Notes:

① All the "accidental" isom. among low dim. Lie alg. lead to interesting isogenies. We've looked so far only at those arising from $SO(4,\mathbb{C})$, $SO(6,\mathbb{C})$ + in this talk have time only for $SO(4,\mathbb{C})$, but this is rich enough to illustrate many pretty features.

② The talk involves reductive Lie groups + the appeal of the subject matter, such as it is, is as much aesthetic as practical or technical; in short its designed to emphasize priorities tailored to Bill's tastes

# Higgs Bundles for Isogenous Lie Groups

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# The main theme

* Relation between G-Higgs bundles and $\mathcal{M}_{Hggs}(G)$ for different Lie groups G....

* .... specifically for $I : G_1 \longrightarrow G_2$ where $G_1 \simeq G_2$ ....

* ... specifically

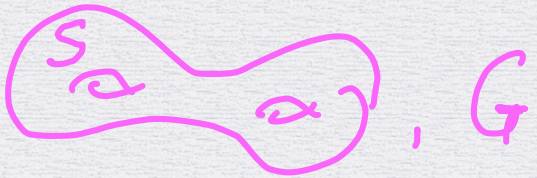
$$SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \longrightarrow SO(4,\mathbb{C})$$

$$SL(4,\mathbb{C}) \longrightarrow SO(6,\mathbb{C})$$

and their real forms.

[Joint with Laura Schaposnik [Res. Math. Sci. 2016]
Laura and Lucas Branco]

# Introduction: the objects.




Given • $S$ genus $g$
• $G$ $SL(n,\mathbb{C})$, $SO(2n,\mathbb{C})$, ...

(Topological) (Smooth) (holomorphic)

$\mathrm{Rep}^+(\pi_1(S), G) \longleftrightarrow \mathcal{F}\mathrm{lat}^+(G) \overset{\text{NAH}^*}{\longleftrightarrow} \mathcal{M}_{\mathrm{Higgs}}(G)$

$\rho : \pi_1(S) \to G$ $(P_G, \nabla)$ **Higgs bundles**

[* Hitchin, Simpson, Corlette, Donaldson]

# Introduction: the maps..

$$\psi : G_1 \longrightarrow G_2$$

... induces

$$\begin{array}{ccccc}
\mathrm{Rep}^+(\pi_1(S), G_1) & \longleftrightarrow & \mathcal{F}\mathrm{lat}^+(G_1) & \longleftrightarrow & \mathcal{M}_{Hig}(G) \\
\downarrow \psi & & \downarrow \psi & & \vdots \psi \\
\mathrm{Rep}^+(\pi_1(S), G_2) & \longleftrightarrow & \mathcal{F}\mathrm{lat}^+(G_2) & \longleftrightarrow & \mathcal{M}_{Hig}(G)
\end{array}$$

$$\begin{array}{ccc}
\pi_1(S) & \xrightarrow{\rho_1} & G_1 \\
& \searrow^{\rho_2} & \downarrow \psi \\
& & G_2
\end{array}$$

$$P_{G_2} = P_{G_1} \times_{\psi} G_2$$

?

# G-Higgs bundles

$(S, J) = \Sigma$

* Holomorphic bundle + Higgs field
* Encodes data for G-local system / flat G-bundle
* We will assume $G \subseteq GL(n, \mathbb{C})$

$G = GL(n, \mathbb{C})$

A $GL(n,\mathbb{C})$-Higgs bundle is a pair $(V, \Phi)$
- $V$ a rank $n$ holomorphic bundle on $\Sigma$
- $\Phi : V \to V \otimes K_\Sigma$, i.e. $\Phi \in H^0(End(V) \otimes K_\Sigma)$

(poly) stable $\iff$ $\exists$ metric $h$, $F_h + [\Phi, \Phi^{*h}] = 0$ $\iff$ $D_h^* \Phi = 0$ and $\nabla = D_h + \Phi + \Phi^{*h}$ is FLAT with $GL(n,\mathbb{C})$-holonomy $\iff$ holonomy $\rho : \pi_1(S) \to G$ reductive

[Hitchin, Simpson, Corlette, Donaldson]

# G-Higgs bundles for $G \subseteq SL(n,\mathbb{C}) \subseteq GL(n,\mathbb{C})$

$(S,J)=\Sigma$

- $V \longrightarrow \Sigma$ rank $n$ holomorphic bundle
- $\Phi : V \longrightarrow V \otimes K$
- Extra conditions so $\nabla$ has holonomy in $G$

| $G$ | $V$ | $\Phi$ |
|---|---|---|
| $SL(n,\mathbb{C})$ | $\det V \simeq \mathcal{O}$ | $\operatorname{Tr}\Phi = 0$ |
| $SO(2n,\mathbb{C})$ | $Q : V \times V \to \mathcal{O}$<br>$\delta : \det V \simeq \mathcal{O}$ | $\Phi^t Q + Q\Phi = 0$ |

(see later for further conditions for real forms of $G$ )

# Moduli Spaces $\mathcal{M}_{Higgs}(G)$.

$$\mathcal{M}_{Higgs}(G) = \{\text{isom. classes of semistable } (V, \Phi)\}$$

* smooth locus admits hyperkahler structure
* Hitchin fibration

$$\mathcal{M}_{Higgs}(G) \xrightarrow{f_H} \mathcal{B} = \bigoplus_i H^0(K^{d_i})$$

