Four-Particle-Scattering Amplitudes in String Theory

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High 1, Feb. 15. 2019

Four-Gauge-Particle Scattering Amplitudes in String Theory

1. String theory and its Low energy dynamics.
2. String theory offers scattering amplitudes.
3. Low energy dynamics of particle physics is described by QFT.
4. Tachyon in open string theory.
5. Realization of Higgs mechanism in string theory
6. Compatibility of string theory in low energy with SM model.
7. Tachyon in closed string theory and its implication.

Calculation of Four-Gauge-Particle Scattering Amplitudes in String Theory

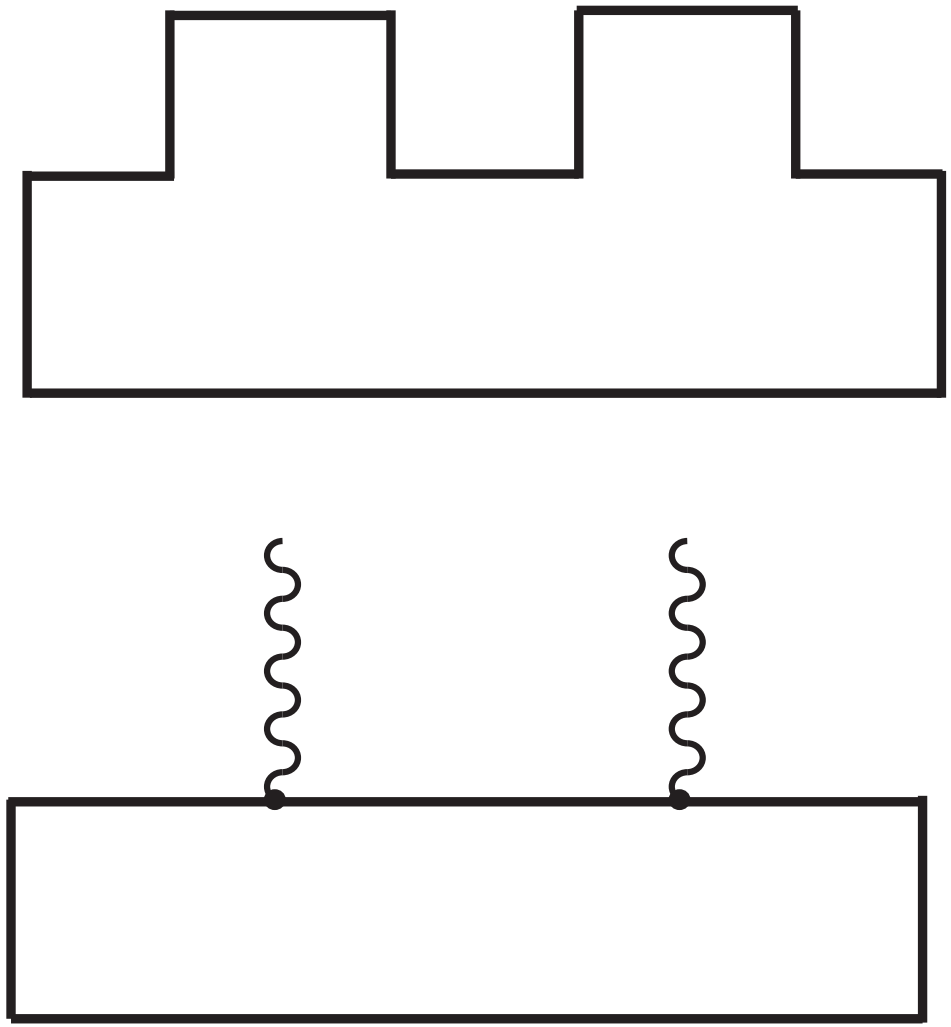
1. Conventional Method: Vertex Operators
2. Polyakov String Path Integral and String Field Theory
3. Three-Particle Scattering
4. Four-Gauge Particle Scattering and Four-Graviton Scattering Amplitudes
5. Comparison: Alpha Prime Corrections and Singularity Strutures
6. Non-locality, Unitarity, ...

