Universality of soft theorem from locality of soft vertex

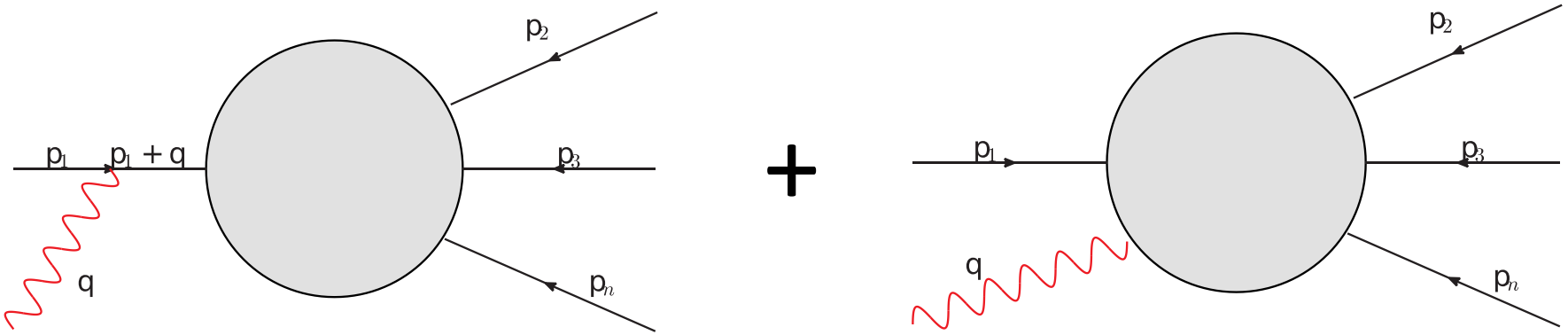
Kyoto University
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Based on SH, H.Kawai, Nucl. Phys. B 936, 400

Talk plan

- Soft theorem in field theory
- Soft theorem in string theory
- Superlocality of soft graviton vertex
- Soft graviton theorem
- Summary

