File 3.

Emergence of Chern-Simons theory in CMT.

(and "statistical transmutation").

Gil Young Cho.

In this part of the lecture, we will derive the Chern-Simons theory from a microscopic theory:

↑ electrons in a fractional quantum Hall effects.

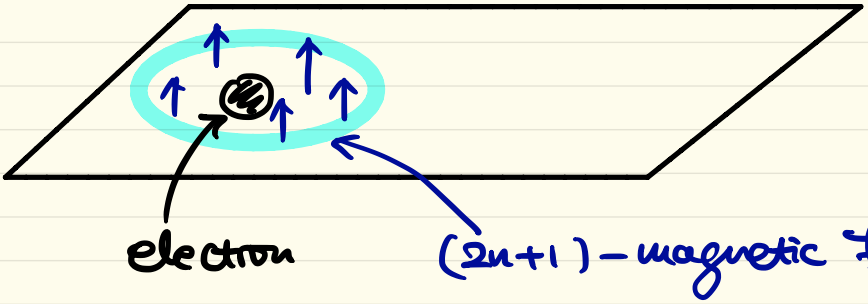
the effective theory that we have extensively studied.

&

we'll provide heuristic pictures of "fractionalizations". (including other examples).

Again, we focus on $\nu = 1/(2n+1)$, $n \in \mathbb{Z}$.

in which...



Then, the theory for "electrons" is ...

$$\mathcal{L} = \int_r \Phi^\dagger (i\partial_t - A_0)\Phi$$

$$- \int_r \frac{1}{2m} \{[-i\vec{\nabla} + \vec{A}_0]\Phi\}^\dagger \cdot \{[-i\vec{\nabla} + \vec{A}_0]\Phi\}$$

$$- \int_{r,r'} V(|r-r'|)(\Phi^\dagger\Phi)(r)\cdot(\Phi^\dagger\Phi)(r')\frac{1}{2}.$$

magnetic fluxes

electron's kinetic energy

interaction btwn the electrons.

We will apply slightly more electric field & magnetic field to measure the conductivities,

i.e., $\vec{A} = \vec{A}_0 + \delta\vec{A}$

$\vec{A}_0$: generating magnetic flux.

$\vec{A}$: full electromagnetic field.

$\delta\vec{A}$: small external field. for "probing" the system.

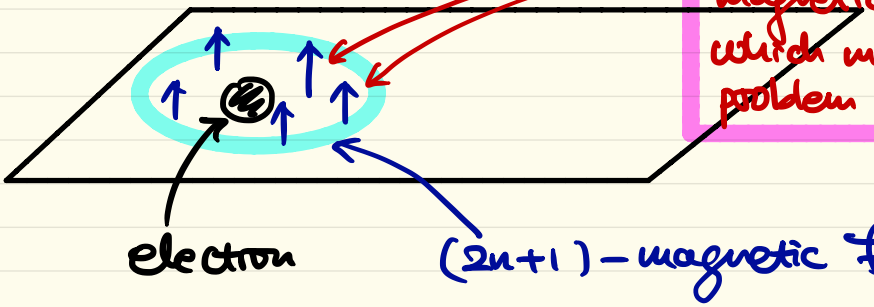
Thus:

$$\mathcal{L} = \int_r \left\{ \Phi^\dagger (i\partial_t - A_0)\Phi - \frac{1}{2m}\left\{(-i\vec{\nabla} + \vec{A})\Phi\right\}^\dagger \cdot \left\{(-i\vec{\nabla} + \vec{A})\Phi\right\} \right\}$$

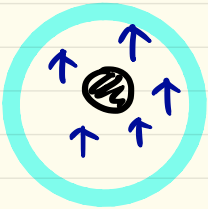
+ interactions.

let's ignore this for a moment (for the clarity?).

From the physical picture...



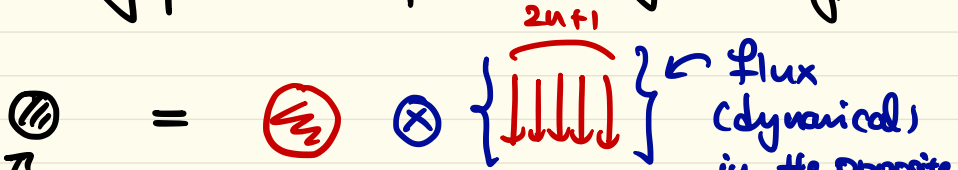
$\Rightarrow$ its tempting to write the theory of