String Scale and String Corrections

1. String scale may not be tied to the Planck scale. It can be low as the Weak scale.
2. J. D. Lykken, hep-th/9603133: *Weak scale superstrings* .
3. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, hep-th/9804398: *New dimenions at a millimeter to a Fermi and superstrings at a TeV*.

We may study the string physics directly at LHC.
Exact calculations of string corrections are urgent and important.

Four-String Scattering: Open String (Vertex Operators)



Four-Tachyon Scattering Amplitudes in String Theory: Veneziano Amplitude

1. Conventional Method using vertex Operators

$$\mathcal{F}^{\mathrm{T}}_{[4]}(s,t) = g^2 \frac{\Gamma\left(-\frac{s}{2}-1\right)\Gamma\left(-\frac{t}{2}-1\right)}{\Gamma\left(-\frac{s}{2}-\frac{t}{2}-2\right)}.$$

2. Expansion with s-channel poles,

$$\begin{aligned}\mathcal{F}^{\mathrm{T}}_{[4]}(s,t) &= -g^2 \sum_{n=0}^{\infty} \frac{(\alpha(t)+1)(\alpha(t)+2)\cdots(\alpha(t)+n)}{n!} \frac{1}{\alpha(s)-n} \\ &= -\frac{2g^2}{s+2} - \frac{g^2(t+4)}{s} - \frac{g^2}{4}\frac{(t+4)(t+6)}{s-2}\cdots\end{aligned}$$

where $\alpha(s) = 1 + s/2$ and $\alpha(t) = 1 + t/2$.

Polyakov String Path Integral

The multiple string scattering amplitude in terms of the Polyakov string path integral defined on the corresponding string world sheet:

$$\begin{aligned}\mathcal{A}_{[N]} &= \int D[X]D[h]\exp\left(iS + i\int_{\partial M}\sum_{r=1}^{N} P^{(r)}\cdot X^{(r)}d\sigma\right), \\ S &= -\frac{1}{4\pi}\int_M d\tau d\sigma\sqrt{-h}h^{IJ}\frac{\partial X^\mu}{\partial\sigma^I}\frac{\partial X^\nu}{\partial\sigma^J}\eta_{IJ}, \qquad I,J=0,\ldots,d-1\end{aligned}$$

where $\sigma^1=\tau$, $\sigma^2=\sigma$ and $d=26$ for open bosonic string and $d=10$ for open super-string.

Two-dimensional world sheet metric in terms of the lapse and shift functions (ADM formulation):

$$\sqrt{-h}h^{\alpha\beta} = \frac{1}{N_1}\begin{pmatrix} -1 & N_2 \\ N_2 & (N_1)^2-(N_2)^2 \end{pmatrix}.$$

Proper-time Gauge

- Two first-class constraints φ^1 and φ^2

$$\begin{aligned} \varphi_1 &= \frac{1}{2}\left(P^\mu P_\mu + \partial_\sigma X^\mu \partial_\sigma X_\mu\right), \\ \varphi_2 &= P^\mu \partial_\sigma X_\mu, \end{aligned}$$

generate the reparameterization

$$\begin{aligned} \delta N_1 &= \partial_\tau \epsilon_1 + \partial_\sigma N_1 \epsilon_2 - N_1 \partial_\sigma \epsilon_2 + \partial_\sigma N_2 \epsilon_1 - N_2 \partial_\sigma \epsilon_1, \\ \delta N_2 &= \partial_\tau \epsilon_2 + \partial_\sigma N_1 \epsilon_1 - N_1 \partial_\sigma \epsilon_1 + \partial_\sigma N_2 \epsilon_2 - N_2 \partial_\sigma \epsilon_2. \end{aligned}$$

- Proper-time gauge condition:

$$\partial_\tau N_{10} = 0, \quad N_{1n} = 0, \quad N_{2n} = 0, \quad n \neq 0,$$

where $N_\alpha = \sum_n N_{\alpha n} e^{in\sigma}$, $\alpha = 1, 2$. ($N_{20} = 0$ for open string.)