- $\Phi$ is locally a matrix of 1-forms
- $p(\Phi) \in H^0(K^d)$ for degree $d$ invariant polynomials
- $f_H(V, \Phi) = (p_1(\Phi), \ldots p_N(\Phi))$ $\{p_1, \ldots p_N\}$ generate ring of invariant polynomials

* A framework for SYZ mirror symmetry

# fibers of the Hitchin fibration: Spectral Curves

$\mathcal{M}_{Higgs} \to \bigoplus H^0(K^{d_i})$

- Take $\{p_1, \dots p_n\}$ coefficients in $|\Phi - \lambda I|$
- $|\phi - \lambda I| = 0$ defines a curve in $|K|$ (total space)

$$[GL(n,\mathbb{C})] \quad \{\lambda^n + p_1\lambda^{n-1} + \dots + p_n = 0\} = S$$

$(p_i \in H^0(K^i))$

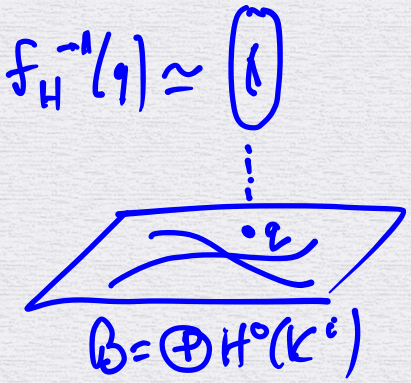
$$\begin{array}{ccc} \pi^*K & & K \\ \downarrow & & \downarrow \\ K & \xrightarrow{\pi} & \Sigma \end{array} \qquad \lambda$$

- $\pi : S \longrightarrow \Sigma$ $n:1$ ramified cover
- $\pi^{-1}(x) =$ Eigenvalues of $\Phi(x)$ at $x \in \Sigma$
- $S =$ Curve on which $|\Phi - \lambda I| = 0$ is single-valued

# Spectral Data* ($G = GL(n, \mathbb{C})$)

On generic fibers:

* $S$ is smooth
* Recover $(V, \Phi)$ from $(S, L)$




$$\begin{array}{ccc} L & \to & S \\ \downarrow \pi_* & & \downarrow \pi \\ V & \longrightarrow & \Sigma \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{\lambda} & L \otimes \pi^* K \\ \pi_* \downarrow & & \downarrow \pi_* \\ V & \xrightarrow{\Phi} & V \otimes K. \end{array}$$

* $h^{-1}(q) \simeq Jac(S)$.

* $G \subseteq GL(n, \mathbb{C})$ : extra conditions on $L$
* singular fibers : $L \rightsquigarrow$ Rank 1 Torsion free sheaf

More on this later

[* Hitchin, Simpson, Beauville-Narasimhan-Ramanan]

# The isogenies

$$SL(2,\mathbb{C})\times SL(2,\mathbb{C}) \xrightarrow{\Psi} SO(4,\mathbb{C})$$

$$(A,B) \longmapsto A\otimes B$$

$$\omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \longmapsto Q = \omega\otimes\omega$$

$$\Psi_*(a,b) = a\otimes I + I\otimes b$$

$$SL(4,\mathbb{C}) \xrightarrow{\Psi} SO(6,\mathbb{C})$$

$$A \longmapsto \Lambda^2 A = A\otimes A\,|_{\Lambda^2\mathbb{C}^4}$$

$$Q : \underbrace{(\Lambda^2\mathbb{C}^4)^*}_{\mathbb{C}^6}\otimes \underbrace{\Lambda^4\mathbb{C}^4}_{\mathbb{C}} \longrightarrow \underbrace{\Lambda^2\mathbb{C}^4}_{\mathbb{C}^6}$$

$$\Psi_*(a) = a\otimes I + I\otimes a$$

Question: Induced maps on Higgs bundles/spectral data ?

$I_2 : SL(4,\mathbb{C}) \times SL(2,\mathbb{C}) \longrightarrow SO(4,\mathbb{C})$

$(A,B) \longmapsto A \otimes B$

## Induced map on Higgs pairs*

$$(V_1, \Phi_1), (V_2, \Phi_2) \longrightarrow ((W, Q), \Phi)$$

rank 2
$\det V_1 \simeq \mathcal{O} \simeq \det V_2$
$\mathrm{Tr}\, \Phi_1 = 0 = \mathrm{Tr}\, \Phi_2$

rank 4, orthogonal
$\Phi^t Q + Q \Phi = 0$

$$W = V_1 \otimes V_2$$

$$\Phi = \Phi_1 \otimes I + I \otimes \Phi_1$$

what about spectral data ?

[* M. Arroyo]

# Spectral Data for $SL(n,\mathbb{C})$

$L$, $V=\pi^* L$, $S=\{\det(\Phi-\lambda\Sigma)=0\}$

$|K| \supset S \xrightarrow{\pi} \Sigma$

* $S = \{\lambda^n + a_2\lambda^{n-2} + a_3\lambda^{n-3} + \cdots + a_n = 0\}$ $\quad [a_i \in H^0(K^i)]$
  * $a_1 = \mathrm{Tr}\,\Phi = 0$
  * generically smooth
* $U = L \otimes [R]^{-1/2} \in \mathrm{Prym}(S,\Sigma)$.

$SL(2,\mathbb{C})$ $\quad S = \{\lambda^2 + a_2 = 0\}$ $\quad a_2 \in H^0(K^2)$

(smooth $\Leftrightarrow$ $a_2$ has simple zeros.)