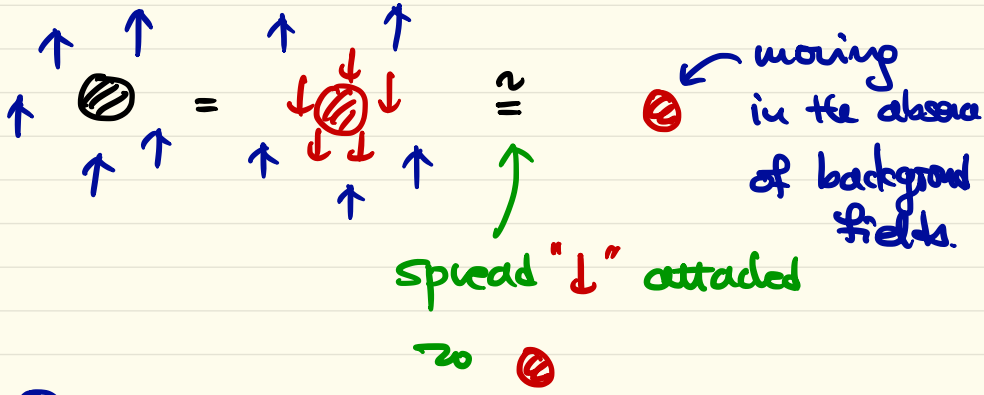
$\equiv$ as a "composite" (w/ internal structure).

However, "magnetic flux" is "external" which is "given", & so we can not attach it simply to electrons.

We by-pass this problem by writing



then



① Q: what is this new particle?
⟹ We will see this is a boson

② How do we introduce the "artificial" magnetic flux "↓" attached to the ?

"statistical transmutation"

Let's start from ②, which we know how to do:

$\Longrightarrow$ we can introduce the fluxes & attach them to the particle by Chern-Simons term.

Consider:

Φ: new particle

$$\mathcal{L} = \int_r \Phi^\dagger (i\partial_t + a_0 - A_0)\Phi - \frac{1}{2m}\{(-i\vec{\nabla} + \vec{a} - \vec{A})\Phi\}^\dagger \cdot \{(-i\vec{\nabla} + \vec{a} - \vec{A})\Phi\} - \frac{1}{4\pi \cdot k}\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

then,

$$\frac{\delta}{\delta a_0}\mathcal{L} = 0$$

$$\Longrightarrow \Phi^\dagger\Phi = \frac{1}{2\pi}\cdot\frac{1}{k}\varepsilon^{ij}\partial_i a_j$$

i.e., $\varepsilon^{ij} \partial_i a_j = 2\pi k \hat{N}_\Phi$.

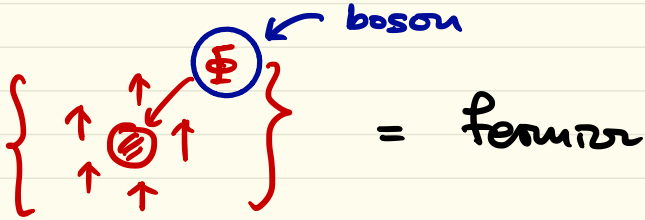
↑ # of ◎

artificial "magnetic" flux

$\Rightarrow$ which is what we hoped for!

Now, we confirm Φ ~ boson.,

for this, we show instead..



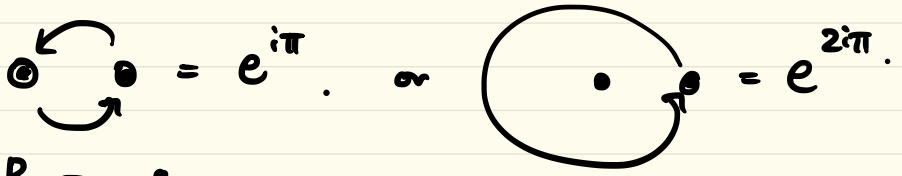
= fermion

Remember that,

{exchange} = $\frac{1}{2}$ {full braiding}

("monodromy" phase).

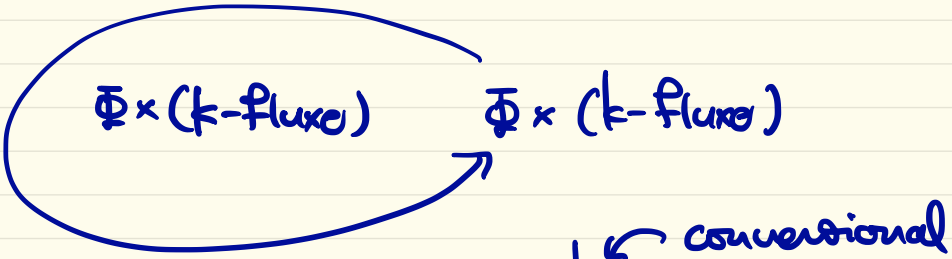
i.e., Fermion :



i.e., Boson :



Now,



$= e^{2i\theta_{ff}} \cdot e^{\pi i k}$ ← conventional AB-phase.

