cdots(4),\\
&i=k<j=l\cdots(5), \quad i=k<j<l\cdots(6), \quad i<k<j=l\cdots(7), \quad i<j=k<l\cdots(8),\\
&k<i=l<j\cdots(9), \quad k<i<l<j\cdots(10), \quad k<i<j=l\cdots(11), \quad k<i<j<l\cdots(12),\\
&k<l<i<j\cdots(13).
\end{aligned}
$$

$$
\Omega_{\rm MHV}^{n\text{-pt 2-loop}} = \sum_{i,j,k,l=2,3,\cdots,n-1} \Omega^1_{ijkl} + \sum_{i<k<j<l} \Omega^2_{ijkl} + \sum_{i<j<k<l} \Omega^3_{ijkl} + \cdots + \sum_{k<l<i<j} \Omega^{13}_{ijkl}
$$

$$
\Omega^1_{ijkl} = \frac{\langle 1ii+1A_i\rangle\langle 1jj+1C_l\rangle\langle ABd^2A\rangle\langle ABd^2B\rangle\langle CDd^2C\rangle\langle CDd^2D\rangle}{\langle AB1i\rangle\langle AB1i+1\rangle\langle AB1j\rangle\langle AB1j+1\rangle\langle ABCD\rangle\langle CD1k\rangle\langle CD1k+1\rangle\langle CD1l\rangle\langle CD1l+1\rangle} \times \frac{\langle ABii+1\rangle\langle A_jC_kC_l1\rangle + \langle A_iA_jC_kC_l\rangle}{\langle ABii+1\rangle\langle ABjj+1\rangle\langle CDkk+1\rangle\langle CDll+1\rangle}
$$

$$
\begin{aligned}
\langle ijkA_l\rangle &\equiv \langle AB1l+1\rangle\langle ijkl\rangle - \langle AB1l\rangle\langle ijkl+1\rangle = \langle ijk(AB)\cap(1ll+1)\rangle\\
\langle ijA_lC_k\rangle &\equiv \langle CD1k+1\rangle\langle ijA_lk\rangle - \langle CD1k\rangle\langle ijA_lk+1\rangle\\
\langle iA_lC_kE_j\rangle &\equiv \langle EF1j+1\rangle\langle iA_lC_kj\rangle - \langle EF1j\rangle\langle iA_lC_kj+1\rangle\\
\langle A_lC_kE_jG_i\rangle &\equiv \langle GH1i+1\rangle\langle A_lC_kE_ji\rangle - \langle GH1i\rangle\langle A_lC_kE_ji+1\rangle.
\end{aligned}
$$

- This is the triangulation of n-pt 2-loop MHV amplitude from sign flip.

- We checked that the sum of these forms is corresponding to the 2-loop amplitude up to 22-pt numerically.

Summary

- The amplituhedron, generalizing the interior of convex polygon, gives a complete definition of scattering amplitudes in planar N=4 SYM.

- From sign flip definition, we can obtain the canonical form of n-point 2-loop MHV amplituhedron from only geometry.

$$\Omega_{\text{MHV}}^{n\text{-pt 2-loop}} = \sum_{\substack{i,j,k,l=2,3,\cdots,n-1 \\ i<k<l<j}} \Omega_{ijkl}^{1} + \sum_{i<k<j<l} \Omega_{ijkl}^{2} + \sum_{i<j<k<l} \Omega_{ijkl}^{3} + \cdots + \sum_{k<l<i<j} \Omega_{ijkl}^{13}$$

- This representation of the 2-loop MHV amplitude is completely new.

- It is not obvious how to derive it from a quantum field theory argument.

- Is there any recursion relation related to this representation?

Summary

- It is possible to generalize this result.

- 3-loop: There are 3 1-loop amplituhedron and

$$\langle ABCD\rangle > 0,\ \langle ABEF\rangle > 0,\ \langle CDEF\rangle > 0$$

- In general, there are L 1-loop amplituhedron and L(L-1)/2 constraints.

Is it possible to obtain the direct triangulation of the L-loop integrand from this triangulation?