Soft theorem in field theory

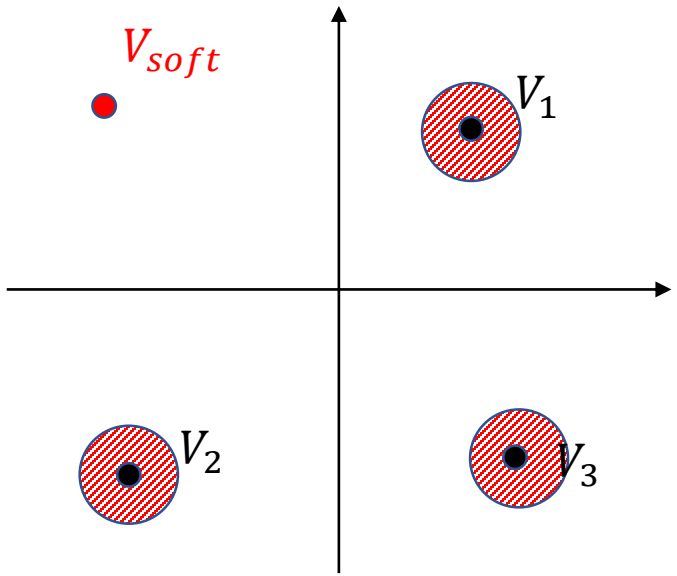


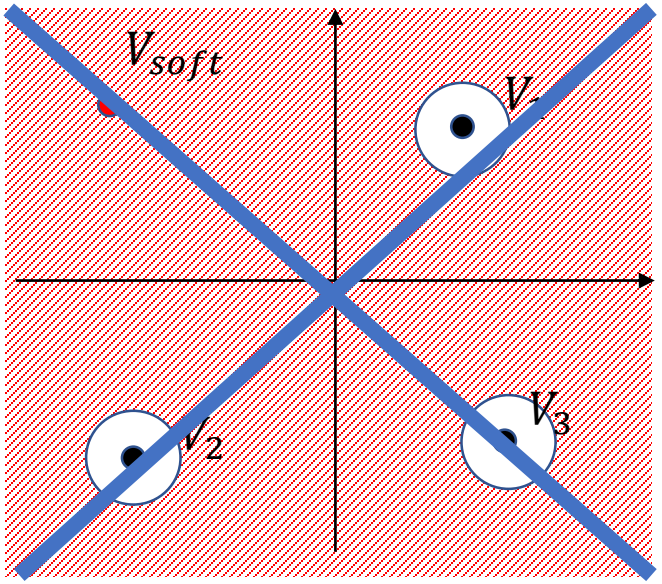
Soft photon theorem

$$M_{n+1}(q; p_1, \cdots, p_n) = \sum_{i=1}^{n} \frac{e_i}{p_i \cdot q} [p_i^\mu - i q_\nu J_i^{\mu\nu}] M_n(p_1, \cdots, p_i, \cdots, p_n) + \mathcal{O}(q\ )$$

Why does the sum of the diagrams look as if the operator act on external particles ?

Soft theorem in string theory





Universality of soft theorem = Superlocality of soft vertex

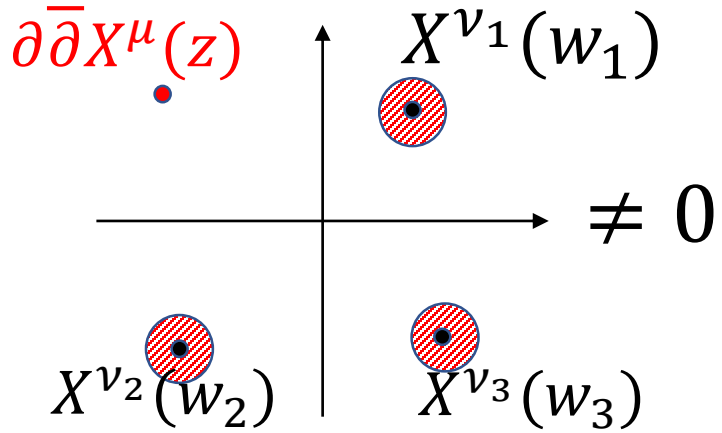
Superlocal operator

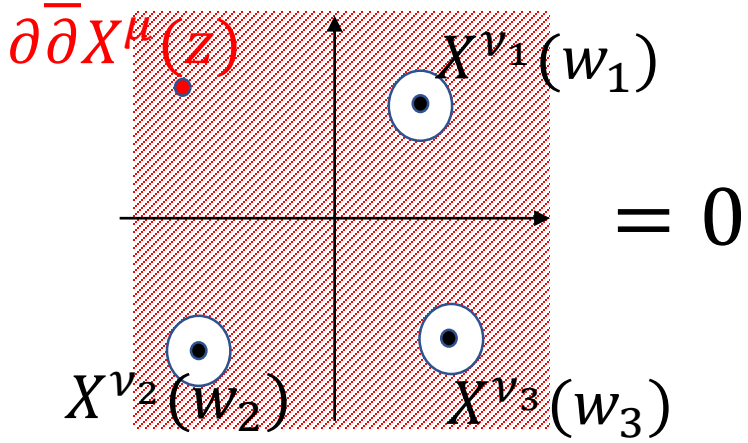
A superlocal operator $O(z)$ is defined as

$$\int d^2z\langle O(z)\phi_1(w_1)\cdots\phi_n(w_n)\rangle = \sum_a\sum_{i=1}^{n} c_a^i\langle O_a^i(w_i)\phi_1(w_1)\cdots\check{\phi}_i(w_i)\cdots\phi_n(w_n)\rangle.$$

Ex. $\partial\bar{\partial}X(z,\bar{z})$ in bosonic closed string

$$\int d^2z\langle\partial\bar{\partial}X^\mu(z)X^{\nu_1}(w_1)\cdots X^{\nu_n}(w_n)\rangle = -\pi\alpha'\sum_{i=1}^{n}\eta^{\mu\nu_i}\langle X^{\nu_1}(w_1)\cdots\check{X}^{\nu_i}(w_i)\cdots X^{\nu_n}(w_n)\rangle.$$





