Topics on Integrability and N=4 SYM

Shota Komatsu

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Topics on Integrability & $\mathcal{N}=4$ SYM

I. Intro to Integrability ($O(N)$ model)

II. From $\mathcal{N}=4$ SYM to Spin Chain

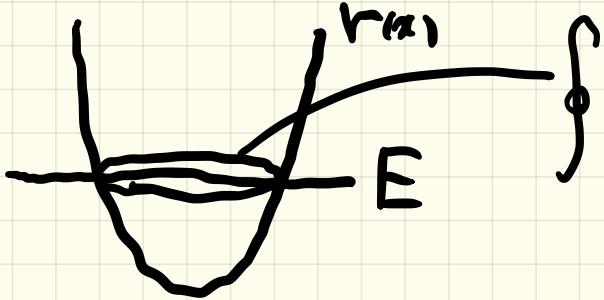
III. ODE/IM correspondence

Lecture IV : "Bootstrapping" Quantum Mechanics

- 1811.04812 [Ito, Maruto, Shu]
- 1002.2459 [Alday etal]

1. Motivation

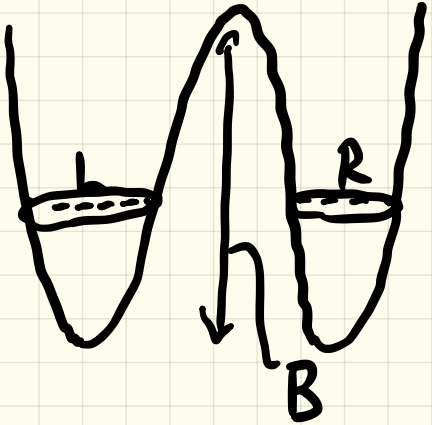
A harmonic oscillator



$$\oint \sqrt{E - V(x)} = \left(n + \frac{1}{2}\right) \hbar$$

↑ Bohr-Sommerfeld Q.C.

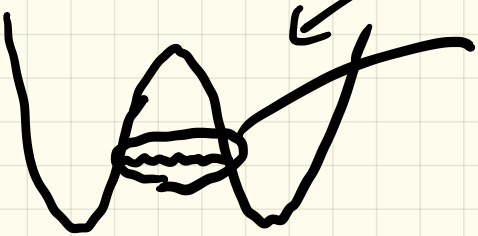
Double-well



$\underline{B \gg 1}$ ⇝ two decoupled h.o.

$$\oint_{L,R} \sqrt{E - V(x)}\, dx = \left(n_{L,R} + \frac{1}{2}\right) h$$

Finite B: Tunneling ⇝ degeneracy lifted



"instanton cycle"

$$\sim e^{-\oint \sqrt{V - E}\, dx}$$

⇝ "Interactions" among cycles.

* Resurgence



WKB exp. → non-Borel summable

instanton ← cycle

Goal: Have a control over interactions among cycles

Method: Analytic properties in Complex x & $\hbar$

2. From Schrödinger to functional rel.

Step 1. Rewriting Schrödinger eq.

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 + V(x) - E\right]\psi(x) = 0$$

{ Rescaling x — polynomial

$$\left[-\zeta^2 \partial_x^2 + \boxed{x^{r+1} + \sum_{k=0}^{r} b_k x^k}\right]\psi(x) = 0$$