↑ self-statistics

$\Rightarrow$ implying,

$\{\text{exchange}\} = e^{i\theta_{ff}} e^{i\pi k}$.

$\Rightarrow$ We want "fermion", $k \in 2u+1$.

i.e., $\theta_{ff} = 0$.

Thus, we see that $\theta_{\Psi\Psi} = 0$,

implying that we have a boson Ψ.

* Statistics is "Transmutated"

from "fermion" to "boson".

Next, we want to have

an Approximation such that

the attached fluxes are spreaded.

to cancel the background fluxes,

i.e., $a_\mu = \bar{a}_\mu + \delta a_\mu$, ← "fluctuations."

such that $-\bar{a}_\mu + A_{\text{ext}\,\mu} = 0$

↑ generating the background magnetic fields.

leading to...

$$\mathcal{L} = \int_r \Phi^\dagger (i\partial_t + \delta a_0 - \delta A_0)\Phi - \frac{1}{2m}\left[(-i\vec{\nabla} + \delta\vec{a} - \delta\vec{A})\Phi\right]^\dagger \left[(-i\vec{\nabla} + \delta\vec{a} - \delta\vec{A})\Phi\right] - \frac{1}{4\pi\cdot k}\,\varepsilon^{\mu\nu\lambda}\,\delta a_\mu \partial_\nu \delta a_\lambda .$$

Here we performed..

$$\frac{1}{4\pi k}\varepsilon^{\mu\nu\lambda}(\bar{a}_\mu + \delta a_\mu)\partial_\nu(\bar{a}_\lambda + \delta a_\lambda)$$

$$= \frac{1}{4\pi k}\varepsilon^{\mu\nu\lambda}\bar{a}_\mu\partial_\nu\bar{a}_\lambda + \frac{2}{4\pi k}\varepsilon^{\mu\nu\lambda}\bar{a}_\mu\partial_\nu\delta a_\lambda + \frac{1}{4\pi k}\varepsilon^{\mu\nu\lambda}\delta a_\mu\partial_\nu\delta a_\lambda ,$$

← just a number. ignored.

only $+\delta a_0 \cdot \frac{1}{2\pi k}\varepsilon^{ij}\partial_i\bar{a}_j$ ← average particle number. ; we keep this.

In total:

$$\mathcal{L} = \int_r \Big\{ \Phi^\dagger (i\partial_t - A_0)\Phi - \frac{1}{2m}\Big\{(-i\vec{\nabla} + \vec{A})\Phi\Big\}^\dagger \cdot \Big\{(-i\vec{\nabla} + \vec{A})\Phi\Big\}\Big\}$$

+ interactions.

"Fermion" under the uniform magnetic fields

$\Downarrow$

"Boson" w/o background field but interacting w/ artificial gauge field

$$\mathcal{L} = \int_r \Phi^\dagger (i\partial_t + \delta a_0 - \delta A_0)\Phi - \frac{1}{2m}\Big[(-i\vec{\nabla} + \delta\vec{a} - \delta\vec{A})\Phi\Big]^\dagger \Big[(-i\vec{\nabla} + \delta\vec{a} - \delta\vec{A})\Phi\Big] - \frac{1}{4\pi \cdot k}\,\varepsilon^{\mu\nu\lambda}\,\delta a_\mu \partial_\nu \delta a_\lambda .$$

What do we know of bosons at the very low-T ? (w/ high densities).

$\Longrightarrow$ We know that they undergo the **Bose-Einstein condensation**.

e.g., superconductor..
superfluid ...
etc.

What is the **BEC**?

$\Rightarrow$ Bosons become coherent.

$$\langle \Phi \rangle = \sqrt{\bar{\rho} + \delta\rho}\, e^{i\theta}$$

θ ← coherent phase.