BRST Structure

- Line element

$$
\begin{aligned}
(dl)^2 &= h_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = -n_1 d\tau d\tau + \frac{1}{n_1} d\sigma d\sigma, \\
h_{\alpha\beta} &= \begin{pmatrix} -n_1 & 0 \\ 0 & 1/n_1 \end{pmatrix}.
\end{aligned}
$$

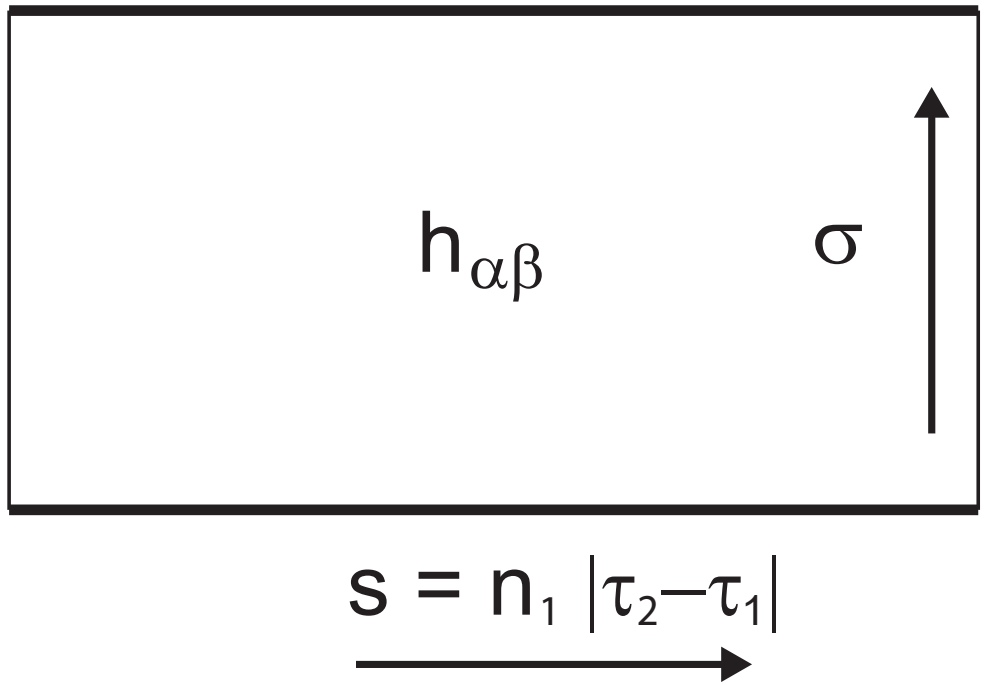
- Faddeev-Popov determinant

$$
\begin{aligned}
\det\left[\frac{\delta N_\alpha}{\delta \epsilon_\beta}\right] &= T \prod_{n=1} \det \begin{bmatrix} \partial_\tau + i n n_1 & 0 \\ 0 & \partial_\tau - i n n_1 \end{bmatrix} \\
&= T \int D[c,d] \exp\left[\int_{\tau_1}^{\tau_2} d\tau \sum_{n=1} \left(c_n \dot{d}_n - i n_1 L_0^{gh}\right)\right], \\
L_0^{gh} &= \sum_{n=1} n\, c_n d_n,
\end{aligned}
$$

where $T = |\tau_2 - \tau_1|$.