$L \otimes [R]^{-1/2} \in \mathrm{Prym}(S,\Sigma)$.

$S \xrightarrow{2:1} \Sigma$

# Spectral Data for $SO(2n,\mathbb{C})$

$L$, $V=\pi_* L$, $S=\{\det(\Phi-\lambda\Sigma)=0\}$

$|K| \supset S \xrightarrow{\pi} \Sigma$

$$S=\{\lambda^{2n}+b_1\lambda^{2n-2}+b_2\lambda^{2n-4}+\cdots+b_n^2=0\}$$

$b_n^2 \longleftarrow \mathrm{Pf}(\Phi)^2=\det\Phi$

* never smooth
* $\sigma: S \to S \quad (\lambda \to -\lambda)$.

Take $\hat{S}$ = Normalization of $S$

$L\otimes[R]^{-1/2} = U \in \mathrm{Prym}(\hat{S}, \hat{S}/\sigma)$

$\hat{S} \xrightarrow{2:1} \hat{S}/\sigma$, $\pi$, $n:1$, $\tilde{\pi}$, $\Sigma$

$SO(4,\mathbb{C})$: $\lambda^4+b_1\lambda^2+b_2^2=0$

$b_1, b_2 \in H^0(K^2)$

$\hat{S} \downarrow 4:1 \ \Sigma$

# Induced map on Spectral Curves

$SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \to SO(4,\mathbb{C})$
$(A,B) \longmapsto A \otimes B$

$\lambda^2 + a_1 = 0 = \lambda^2 + a_2$ $\quad [\,|\Phi - \lambda \mathbb{I}| = 0\,]$ $\quad \lambda^4 + b_1 \lambda^2 + b_2^2 = 0$



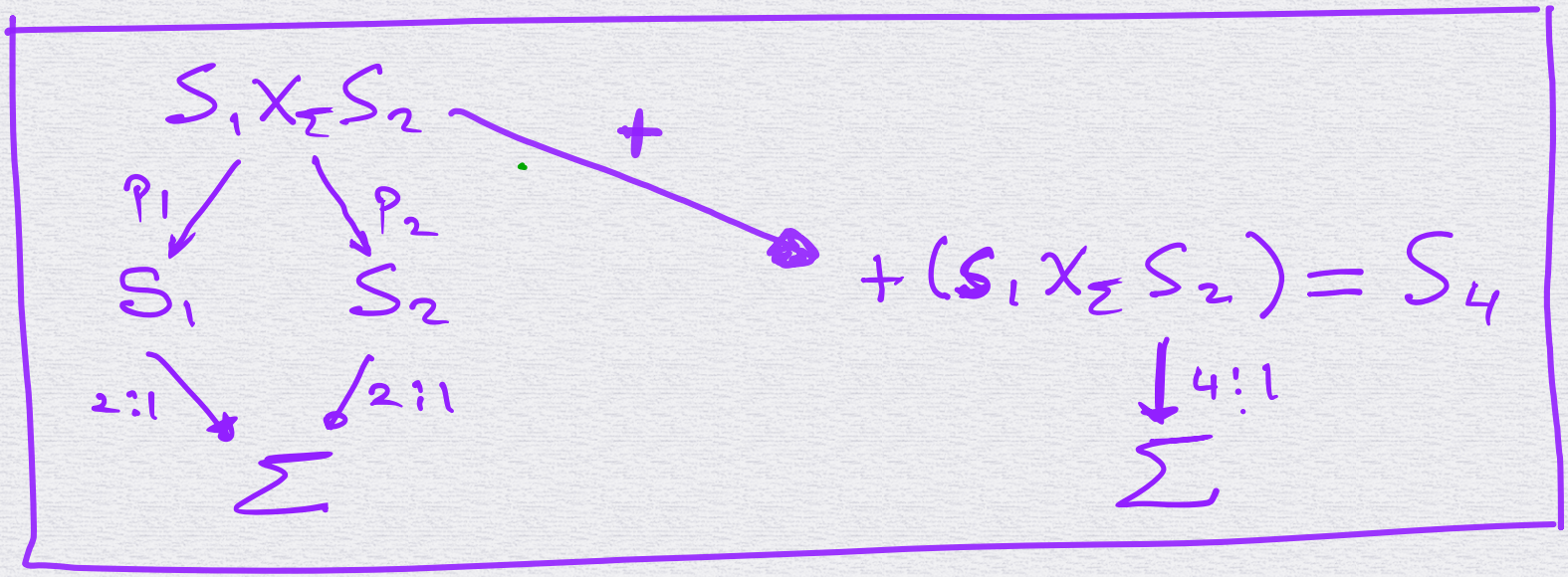

$(V_1, \Phi_1), (V_2, \Phi_2) \dashrightarrow (V_1 \otimes V_2, \Phi_1 \otimes 1 + 1 \otimes \Phi_2)$

"Low-tech"
- Fiber over $x \in \Sigma$ = eigenvalues of $\Phi$
- Under $(\Phi_1, \Phi_2) \to \Phi_1 \otimes 1 + 1 \otimes \Phi_2$ eigenvalues add.