ζ: new $\hbar$

$V_{new}(x)$

We further assume

1. r : odd



2. All zeros of $V(x)$ → real axis
(technical) (reality condition)



$r+1=6$

Step 2. solution @ $|x| \to \infty$

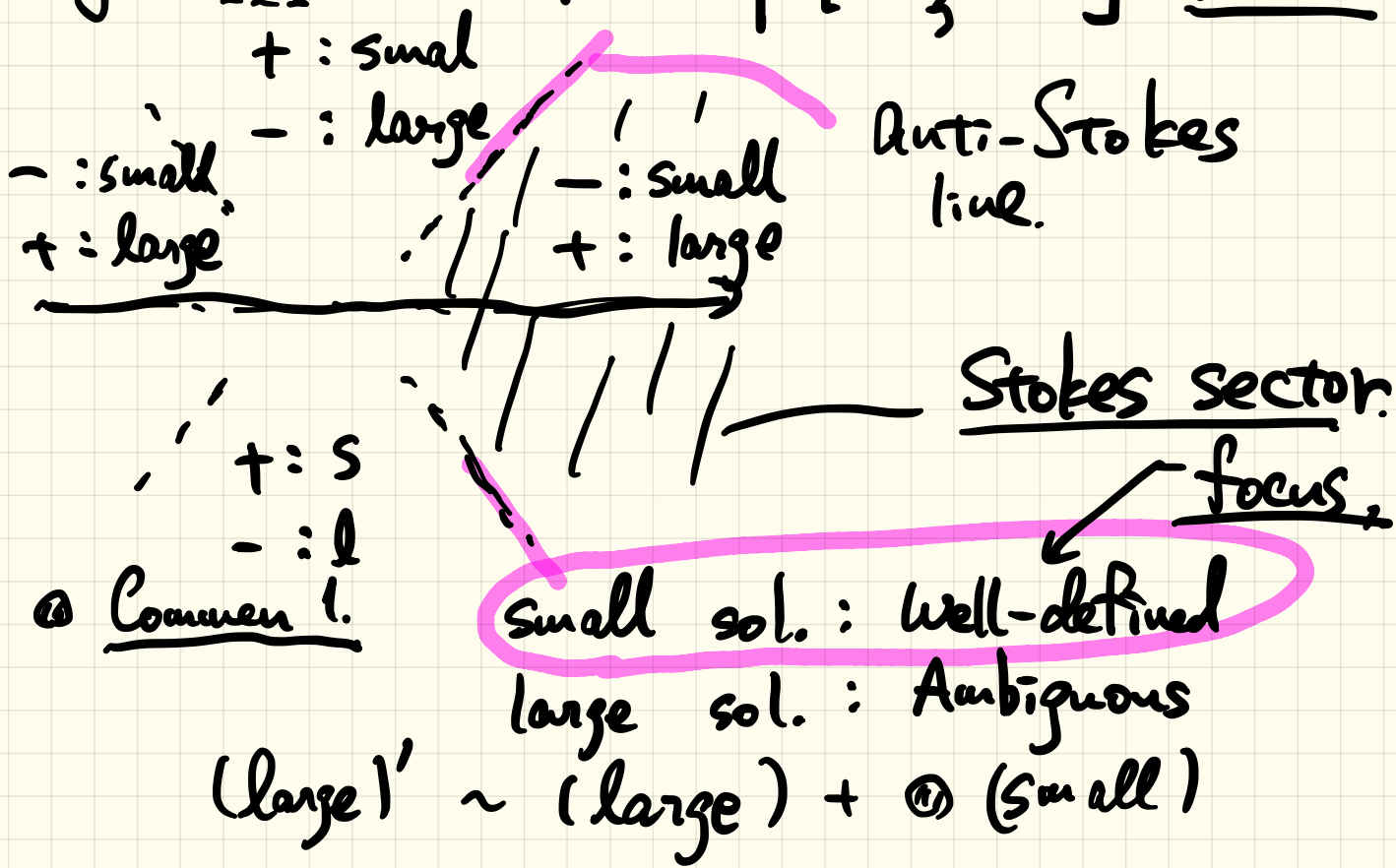
$|x| \to \infty \Rightarrow \left[-\zeta^2 \partial_x^2 + x^{r+1} + \underbrace{\cdots}_{\text{subleading}}\right] \psi(x) = 0$

$$\psi(x) \sim \exp\left[\begin{matrix}+\\-\end{matrix} \frac{1}{\zeta} \frac{2}{r+3} x^{\frac{r+3}{2}}\right]$$

2 different sol.

2 sol: exponentially $\begin{cases}\text{large}\\ \text{small}\end{cases}$

e.g. $r=1$ $\quad \psi \sim \exp\left[\pm \frac{1}{\zeta}\frac{x^2}{2}\right]$ $\quad \zeta$: real



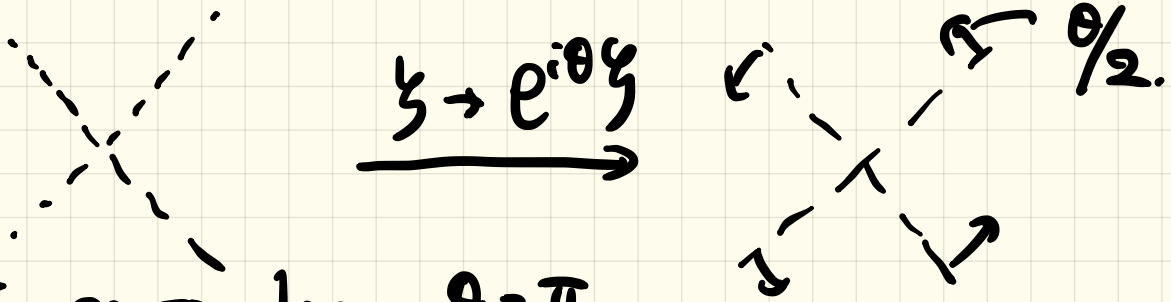
⊛ Comment.

