Differential Brauer Group

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Outline

1. Differential Brauer group - affine case
2. Δ-flat cohomology
3. Smooth, projective case

1) Basics

- All rings/schemes are defined over a field K of characteristic 0. To simplify notation we assume K is algebraically closed.
- A derivation $\delta : A \longrightarrow A$ is an additive map satisfying $\delta(ab) = a\delta(b) + \delta(a)b$ if the ring A is non-abelien.
- Maps are required to preserve all derivations unless otherwise stated.
- The partial case is indicated by $\Delta = \{\delta_1, \ldots, \delta_n\}$.
- If M and N are Δ-A modules, so are $M \otimes_A N$ and $Hom_A(M, N)$ the latter being $M^\vee \otimes_A N$ if M and N are locally free A-modules.
- If M is a Δ module, then $Hom_A(M, M)$ is a Δ-A algebra.
- A $\Delta-$Azumaya algebra is an Azumaya algebra equipped with Δ derivations.

Definitions (Magid and Juan 2008 - ordinary case)

We let

$$Az_{\Delta}(A) = \{\Lambda / \Lambda \text{ is a } \Delta - A \text{ Azumaya algebra}\}$$

and introduce an equivalence relation $\sim$ by defining $\Lambda \sim \Gamma$ if there are faithful, locally free Δ modules E and F and an $\Delta - A$ algebra isomorphism

$$\Lambda \otimes_A End_A(E) \approxeq \Gamma \otimes_A End_A(F).$$

Definition

$Br_{\Delta}(A) := Az_{\Delta}(A) / \sim$

It is clear, of course, that $0 = [M_n] \in Br_{\Delta}(A)$ where M_n has coordinate wise differentiation for all $\delta \in \Delta$. We indicate this by (M_n, d). There is also a map $p : Br_{\Delta}(A) \to Br(A)$ that we wish to understand. We should also note that any other Δ structure on Λ is given by adding $d_{\lambda} = \lambda \cdot () - () \cdot \lambda = [\lambda, \ ]$ to the given Δ structure.

Theorem

(Barr − Knus, 1971) Any Azumaya A algebra Λ possesses a Δ structure.

Idea of Proof Let $\left\{ U_i = Spec\left(B_i\right) \xrightarrow{p_i} Spec\left(A\right) \right\}$ be an etale cover with algebra isomorphisms $\alpha_i : \Lambda_i := \Lambda \otimes B_i \underset{\approx}{\rightarrow} M_n\left(B_i\right)$. For any derivation $\delta \in \Delta$, let D_i be the extension of δ to Λ_i using the canonical extension of δ to B_i to act on each entry of $M_n\left(B_i\right)$ and then transferring it to Λ_i using α_i. On $\Lambda_{ij} := \Lambda \otimes B_i \otimes B_j$ we then have two derivations extending δ and so they must differ by $d_{\lambda_{ij}} := [\lambda_{ij}, -]$ for some $\lambda_{ij} \in \Lambda_{ij}$ which is unique if we require that $tr\left(\lambda_{ij}\right) = 0$. Since $\left(D_i - D_j\right) + \left(D_j - D_k\right) = \left(D_i - D_k\right)$, $\left[\{\lambda_{ij}\}\right] \in \check{H}^1\left(A_{Zar}, \Lambda\right)$. But A, being affine, has vanishing Zariski cohomology. Thus there are, possible after refining our covering, elements $\lambda_i \in \Lambda_i$ such that the derivations $D_i + d_{\lambda_i}$ agree on Λ_{ij} as desired.

Example

Example

When can there be $\alpha : (M_n, d + d_\lambda) \underset{\approx}{\to} (M_n, d)$?

If A is a local ring, α must be conjugation by a unique unit $u = (u_{ij})$ of trace 0.Then $d\left(umu^{-1}\right) = u\left(dm + [\lambda, m]\right)u^{-1}$ if and only if $u^{-1}du \cdot m - m \cdot u^{-1}du = \lambda \cdot m - m \cdot \lambda$. So u must solve the differential equation $u^{-1}du = \lambda$. So, for instance, we could consider the polynomial extension with det inverted, $B = A\left[U_{ij}\right]\det\left(U_{ij}\right)^{-1}$. This becomes a Δ extension by defining $\delta\left(U_{ij}\right) = \left(U \cdot \lambda\right)_{ij}$ where $U = \left(U_{ij}\right)$.

Example

Suppose B is a Galois extension with $\mathbb{Z}/n = \langle\sigma\rangle$ action and $C = \oplus_0^{n-1} L^{\otimes i} / \left(\ell_1 \otimes \ell_{n-1} - f\left(\ell_1 \otimes \ell_{n-1}\right)\right)$ where $f : L^{\otimes n} \to A$ is an isomorphism. Then let $\Lambda := B \otimes C$ with multiplication $\ell \cdot b = \sigma\left(b\right) \cdot \ell$.