$\Longrightarrow$

$$b_1 = 2(a_1 + a_2)$$
$$b_2^2 = (a_1 - a_2)^2$$

# "High-Tech" description




$K \oplus K$ $\xrightarrow{+}$ $K$

**Prop**: [Schaposnik, _] On generic fibers

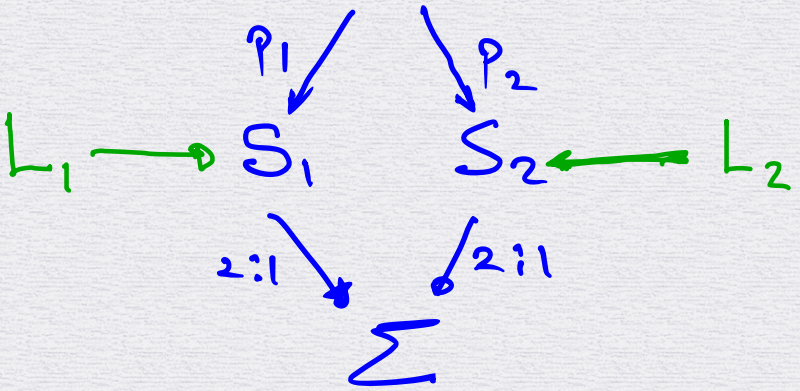
- $+(S_1 \times_\Sigma S_2) = S_4$
- $S_1 \times_\Sigma S_2 = \hat{S}_4$

Q: What about the line bundles?

# Induced map on the spectral line bundles

$\mathcal{L} = p_1^*(L_1) \otimes p_2^*(L_2)$

$\hat{S}_4 = S_1 \times_\Sigma S_2 \xrightarrow{2:1} (S_1 \times_\Sigma S_2)/\sigma_1 \times \sigma_2 = \hat{S}_4/\sigma$




$[\sigma_i : S_i \to S_i$ involution$]$

**Prop** [sch., –] on generic fibers

- $L_i [R_i]^{-1/2} \in Prym(S_i, \Sigma) \Rightarrow \mathcal{L}[R]^{-1/2} \in Prym(\hat{S}_4, \hat{S}_4/\sigma)$

($[R_i]$: Ramification Divisors; $[R]$: Divisors)

Q: what's the picture for real forms? what about non-generic fibers?

# Real forms.

| | $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ | $SO(4,\mathbb{C})$. |
|---|---|---|
| Compact | $SU(2) \times SU(2)$ | $SO(4)$ |
| Split | $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ | $SO(2,2)$ |
| | $SU(2) \times SL(2,\mathbb{R})$ | $SO^*(4)$ |
| | $SL(2,\mathbb{C})$ | $SO(1,3)$. |

- Defined by anti holomorphic involutions

Q: * What are the Higgs bundles?
* How do the isogenies/induced maps behave?

# Higgs bundles for real forms $G_{\mathbb{R}} \subset G_{\mathbb{C}}$

Recall $\quad \mathrm{Rep}^{+}(\pi(S), G_{\mathbb{C}}) \xleftrightarrow{\text{NAH}} \mathcal{M}_{\text{Hitgp}}(G_{\mathbb{C}})$

$\pi_1(S) \longrightarrow G_{\mathbb{C}} \qquad (V, \Phi)$ stable

$\nabla^h$ = Flat with $G_{\mathbb{C}}$-holonomy $\qquad (V, \nabla^h)$ $\qquad h$ = metric

"Defn" A $G_{\mathbb{R}}$-Higgs bundle is a $G_{\mathbb{C}}$-Higgs bundle with additional conditions to ensure holonomy lies in $G_{\mathbb{R}}$

* Get $\mathcal{M}_{\text{Hitgp}}(G_{\mathbb{R}}) \subset \mathcal{M}_{\text{Higgs}}(G_{\mathbb{C}})$.

# Why consider $\mathcal{M}_{Hggs}(G_r) = \mathrm{Rep}^+(\pi_1(S), G_r)$?

$(S, J) = \Sigma$

① Special Components:

* $\pi_0$ contains surprises [Goldman Hitchin, Garcia-Prada Gothen, Collier, Oliveira, Burger-Iozzi-Wienhard, Bradlow, Schaposnik, ...]
* $SL(2,\mathbb{R})$: Teichmuller space
* In some cases [Gr split, Hermitian, ...] "Higher Teichmuller" components

[Labourie, Guichard, Wienhard, Canary, Bridgeman, Sambarino, ...]

② $\mathcal{M}_{Hggs}(G_r) \subset \mathcal{M}_{Hggs}(G_{\mathbb{C}})$ give BAA branes

[Gukov, Schaposnik, Hitchin, Branco, Baraglia, ...]

# Compact Real Forms $G_0 \subseteq G_{\mathbb{C}}$

| $SL(n,\mathbb{C})$ | $SO(2n,\mathbb{C})$ |
|---|---|
| $SU(n)$ | $SO(2n)$ |

* Narasimhan-Seshadri:

$$\mathrm{Rep}^{\dagger}(\pi_1(S), G_0) \longleftrightarrow M^{ss}(G_0)$$ (Mod. space of semistable bundles)

$$= \{\text{locus in } \mathcal{M}_{Higgs}(G_0) \text{ where } \mathcal{E} = 0\}$$

* In $\mathcal{M}_{Higgs}(G_{\mathbb{C}}) \xrightarrow{f_H} \bigoplus H^0(K^{d_i})$

$M^{ss}(G_0) \subseteq f_H^{-1}(0)$ (Nilpotent Cone)

Spectral data very non-generic!