String Propagator

- Path integral over a strip



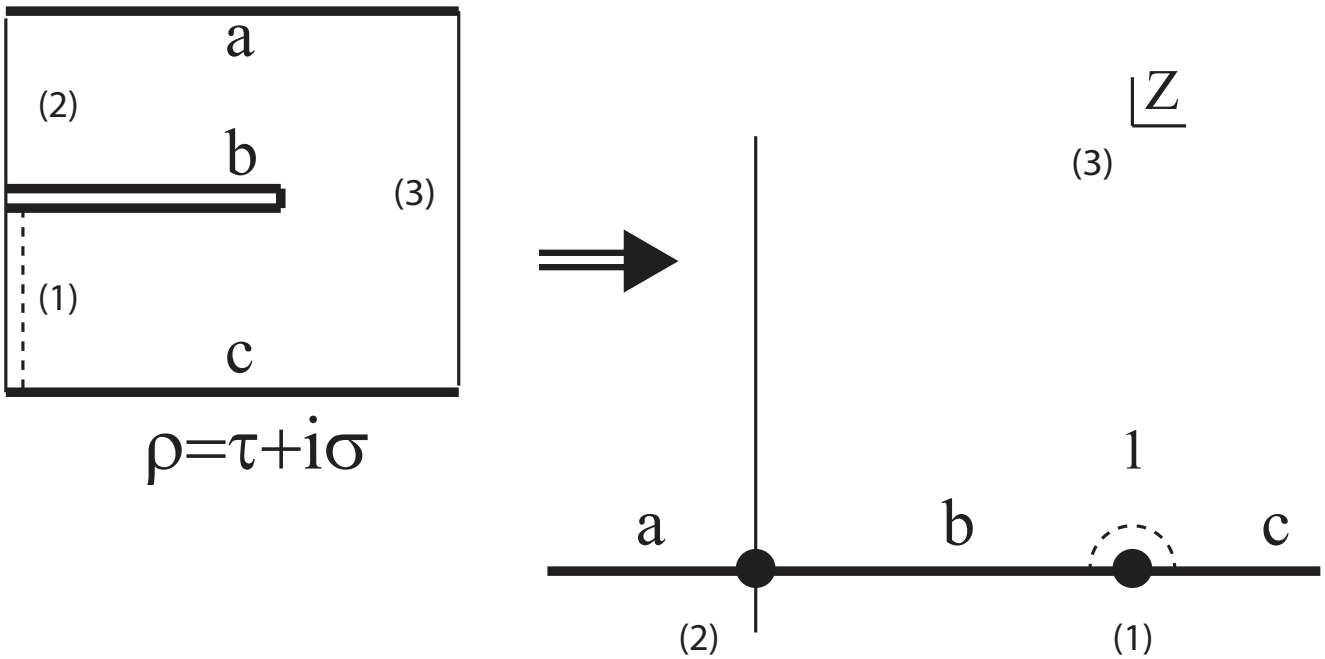
- String propagator:

$$
\begin{aligned}
& G[X_1^\mu, c_1, d_1; X_2^\nu, c_2, d_2] \\
= & \int_0^\infty ds \langle X_1^\mu, c_i, d_1 | \exp\left[-is\left(L_0 + L_{gh} - i\epsilon\right)\right] | X_2^\nu, c_2, d_2 \rangle \\
= & \langle X_1^\mu, c_1, d_1 | \frac{1}{L_0 + L_0^{gh} - i\epsilon} | X_2^\nu, c_2, d_2 \rangle, \\
L_0 = & \sum_{n=1} \left(P_n^2 + n^2 X_n^2\right),
\end{aligned}
$$

where $s = Tn_1$ is the proper-time.

Three-String Scattering Diagram

Mapping of the three-string vertex diagram onto the upper half complex plane



- Schwarz-Christoffel trnasformation (Mandelstam1973, Cremmer74, Cremmer75)

$$\rho = \ln(z-1) + \ln z.$$

- The local coordinates $\zeta_r = \xi_r + i\eta_r$, $r = 1, 2, 3$:

$$\begin{aligned} e^{-\zeta_1} &= e^{\tau_0} \frac{1}{z(z-1)}, \\ e^{-\zeta_2} &= -e^{\tau_0} \frac{1}{z(z-1)} \\ e^{-\zeta_3} &= -e^{-\frac{\tau_0}{2}} \sqrt{z(z-1)} \end{aligned}$$