2) Differential Cohomology

The basic model for a Δ Azumaya algebra of rank n^2 is $M_n(A)$ with coordinate wise differentiation which we indicate by (M_n, d). Given Λ we can use etale covers to make $\Lambda\mid_{U_i} \approx M_n(\mathcal{O}_{U_i})$ but then we must also solve a linear differential equation in order to find a basis $\{e_{ij}\}$ such that $\delta(e_{ij}) = 0$. This leads to the Δ-flat pretopology.

Definition

The Δ-flat pretopology is defined by coverings of $Spec(A)$ given by

$$Cov(A) = \left\{ \left(Spec(B_i) \xrightarrow{\alpha_i} Spec(A) \right)_{i \in I} \mid \begin{array}{r} \alpha_i \text{ is a flat, } \Delta \text{ map} \\ \text{locally of finite presentation with} \\ \cup_i \alpha_i(Spec(B_i)) = Spec(A) \end{array} \right\}$$

Coverings then are easily shown to include all isomorphisms, base extension, and composites. We let X_Δ denote the scheme X with the Δ-flat topology.

Differential Cohomology

There are obvious morphisms of sites for $X = Spec\,(A)$,

$$X_{pl} \xrightarrow{\tau} X_{\Delta} \xrightarrow{\varepsilon} X_{et} \text{ and}$$
$$X_{\Delta} \xrightarrow{\pi} X_{Zar}$$

since any etale extension of A is uniquely a Δ extension, any fppf Δ morphism is a fppf morphism, and any Zariski open can be covered by affine Zariski opens. Since etale and fppf cohomology with coefficients in a smooth group scheme G (such as G_m) agree, we see that $R^i\tau_*(G) = 0$ for $i > 0$ and so

$$H^i(X_{\Delta}, \tau_* G) = H^i(X_{et}, G)$$

for all i. Similarly it is easy to see that for any coherent sheaf $\mathcal{F}$,

$$H^i(X_{\Delta}, \mathcal{F}) = H^i(X_{Zar}, \mathcal{F})$$

.

Differential Cohomology

We use this to cohomologically characterize

$$Az_{\Delta,n}(A) = \left\{\Lambda \in Az_{\Delta}(A) \mid rk(\Lambda) = n^2\right\}.$$

The example shows that any $\Lambda \in Az_{\Delta,n}(A)$ is locally in the $\Delta-$flat topology isomorphic to (M_n, d). Thus we need to understand $Aut_{\Delta}((M_n, d))$, the sheaf of $\Delta-$automorphisms of (M_n, d). $Aut_{\Delta}((M_n, d)) \subset Aut(M_n) = PGl_n$ consists of automorphisms that preserve coefficient derivations. So, for instance, $Aut_{\Delta}(M_n)(A) = PGl_n(A)^{\Delta}$. Once we have identified $Aut_{\Delta}(M_n) = PGl_n^{\Delta}$, we find that $Az_{\Delta,n}(A) = H^1\left(A_{\Delta}, PGl_n^{\Delta}\right)$. Finally we get a non-commutative central extension for the site A_{Δ}

$$1 \to G_m^{\Delta} \to Gl_n^{\Delta} \to PGl_n^{\Delta} \to 0$$

where the superscript Δ indicates the subsheaf of Δ constants which, by taking limits over n, may be used to define an injection, $Br_{\Delta}(A) \hookrightarrow H^2\left(A_{\Delta}, G_m^{\Delta}\right)$.

3) Smooth projective case

Andre introduced an alternate description of a Δ ring which we will now use. So a Δ scheme is a pair $\left(X, \Omega^1_X\right)$ consisting of a scheme X, a locally free, coherent sheaf Ω^1_X, and a differential map $d_X : \mathcal{O}_X \to \Omega^1_X$. That is we have a commutative triangle

$$\begin{array}{ccc} & & \Omega^1_{X/S} \\ & d_{X/S} \nearrow & \downarrow \phi_{X_\Delta} \\ \mathcal{O}_X & \xrightarrow{d_X} & \Omega^1_X \end{array}$$

where we also require ϕ_{X_Δ} to be surjective. A Δ map $Y \xrightarrow{f} X$ is then a morphism of schemes such that $f^*\left(\Omega^1_X\right) \underset{\cong}{\to} \Omega^1_Y$ with a commutative square

$$\begin{array}{ccc} f^*\left(\Omega^1_{X/S}\right) & \to & \Omega^1_{Y/S} \\ f^*\left(\phi_{X_\Delta}\right) \downarrow & & \phi_{Y_\Delta} \downarrow \\ f^*\left(\Omega^1_X\right) & \to & \Omega^1_Y \end{array}.$$

This extends in an obvious manner to define Δ modules, etc.