$(\text{large})' \sim (\text{large}) + ⊛\,(\text{small})$

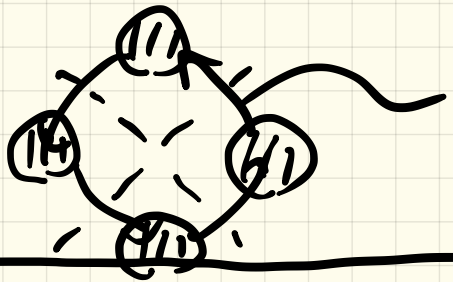
② Comment 2 : $\zeta \to e^{i\theta}\zeta$.

$\Rightarrow$ Compensated by $x \to e^{i\theta/2} x$

$\left[\psi \sim \exp\left[\pm \frac{1}{\zeta} \frac{x^2}{2} \right] \right.$



In particular, $\theta = \pi$



$\zeta \to e^{i\pi}\zeta$

• More generally, $\zeta \to e^{i\pi}\zeta$ rotates sectors by 1 unit

ζ: real



$$S_k(e^{i\pi}\zeta) \propto S_{k+1}(\zeta)$$

$$\Rightarrow \underline{S_0(e^{i\pi k}\zeta) \propto S_k(\zeta)}$$

Define. $S_k(\zeta) := S_0(e^{i\pi k}\zeta)$

Quantization Condition

Energy eigenstate on $x \in \mathbb{R}$

$\rightarrow$ ψ must decay both $x \to \pm\infty$

$\rightarrow$ $\boxed{S_0(\zeta) \otimes S_{\frac{\nu+3}{2}}(\zeta)}$

($V(x) = x^{\nu+4} + \sum_{k=0}^{\nu+3} b_k x^k$, $b_0 \leftarrow$ Energy

$\rightsquigarrow$ Q.C. determines b_0)

Step 3. Wronskian.

For $\left[-\hbar^2 \partial_x^2 + V(x)\right] \psi_{1,2} = 0$

Wronskian:

$$\langle \psi_1, \psi_2 \rangle \equiv \psi_1 \partial_x \psi_2 - \psi_2 \partial_x \psi_1 \qquad \hat{\psi} = \begin{pmatrix} \psi \\ \partial_x \psi \end{pmatrix}$$

$$= \epsilon_{ab} \hat{\psi}_1^a \hat{\psi}_2^b$$

- $\langle \psi_1, \psi_2 \rangle(\hbar)$ ← independent of x
- $\langle \psi, \psi \rangle = 0$

Q.C.: $\langle s_0, s_{\frac{n-3}{2}} \rangle(y) = 0$.

Step 4. Functional Eq.

Identity: $\epsilon_{ab}\epsilon_{cd} + \epsilon_{ad}\epsilon_{bc} + \epsilon_{ac}\epsilon_{db} = 0$

$\rightsquigarrow$ $\langle s_i, s_j \rangle \langle s_k, s_l \rangle + \langle s_i, s_l \rangle \langle s_j, s_k \rangle + \langle s_i, s_k \rangle \langle s_l, s_j \rangle = 0$.

$\leftarrow$ Plücker rel.

Def. Υ-function

- $\Upsilon_{2j}(\zeta) = \dfrac{\langle S_{-j}, S_j\rangle \langle S_{-j-1}, S_{j+1}\rangle}{\langle S_{-j-1}, S_{-j}\rangle \langle S_j, S_{j+1}\rangle}(\zeta)$
- $\Upsilon_{2j+1}(\zeta) = \dfrac{\langle S_{-j-1}, S_j\rangle \langle S_{-j-2}, S_{j+1}\rangle}{\langle S_{-j-2}, S_{-j-1}\rangle \langle S_j, S_{j+1}\rangle}$

$(e^{i\pi/2}\zeta)$

Thm. $Y_s^{[+]} Y_s^{[-]} = (1 + Y_{s+1})(1 + Y_{s-1})$

$$f^{[\pm]} := f(e^{\pm i\pi/2}\zeta)$$

Proof. Plücker rel.