$\bar{\rho}$ ← average density

$\delta\rho$ ← modulation in density

Plug,

$$\mathcal{L} = \underbrace{(\bar\rho + \delta\rho)(+\delta a_0 - \delta A_0) + \frac{1}{2m}\bar\rho\,(\vec\nabla\theta - \delta\vec a + \delta\vec A)^2}_{\text{no gradient}} + \underbrace{(\cdots + \bar\rho\,\partial_t\theta + \cdots)}_{\text{(w/gradient)}}$$

$$+ \underbrace{(\cdots + \bar\rho\,\vec\nabla\delta\rho\cdot\delta\vec a + \cdots)}_{\text{involving } \nabla\delta\rho}$$

"Not important"

$$- \frac{1}{4\pi\cdot k}\,\varepsilon^{\mu\nu\lambda}\,\delta a_\mu \partial_\nu \delta a_\lambda$$

$$- \delta a_0 \cdot \bar\rho \quad \leftarrow \text{from the expansion}$$

$$= \underbrace{-\bar\rho\,\delta A_0}$$

$$+ (\delta a_0 - \delta A_0)\,\delta\rho + \frac{\bar\rho}{m}(\delta\vec A - \delta\vec a)\cdot\nabla\theta$$

if you look at this...

$$J^\mu \equiv (\delta\rho, -\tfrac{\bar\rho}{m}\nabla\theta)$$

and

$$\equiv J^\mu(\delta a_\mu - \delta A_\mu)$$

$$- \frac{1}{4\pi k}\,\varepsilon^{\mu\nu\lambda}\,\delta a_\mu \partial_\nu \delta a_\lambda + (\cdots \text{Higher gradients} \cdots).$$

rewriting

$$\mathcal{L} = -\delta A_0 \cdot \bar{\rho}$$

$$+ \; J^\mu \delta a_\mu - \frac{1}{4\pi k} \varepsilon^{\mu\nu\lambda} \delta a_\mu \partial_\nu \delta a_\lambda .$$

$$- J^\mu \delta A_\mu \; (+ \cdots) \leftarrow \text{not important.}$$

next, using $J^\mu = \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \partial_\nu b_\lambda$.

$$= -\delta A_0 \bar{\rho} - \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} \delta A_\mu \partial_\nu b_\lambda$$

$$+ \frac{1}{2\pi} \varepsilon^{\mu\nu\lambda} b_\mu \partial_\nu \delta a_\lambda$$

$$- \frac{1}{4\pi k} \varepsilon^{\mu\nu\lambda} \delta a_\mu \partial_\nu \delta a_\lambda .$$

and then solve the eqn. of motion for δa_μ, we obtain..

$$\mathcal{L} = -\delta A_0 \bar{\rho} + \frac{k}{4\pi}\varepsilon^{\mu\nu\lambda} b_\mu \partial_\nu b_\lambda$$

"voltage drop"

$$- \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda} b_\mu \partial_\nu \delta A_\lambda .$$

$\Longrightarrow$ completing the derivation!

Next, we can ask what's the physical meaning of $j^\mu b_\mu$

(j^μ: current of b_μ.)

imagine that

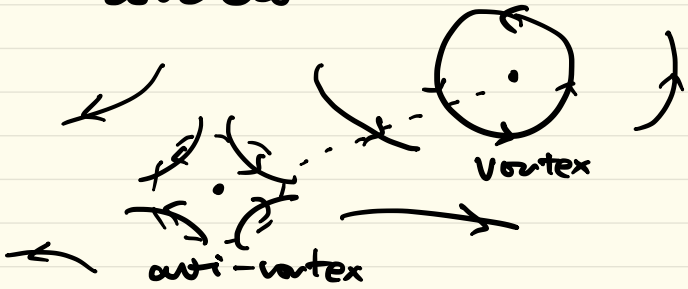
$$\langle \Phi \rangle = \sqrt{\bar{\rho} + \delta\rho}\, e^{i\theta},$$

this becomes "j^μ"

$\theta = \theta + \theta_{vm}$ ← in space,

↑ coherent

then,



Remarks

1. This approach is known as the composite boson theory

$\Longrightarrow$ we can also use the so-called composite fermion theory.

Here we attach only the even number of fluxes to the electron

Remarkably, whichever the approach is used, the result remains the same.

2. Appearance of Chern-Simons theory is somewhat universal.

$\Rightarrow$ When you see a particle, which is neither boson nor fermion, it is very likely you can describe them by Chern-Simons theory.

e.g., $\mathbb{Z}_2$ spin liquid.

Additional Reference:

Aspects of Chern–Simons theory, GV Dunnes, arxiv: 9902115

Shou-Cheng Zhang, The Chern–Simons–Landau–Ginzburg theory for the fractional quantum Hall effect, Int. J. Mod. Phys. B 1992