Smooth projective case

We can get a better understanding of the relationship between $Br(A)$ and $Br_{\Delta}(A)$ by looking at the short exact sequence on X_{Δ},

$$0 \to G_m^{\Delta} \xrightarrow{i} G_m \xrightarrow{d\ln} \Omega_X^1 \to 0$$

where the superscript Δ indicates the subsheaf of Δ constants so that, for instance, $G_m^{\Delta}(B) := (B^*)^{\Delta}$.
The various isomorphisms between Zariski, etale and Δ-flat cohomology then show that on X_{Δ}, we have a long exact sequence

$$\begin{aligned} \Gamma\left(X, \Omega_X^1\right) &\to Pic_{\Delta}(X) \to Pic(X) \to H^1\left(X_{Zar}, \Omega_X^1\right) \\ &\to H^2\left(X_{\Delta}, G_m^{\Delta}\right) \xrightarrow{i^2} H^2\left(X_{et}, G_m\right) \end{aligned}$$

with i^2 being surjective if X is regular. Note that in the affine case we have $H^1\left(A_{Zar}, \Omega^1\right) = 0$ which shows that $Br_{\Delta}(A) = H^2\left(A, G_m^{\Delta}\right) = Br(A)$ and, in the smooth, projective case we have $Pic_{\Delta}(X) = Pic^0(X)$.

Smooth projective case

Here the question is how to produce a connection on a sheaf of Azumaya algebras Λ on a smooth, projective variety X.

Lemma

Let $x \in H^2\left(X, G_m^{\Delta}\right)$ and suppose there is $\pi : Y \to X$ such that $S := \pi_* \mathcal{O}_Y$ is a vector bundle with a connection, e.g. Y is a finite etale covering of X, and $\pi^*(x) = 0$. Then there is a sheaf of Azumaya algebras Λ which is a twisted form of $End(S)$ twisted by Δ constants that represents $x \in Br(X)$. In particular Λ comes with a connection $\nabla : \Lambda \to \Lambda \otimes \Omega^1_{X/K}$.

Example

If A is an abelian variety over K, then any element $x \in H^2\left(A_{et}, \mu_n\right)$ is split by $A \xrightarrow{n} A$.

Smooth projective case

Biswas conjectured in 2009 that finite etale quotients of abelian varieties were those smooth, projective varieties X of dimension ≥ 2 such that $j^*(T_X)$ is semi-stable for $j : C \to X$ where C is any compact Riemann surface. This was proved by Jahnke and Radloff in 2013 and so all elements of $Br(X)$ can be represented by sheaves of Azumaya algebras with connections.

Another possibility is a conjecture of Iitaka that a smooth complex projective variety X with a universal covering space biholomorphic to $\mathbb{C}^n$ admits a finite, etale cover by an abelian variety. If true such varieties would also have $Br_\Delta(X)$ mapping onto $Br(X)$.

Caution: Biswas has outlined a proof that no sheaf of Azumaya algebras on a complex K3 surface has a connection.

Smooth projective case

General Requirement for a bundle E on a projective variety to have a connection:

- $c_i(E) = 0$ for all i.
- $\Delta(E) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 = 0$
- If E is a Hermitian-Einstein bundle (=Hermitian metric) then E must be semi-stable.

Biswas, Bruzzo, Gurjar show that for all bundles E of rank r on a smooth projective manifold X, $j^*(E)$ is semi-stable for all maps $j : C \to X$ from any smooth, irreducible projective curve iff $\Delta(E) = 0$ and E is semi-stable for some H. (If $\pi : Y \to X$ is a finite, etale cover, then $\pi_* \mathcal{O}_Y$ is semi-stable since it is self dual and so the Harder-Narasimhan filtration can only have a weight 0 piece.)

Smooth projective case

One last remark using an old result of Sumihiro.

Theorem

Let S be the Severi-Brauer variety associated to an Azumaya algebra Λ on X, a projective smooth variety over K. Then there is a $Y \subseteq S$ such that Y is a finite, flat cover of X.

Idea of Proof S is a projective scheme over X and may be embedded in a $\mathbb{P}^N_K$. Sumihiro shows that in this setting there is an open set of hyperplanes in the dual projective space that intersect every fibre of $\pi : S \longrightarrow X$. Repeat this until the fibre dimension of π is 0.

Note:The argument in the Lemma then shows that Λ is a twisted form of $End\left(\pi_* \mathcal{O}_Y\right)$.

Haboush at 75

Congratulations, Bill!