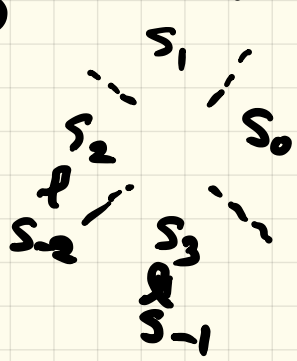
Proof $(r=1)$

$$Y_0 = \frac{\overset{0}{\overset{\|}{\langle S_{-0}, S_0\rangle}} \overset{0}{=} \cdots}{\cdots \qquad \cdots} = 0$$

$$Y_1 = \frac{\langle S_{-1}, S_0\rangle \langle S_{-2}, S_1\rangle}{\langle S_{-2}, S_{-1} \times S_0, S_1\rangle} \quad \left(\ell^{i\pi/2}\,\zeta\right)$$

$$Y_2 = \frac{\langle S_{-1}, S_1\rangle \langle S_{-2}, S_2\rangle \overset{0}{\overset{\|}{\cdots}}}{\cdots\cdots}$$

$$= 0$$



$$\gamma_1^{[+]}\,\gamma_1^{[-]} = 1.$$

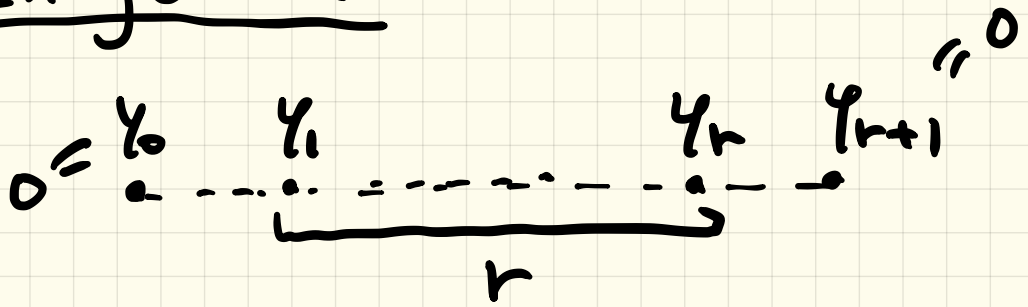
$$s_2 = \alpha\, s_{-2}$$

$e^{i\pi}$ $\quad -\alpha$

$$\Leftrightarrow \frac{\langle s_{+1}, s_0\rangle \langle s_{-2}, s_1\rangle}{\langle s_{-2}, s_{-1}\rangle \langle s_0, s_{+2}\rangle} \frac{\langle s_0, s_1\rangle \langle s_{-1}, s_{+2}\rangle}{\langle s_{-1}, s_0\rangle \langle s_1, s_2\rangle} \quad (-\alpha)$$

$$= 1$$

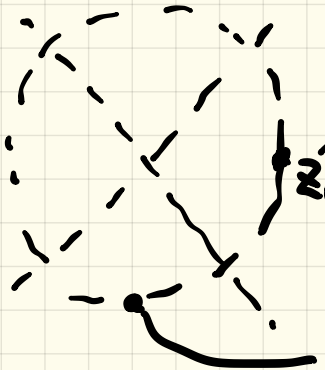
In general



⑪ Physical Meaning of Y-function

A. WKB approx

r=1 $\quad Y_1^{[-]} = \dfrac{\langle S_{-1}, S_0\rangle \langle S_{-2}, S_1\rangle}{\langle S_{-2}, S_{-1}\rangle \langle S_0, S_1\rangle} \longrightarrow$ WKB



$S_0 \sim \exp\left[-\frac{1}{\hbar}\int_{z_0}^{z}\sqrt{V(x')}\,dx'\right]$

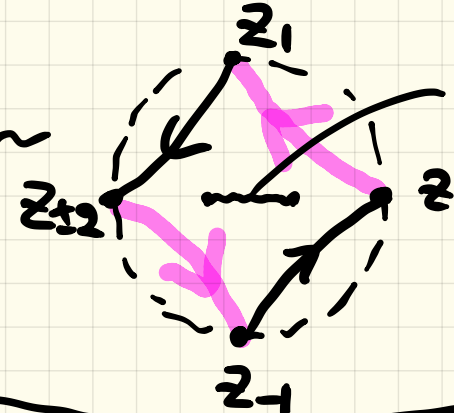
$|z_i| \gg 1$

"$\frac{1}{x^2-E}$"

$S_{-1} \sim \exp\left[+\frac{1}{\hbar}\int_{z_1}^{z}\sqrt{V(x')}\,dx'\right]$

$$\langle S_{-1}, S_0 \rangle \sim \exp\left[+\frac{1}{\hbar}\int_{z_{-1}}^{z_0} \frac{\sqrt{r(x')}\,dx'}{x^2-E.}\right]$$

$Y_1^{[-]}$ $\sim$



$\sim \exp\left[+\frac{1}{\hbar}\oint_\gamma \sqrt{r(x')}\,dx'\right]$

$\leftarrow r = x^2 - E$

$$\exp\left[-\frac{1}{\hbar}\oint_\gamma \sqrt{E-(x')^2}\,dx'\right]$$

B.S. period.