- Green's functions on the upper half plane

$$\begin{aligned} G_N(z, z') &= \ln|z - z'| + \ln|z - z'^*|, \quad \text{for Neumann BC}, \\ G_D(z, z') &= \ln|z - z'| - \ln|z - z'^*|, \quad \text{for Dirichlet BC}. \end{aligned}$$

Neumann functions

$$
\begin{aligned}
N(\rho_r, \rho'_s) &= \ln|z - z'| + \ln|z - z'^*| \\
&= -\delta_{rs}\left\{\sum_{n\geq 1}\frac{2}{n}e^{-n|\xi_r - \xi'_s|}\cos(n\eta_r)\cos(n\eta'_s) - 2\max\left(\xi_r, \xi'_s\right)\right\} \\
&\quad +2\sum_{n,m\geq 0}\bar{N}^{rs}_{nm}e^{n\xi_r + m\xi'_s}\cos(n\eta_r)\cos(m\eta'_s).
\end{aligned}
$$

where ρ_r and ρ'_s lie in the region of the r-th and s-th string strip respectively,

Vertex for Multiple Strings

$$
\begin{aligned}
\mathcal{A}_{[N]} &= \langle P^{(1)}, \ldots, P^{(N)}| e^{\sum_r \frac{1}{2}\xi_r \left(p^{(r)}\right)^2} \\
&\quad \exp\left\{ \frac{1}{2} \sum_{r,s} \sum_{n,m\geq 0} \bar{N}^{rs}_{nm} e^{n\xi_r + m\xi_s}\, P^{(r)}_n \cdot P^{(s)}_m \right\} |0\rangle \\
&= \langle \mathbf{P}| \exp\left[\sum_r \xi_r L^{(r)}_0 \right] |V[N]\rangle .
\end{aligned}
$$

$$
|V[N]\rangle = e^{-\sum_r \bar{N}^{rr}_{00}} \exp\left\{ \frac{1}{2} \sum_{r,s} \sum_{n,m\geq 0} \bar{N}^{rs}_{nm}\, \alpha^{(r)\dagger}_n \cdot \alpha^{(r)\dagger}_n \right\} |0\rangle .
$$