Superlocality of soft graviton

Deform soft graviton by partial integral

$$
\begin{aligned}
&: \partial X^\mu(z)\bar{\partial}X^\nu \exp\left(iq\cdot X(z,\bar{z})\right) : \\
&= \partial : X^\mu(z,\bar{z})\bar{\partial}X^\nu(\bar{z}) \exp\left(iq\cdot X(z,\bar{z})\right) : - : X^\mu(z,\bar{z})\bar{\partial}X^\nu(z,\bar{z})\partial_z \exp\left(iq\cdot X(z,\bar{z})\right) : \\
&= \partial : X^\mu(z,\bar{z})\bar{\partial}X^\nu(\bar{z}) \exp\left(iq\cdot X(z,\bar{z})\right) : -\frac{\bar{\partial}}{2} : X^\mu(z,\bar{z})X^\nu(z,\bar{z})\partial_z \exp\left(iq\cdot X(z,\bar{z})\right) : \\
&+ \frac{1}{2} : X^\mu(z,\bar{z})X^\nu(z,\bar{z})\partial_z\bar{\partial}_z \exp\left(iq\cdot X(z,\bar{z})\right) :
\end{aligned}
$$



Drop total derivatives

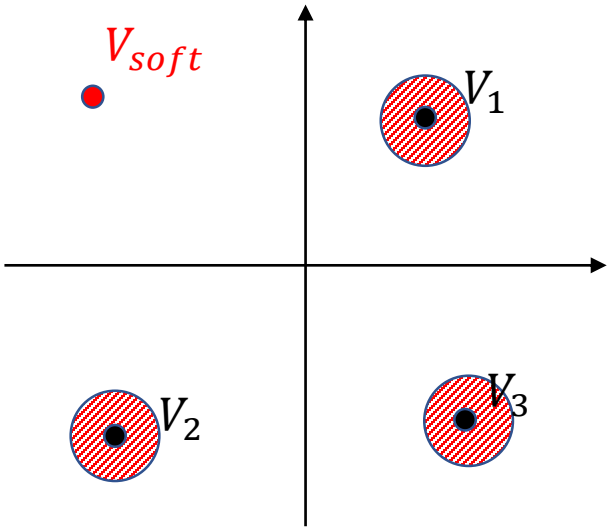
$$
\begin{aligned}
&\int d^2z \langle : h_{\mu\nu}\partial X^\mu(z)\bar{\partial}X^\nu \exp\left(iq\cdot X(z,\bar{z})\right) : \cdots \rangle \\
&= \int d^2z \langle \frac{1}{2} : h_{\mu\nu}X^\mu(z,\bar{z})X^\nu(z,\bar{z})\partial_z\bar{\partial}_z \exp\left(iq\cdot X(z,\bar{z})\right) : \cdots \rangle .
\end{aligned}
$$

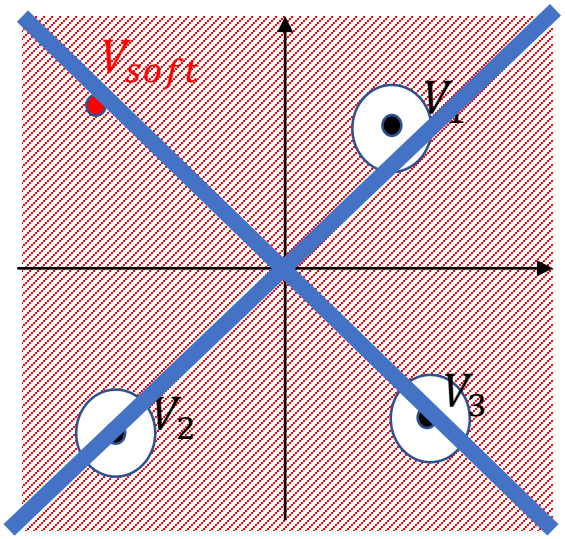
Superlocality of soft graviton

$$\int d^2z\langle \frac{1}{2} : h_{\mu\nu}X^\mu(z,\bar{z})X^\nu(z,\bar{z})\partial_z\bar{\partial}_z \exp\left(iq\cdot X(z,\bar{z})\right) : \cdots \rangle.$$