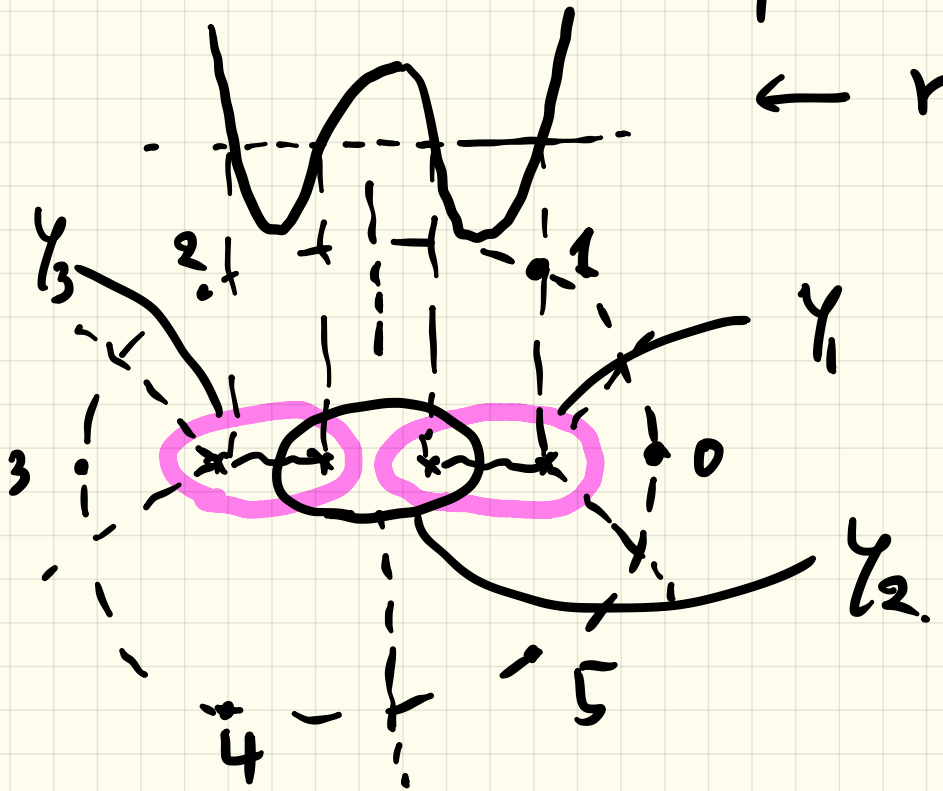
$Y_1 = \exp\left[-\frac{1}{\hbar}\oint_{\gamma_1} \cdots\right]$

positive.

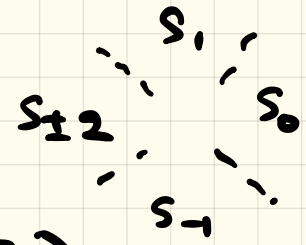
γ's $\rightsquigarrow$ Non perturbative / Quantum

B.S. period.

$\leftarrow r = 3$



B. Q.C.



@ $r=1$

$$\gamma_1^{[-]} = \frac{\langle s_{-1}, s_0 \rangle \langle s_2, s_1 \rangle}{\langle s_2, s_{-1} \rangle \langle s_0, s_1 \rangle}$$

(s_2 over $\langle s_{-1}, s_0 \rangle$, s_2 over $\langle s_2, s_1 \rangle$, s_2 over $\langle s_2, s_{-1} \rangle$, s_1 over $\langle s_0, s_1 \rangle$)

Q.C. : $s_{-2} \overset{s_2}{\prec} s_0 \overset{s_1}{\succ} s_{+2}$

$$\rightarrow \boxed{\gamma_1^{[-]} = -1} \Leftrightarrow 1 + \gamma_1^{[-]} = 0$$

④ $r = 3$ (double-well)



$$\frac{(1+\gamma_1)^{[-]}\,(1+\gamma_3)^{[+]}}{\gamma_2} = \frac{\langle S_{-1}, S_3\rangle \langle S_0, S_2\rangle}{\langle S_{-1}, S_0\rangle \langle S_2, S_3\rangle}$$

Q.C. $S_0 \leftrightarrow S_3$

Plücker

$= -1$

Q.C. $Y_2 + (1+Y_1)^{[-]}(1+Y_3)^{[-]} = 0$

Y_2 ↓ instanton

$e^{-\frac{1}{\hbar}\cdots}$



* I derived this this morning.