# Split Real Forms

| | |
|---|---|
| $SL(n,\mathbb{C})$ | $\det(V) \simeq \mathcal{O}$<br>$Tr\,\Phi = 0$ |
| $SL(n,\mathbb{R})$ | $Q: V \xrightarrow{\sim} V^*$ sym<br>$\Phi^t Q - Q\Phi = 0$ |

$(V, \Phi)$

| | |
|---|---|
| $SO(2n,\mathbb{C})$ | $Q: V \xrightarrow{\sim} V^*$ sym<br>$\Phi^t Q + Q\Phi = 0$ |
| $SO(n,n)$ | $V = V_1 \oplus V_2$<br>$Q = Q_1 \oplus Q_2$<br>$\Phi = \begin{bmatrix} 0 & \eta \\ -\eta^T & 0 \end{bmatrix}$ |

$((V, Q), \Phi)$

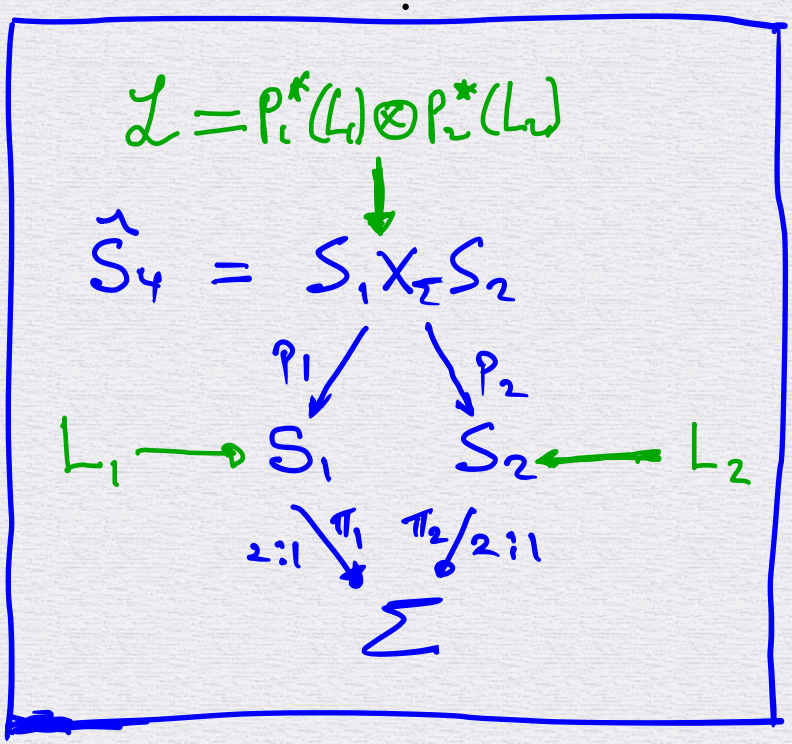
* $\mathcal{M}_{Higgs}(G_{split})$ intersects all fibers of

$$f_H : \mathcal{M}_{Higgs}(G_{\mathbb{C}}) \longrightarrow \bigoplus H^0(K^{d_i})$$

* [Schaposnik] • Spectral curve same as for $G_{\mathbb{C}}$

• $\mathcal{M}_{Higgs}(G_{split}) \cap \{$ smooth fiber of $f_H \} = \{$ points of order 2 $\}$

# $SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \longrightarrow SO(2,2)$




$(S_i, L_i) \sim (V, \Phi) \in \mathcal{M}_{Higgs}^{(SL(2,\mathbb{R}))}$

$\Leftrightarrow L_i [R_i]^{-1/2} \in Prym(S_i, \Sigma)$

and $(L_i [R_i]^{-1/2})^2 \simeq \Theta_{S_i}$.

$(\mathcal{L}, \hat{S}_4) \sim ((V_6, Q), \Phi_6) \in \mathcal{M}_{Higgs}^{(SO(2,2))}$

$\Leftrightarrow \mathcal{L} [R_4]^{-1/2} \in Prym(\hat{S}_4, \hat{S}_4/\sigma)$

and $(\mathcal{L} [R_4]^{-1/2})^2 \simeq \Theta_{\hat{S}_4}$

**Thm:** $L_i [R_i]^{-1/2} \in Prym(S_i, \Sigma)[2]$ for $i = 1, 2$

$\Rightarrow \mathcal{L} [R_4]^{-1/2} \in Prym(\hat{S}_4, \hat{S}_4/\sigma)[2]$

[... simly for $SL(4,\mathbb{R}) \longrightarrow SO(3,3)$ ]

# Other non-compact real forms

$$\mathcal{M}_{Hggs}(G_r) \subset \mathcal{M}_{Hggs}(G_c)$$

$$\downarrow f_H \qquad \downarrow f_H$$

$$\mathcal{B}_r \subsetneq \mathcal{B}_{\mathbb{C}}$$

$$\boxed{\mathcal{B}_r \subseteq (\mathcal{B}_{\mathbb{C}})_{singular}}$$

* Spectral curves not smooth
  not irreducible
  not reduced.