This is superlocal through the linear order in q, because on the bulk region

$$\partial\bar{\partial}\exp(iq\cdot X(z)) = \partial\bar{\partial}\left(1+iq\cdot X(z)+\mathcal{O}(q^2)\right) \sim \mathcal{O}(q^2).$$





$\sim \mathcal{O}(q^2)$

Soft graviton theorem

All that we need to do is evaluating the integral over the disk.

$$\int d^2z : V_{soft}(z) :: V_{hard}(w) :$$

$$= \int d^2z : h_\mu h_\nu X^\mu X^\nu \partial_z \bar{\partial}_z \mathrm{e}^{iq\cdot X}(z) :: O\mathrm{e}^{ip\cdot X}(w) :$$

$$= \int d^2z : h_\mu h_\nu \left(X^\mu(z) - i\alpha' p^\mu \ln|z-w|\right)\left(X^\nu(z) - i\alpha' p^\nu \ln|z-w|\right)$$

$$\times \partial_z \bar{\partial}_z \left(\mathrm{e}^{iq\cdot X(z)+ip\cdot X(w)}|z-w|^{\alpha' p\cdot q}\right) :: O(w) :$$

Focus on only $|z-w|^{\alpha' p\cdot q-2}$ terms.

Soft graviton theorem

$$\int d^2z : V_{soft}(z) :: V_{hard}(w) :$$

$$\sim -\frac{4\pi\alpha'(h\cdot p)^2}{p\cdot q} : \exp\left(i\frac{p\cdot q}{h\cdot p}h\cdot X(z) + iq\cdot X(z) + ip\cdot X(w)\right)$$

$$\times\left(1+\frac{z-w}{\alpha' p\cdot q}iq\cdot\partial X(z)\right)\left(1+\frac{\bar{z}-\bar{w}}{\alpha' p\cdot q}iq\cdot\bar{\partial}X(\bar{z})\right) :: O(w) :\Big|_{|z-w|^0}$$

$$\sim -\frac{4\pi\alpha'(h\cdot p)^2}{p\cdot q} : e^{ia\cdot X(z)+ip\cdot X(w)}(1+(z-w)b\cdot\partial X(z))\left(1+(\bar{z}-\bar{w})b\cdot\bar{\partial}X(\bar{z})\right) :$$

$$\times : O(w) :|_{|z-w|^0} .$$

where $a^\mu = q^\mu - \frac{p\cdot q}{h\cdot p}h^\mu, \quad b^\mu = \frac{iq^\mu}{\alpha' p\cdot q}.$

The 0-th, 1st, 2nd order terms in a^μ correspond to the leading, subleading, subsubleading.

Soft graviton theorem

We can obtain compact form of soft graviton theorem for any hard vertex.

$$\int d^2z : V_{soft} :: V_{hard}(w) :$$

$$\to \frac{2\pi\alpha'}{p\cdot q}\left[-2(h\cdot p)(h\cdot p) - 2i(h\cdot p)h_\mu q_a J^{\mu a}\right.$$
$$\left. + q_a q_b h_\mu h_\nu \left(L^{\mu a}L^{\nu b} + S^{\mu a}S^{\bar{\nu} b}\right) + Q + \bar{Q}\right] |\text{hard massive state}; k\rangle + \mathcal{O}(q^2)$$

where
$$Q \equiv \left(\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{(a\cdot\alpha_k)}{kz^k}\right)^2 \left(1+\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{b\cdot\alpha_k}{iz^k}\right)\Bigg|_{z^0},$$

$$\bar{Q} \equiv \left(\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{(a\cdot\alpha_k)}{k\bar{z}^k}\right)^2 \left(1+\sqrt{\frac{\alpha'}{2}}\sum_{k\neq 0}\frac{b\cdot\alpha_k}{i\bar{z}^k}\right)\Bigg|_{\bar{z}^0}.$$

Summary

- After some partial integration, we can find that soft vertex is superlocal at a certain order in q.
- As a result, the scattering amplitude can be evaluated from only the disk region, which leads to the universality of the soft theorem.
- We can derive the compact form of the soft graviton theorem for any hard vertex.