3. Solving func. eq. [Y-system

$$Y_s^{[+]}\, Y_s^{[-]} = (1+Y_{s-1})(1+Y_{s+1})$$

$$\log Y_s^{[+]} + \log Y_s^{[-]} = \log(1+Y_{s-1}) + \log[1+Y_{s+1}]$$

$$Y_s \sim \exp\left[-\frac{1}{\zeta} M_s\right]$$

B.S. period >0

- $l_s(\zeta) := \log Y_s(\zeta) + \frac{1}{\zeta} M_s$

$$\therefore \underbrace{l_s^{[+]}}_{+i} + \underbrace{l_s^{[-]}}_{-i} = \log(1+\gamma_{s-1}) + \log(1+\gamma_{s+1})$$

$\xi = e^{-\theta}$ θ: rapidity.

- $K(\theta) = \dfrac{1}{2\pi \cosh\theta}$ ← Kernel

& Consider convolution

$$K * (l_s^{[+]} + l_s^{[-]})$$

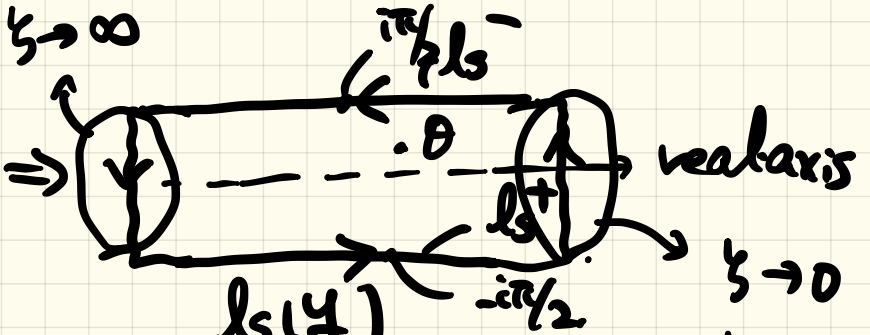
$$K_s * \left(l_s^{[+]} + l_s^{[-]}\right)$$

$$= \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{l_s\left(y+\frac{i\pi}{2}\right) + l_s\left(y-\frac{i\pi}{2}\right)}{\cosh(\theta - y)}$$

θ

y

$$l_s^+ + l_s^-$$



$$= \int_{\gamma} \frac{dy}{2\pi i} \frac{l_s(y)}{\sinh(\theta - y)}$$

$$= \boxed{l_s(\theta)}$$

- I assumed that $\ell s(y)$ doesn't have poles inside the strip.
  ← · reality condition.
  └ $\xi \to 0$. You can show it by WKB.

$$l_s = K * (l_s^+ + l_s^-)$$

$$= K * \left[\log(1+\gamma_{s-1}) + \log(1+\gamma_{s+1}) \right]$$

$$\log \gamma_s = -M_s e^{+\theta} + \int_{-\infty}^{\infty} \frac{dy}{2\pi} \frac{\log(1+\gamma_{s-1}) + \log(\cdots)}{\cosh(\theta - y)}$$

"TBA eq."

WKB $\frac{M_s}{\lambda}$

$\gamma \sim \exp\left[-\frac{\omega}{\lambda}\right]$

$r=1$, Harmonic Osc.

$$\log Y_1 = -M_S \, e^{+\theta}$$

M_S ↳ B.S. period.

Q.C. $Y_1^{[-]} \pm 1 = 0$

$$\Rightarrow \oint \sqrt{E - V(x)} = 2\pi \left(n + \frac{1}{2}\right)$$

↳ B.S. quantization

Future directions

- SUSY QM
  $\{Q, Q^\dagger\} = H$
  ← N-fold SUSY QM
  $\{Q, Q^\dagger\} = f(H)$
  ground st.
  → N-th excited state

- large c conformal block.
  $V(x) \leftrightarrow T(x)$
  ← irregular conformal block.