* Spectral data is tricky!
  (Rank 1 Torsion free sheaves on non-integral curves)

# $SL(2,\mathbb{C})\times SL(2,\mathbb{C}) \longrightarrow SO(4,\mathbb{C})$ : Spectral curves

| $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ | $(s_1, s_2)$ <br> $\lambda^2+a_1,\ \lambda^2+a_2$ |
|---|---|
| $SU(2)\times SL(2,\mathbb{R})$ | $(\lambda^2,\ \lambda^2+a_2)$ <br> $(2\Sigma, S)$ |
| $SL(2,\mathbb{C})$ | $(\lambda^2+a,\ \lambda^2+a)$ <br> $(S, S)$ |

$b_1 = 2(a_1+a_2)$
$b_2 = (a_1-a_2)^2$

$a_1 = 0$

$a_1 = a_2$

| $S_4$ <br> $\lambda^4 + b_1\lambda^2 + b_2^2$ | $SO(4,\mathbb{C})$ |
|---|---|
| $(\lambda^2+b_1)^2$ <br> $2S$ | $SO^*(4)$ |
| $\lambda^2(\lambda^2+b_1)$ <br> $2\Sigma \cup S$ | $SO(1,3)$ |

ribbons , reducible

* Spectral data for $SO^*(4)$ : Hitchin-Schaposnik
  $SO(p,q)$ : Schaposnik-Baraglia
  on ribbons ! Branco

# The real form $SL(2,\mathbb{C}) \subset SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$

Defining involution: $(A,B) \to (B^{*-1}, A^{*-1})$

$$SL(2,\mathbb{C}) = \{(A, A^{*-1})\} \subset SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$$

$$SU(2) = \{(U,U) \mid U \in SU(2)\}$$

$$SU(2)\otimes\mathbb{C} = \{(A,A) \mid A \in SL(2,\mathbb{C})\}$$

| $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ | $(V_1, \Phi_1), (V_2, \Phi_2)$ | $(S_1, L_1), (S_2, L_2)$ |
|---|---|---|
| $SL(2,\mathbb{C})$ | $V_1 = V_2,\ \Phi_2 = -\Phi_1$ | $S_1 = S_2,\ L_2 = L_1^{-1}$ |

Under $\Sigma: SL(2,\mathbb{C}) \times SL(2,\mathbb{C}) \longrightarrow SO(4,\mathbb{C})$

$(V_1, \Phi_1), (V_2, \Phi_2)$
$\downarrow$
$(V_1 \otimes V_2, \Phi_1 \otimes 1 + 1 \otimes \Phi_2)$

$(V, \Phi), (V, -\Phi) \longrightarrow$

$V \otimes V \simeq \theta \oplus Sym^2 V$
$\Phi \otimes 1 - 1 \otimes \Phi = \begin{bmatrix} 0 & \eta \\ -\eta^T & 0 \end{bmatrix}$

$SO_0(1,3)$
Higgs
bundle

$S \times_\Sigma S \xrightarrow{+} +(S \times_\Sigma S) = S_{1,3}$ $\quad \{\lambda^2(\lambda^2 + 4a) = 0\}$

$S_1 \simeq S \simeq S_2$
$\{\lambda^2 + a = 0\}$

Under $\Sigma: SL(2,\mathbb{C})\times SL(2,\mathbb{C}) \longrightarrow SO(4,\mathbb{C})$

$(V_1,\Phi_1),(V_2,\Phi_2)$
$\downarrow$
$(V_1\otimes V_2,\ \Phi_1\otimes 1+1\otimes\Phi_2)$

$(V,\Phi),(V,-\Phi) \longrightarrow$

$V\otimes V \simeq \theta \oplus Sym^2 V$
$\Phi\otimes 1-1\otimes\Phi = \begin{bmatrix} 0 & \eta \\ -\eta^T & 0 \end{bmatrix}$
$SO_0(1,3)$ Higgs bundle

$S \times_\Sigma S \xrightarrow{+} +(S\times_\Sigma S) = S_{1,3}$ $\{\lambda^2(\lambda^2+4a)=0\}$

$\Sigma(x,y)=(y,x)$

$S_1 \simeq S \simeq S_2$
$\{\lambda^2+a=0\}$

$\dfrac{(S\times_\Sigma S)}{\Sigma} \simeq (S_{1,3})_{red}$ $\{\lambda(\lambda^2+4a)=0\}$

.... the significance of $(S_{13})_{red}$ ---->

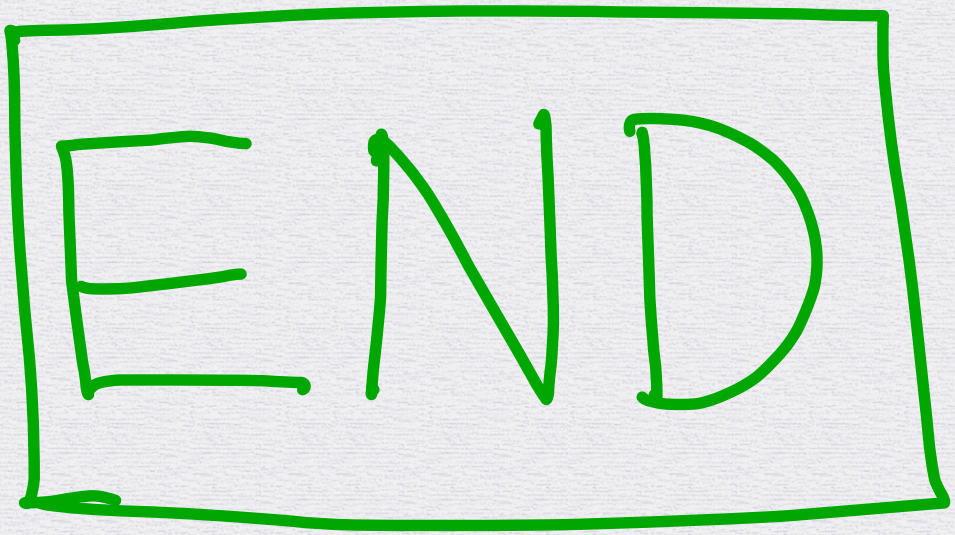
# Fibers of Hitchin Fibration

$$\begin{array}{ccc} \mathcal{M}_{Higgs}(SO(1,3)) & \hookrightarrow & \mathcal{M}_{Higgs}(SO(4,\mathbb{C})) \\ \downarrow & & \downarrow f_H \\ H^0(K^2) & \hookrightarrow & H^0(K^2)\oplus H^0(K^2) \\ a & \longmapsto & (4a, 0) \end{array}$$

$$f_H^{-1}(S_{13}) = \{\text{RITFS on } S_{13}\}$$

$$f_H^{-1}(S_{13}) \cap \mathcal{M}_{Higgs}(SO(1,3)) = \left\{ \begin{array}{ccc} \tilde{S} & & i_*\tilde{S} \\ \downarrow & & \downarrow \\ (S_{1,3})_{red} & \xhookrightarrow{i} & S_{1,3} \end{array} \right\}.$$

[... in progress with Laura Schaposnik and Lucas Branco]


END

^

# The other isogeny, i.e. $I_4 : SL(4,\mathbb{C}) \longrightarrow SO(6,\mathbb{C})$

$A \longrightarrow \Lambda^2 A$

On Higgs pairs:

$$(V_4, \Phi_4) \xrightarrow{\ I\ } (V_6, Q), \Phi_6$$

rank 4, $\det V \simeq \mathcal{O}$

$\mathrm{Tr}\, \Phi_4 = 0$

orthogonal rank 6

$\Phi_6^t Q + Q \Phi_6 = 0$

$$V_6 = \Lambda^2 V_4$$

$$Q : (\Lambda^2 V_4)^* \times (\Lambda^4 V_4) \xrightarrow{\sim} \Lambda^2 V_4$$

$$\Phi_6 = \Phi_4 \otimes I + I \otimes \Phi_4$$

# Induced map on Spectral Curves



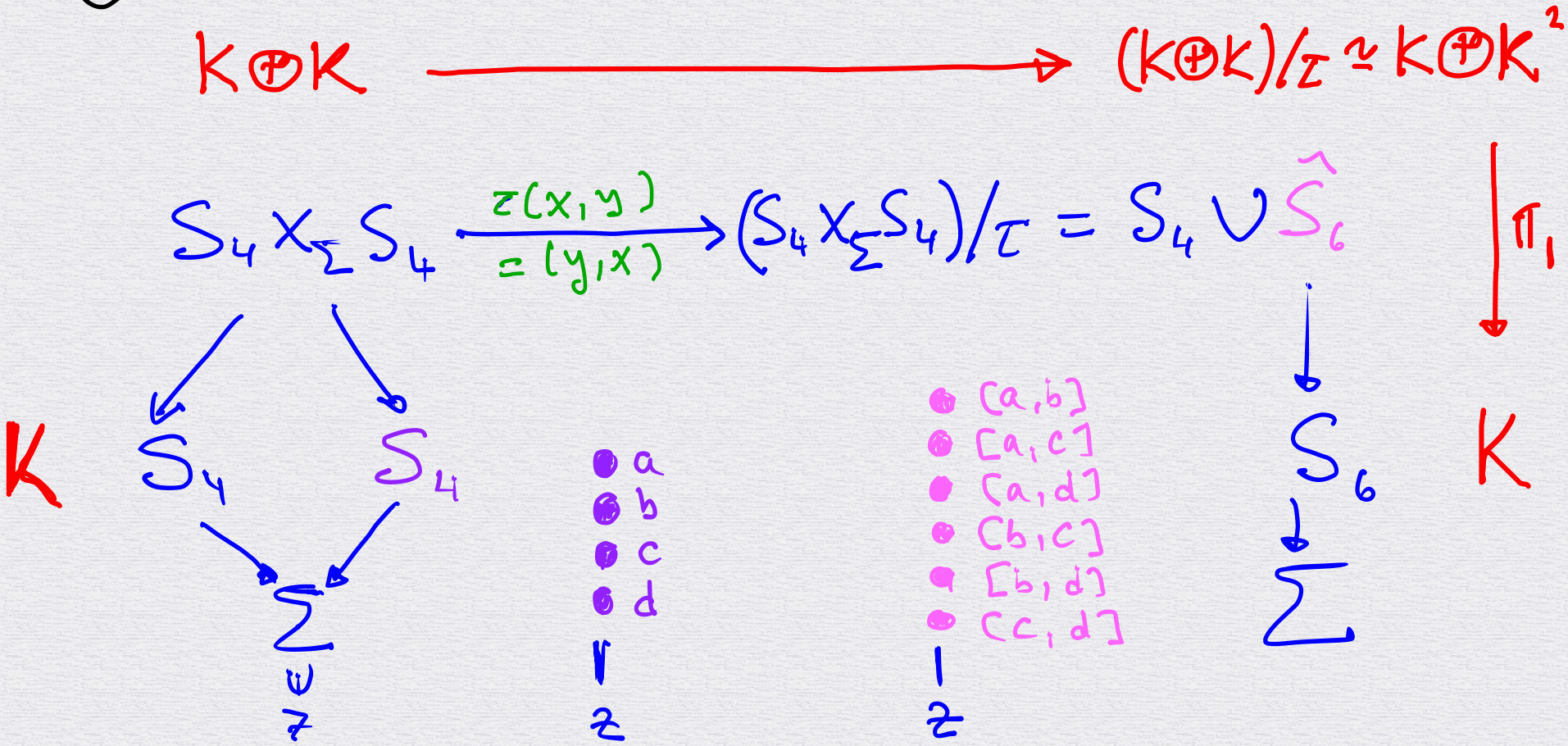

$$\lambda^4 + a_2\lambda^2 + a_3\lambda + a_4 = 0 \qquad \lambda^6 + b_2\lambda^4 + b_4\lambda^2 + b_3^2 = 0$$

Low-tech : $\Phi_4 = \prod_{i=1}^{4} (\lambda - \gamma_i)$ $\qquad \Phi_6 = \prod_{i \neq j} (\lambda - \mu_{ij})$

$\mu_{ij} = \gamma_i + \gamma_j$

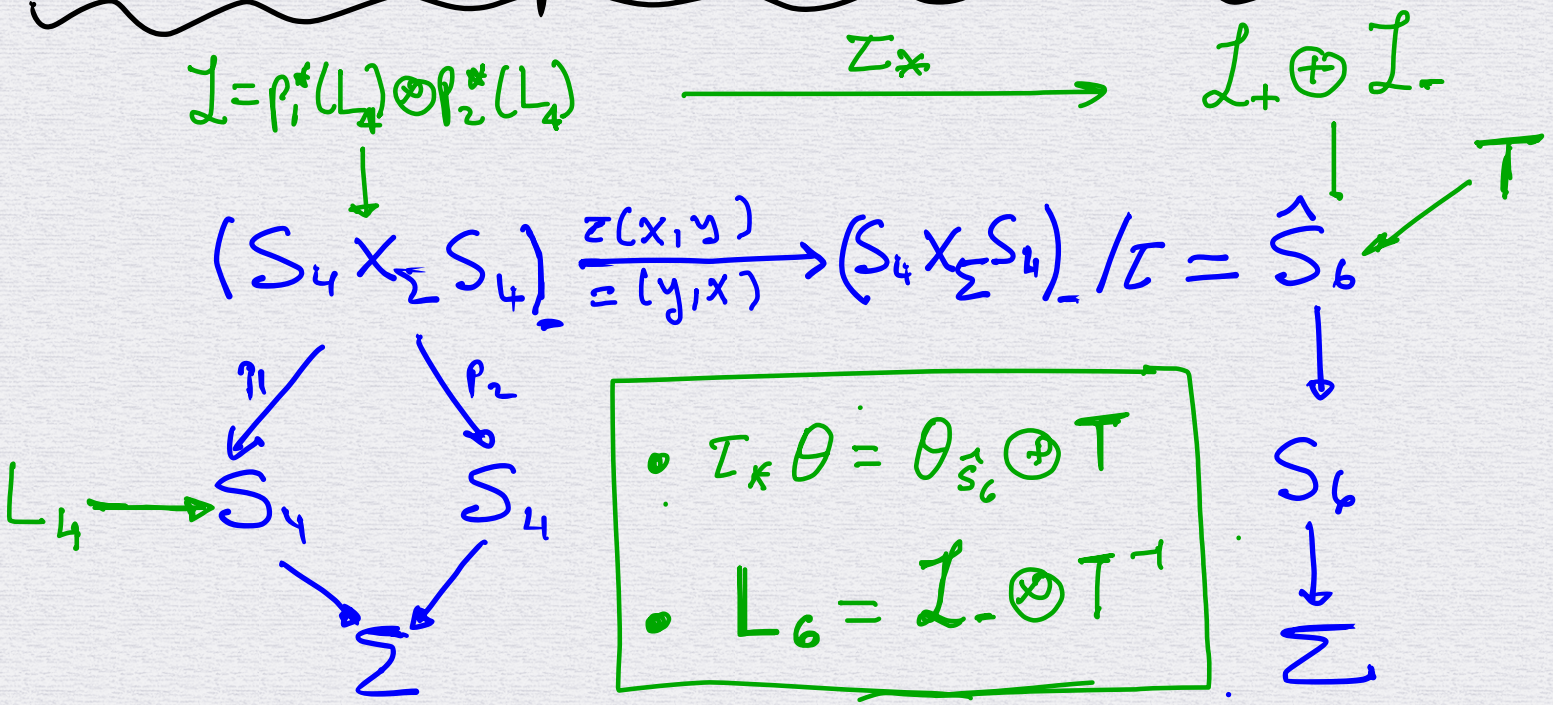
$$b_2 = -2a_2$$
$$b_3^2 = a_3^2$$
$$b_4 = a_2^2 - 4a_4$$

# "High-tech" (fiber product) description




- $\hat{S}_6$ generically smooth with involution $\sigma$: $[a,b] = [c,d]$
- $\hat{S}_6$ = normalization of $S_6$.

# The induced map on spectral line bundles




* Defines $\quad Prym(S_4, \Sigma) \longrightarrow Prym(\hat{S}_6, \hat{S}_6/\sigma)$

$$L_4 \otimes [R_4]^{-1/2} \longmapsto L_6 \otimes [R_6]^{-1/2}.$$

* a.k.a. Recillas Trigonal Construction !