Three-Open-String Scattering Amplitude

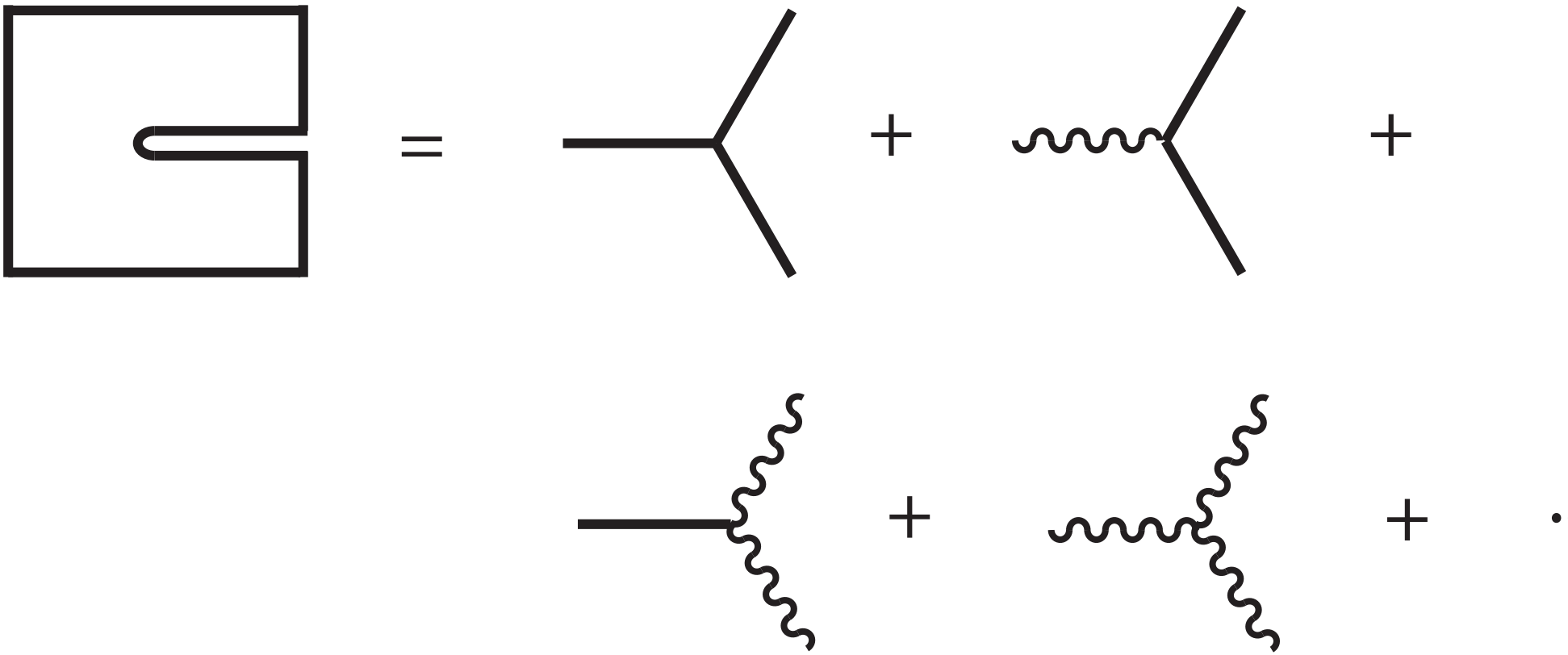
$$\begin{aligned}
\mathcal{I}_{[3]} &= \frac{2g}{3}\int\prod_{r=1}^{3}dp^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)\ \mathrm{tr}\,\langle\Psi^{(1)},\Psi^{(2)},\Psi^{(3)}| \\
&\quad \exp\Big\{E_{[3]}[1,2,3]\Big\}|0;p\rangle \\
E[1,2,3] &= \frac{1}{2}\sum_{n,m=1}^{\infty}\sum_{r,s=1}^{3}\bar{N}^{rs}_{nm}\alpha^{(r)}_{-n}\cdot\alpha^{(s)}_{-m}+\sum_{n=1}^{\infty}\sum_{r=1}^{3}\bar{N}^{r}_{n}\alpha^{(r)}_{-n}\cdot\mathbf{P}+ \\
&\quad \tau_0\sum_{r=1}^{3}\frac{1}{\alpha_r}\left(\frac{(p^{(r)})^2}{2}-1\right),
\end{aligned}$$

where $\mathbf{P}=p^{(2)}-p^{(1)}$ and $|\Psi^{(r)}\rangle$, $r=1,2,3$ denote the external string states which carry $U(N)$ group indices. For the proper-time gauge,

$$\alpha_1=1,\alpha_2=1,\alpha_3=-2,\text{ and }\tau_0=-2\ln 2.$$

Three-Open-String Scattering Amplitude

If we expand the external string states in terms of mass eigen-states,



Three-Open-String Scattering Amplitude

Choosing the external string state,

$$\langle \Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}| = \langle 0| \prod_{r}^{3} (\phi(r) + \mathbf{A}(r))$$

where $\phi(r) = \phi(p^{(r)})$, $\mathbf{A}(r) = A_\mu(p^{(r)}) a_1^{(r)\mu}$, $r = 1, 2, 3$.

$$\mathcal{I}_{[3]} = \mathcal{I}_{\phi\phi\phi} + \mathcal{I}_{\phi\phi A} + \mathcal{I}_{\phi AA} + \mathcal{I}_{AAA}.$$

$$\mathcal{I}_{\phi\phi\phi} = \frac{2g}{3} \int \prod_{r=1}^{3} dp^{(r)} \delta \left(\sum_{r=1}^{3} p^{(r)} \right) \operatorname{tr} \phi(1)\phi(2)\phi(3).$$

Three-Open-String Scattering Amplitude

$$
\begin{aligned}
\mathcal{I}_{\phi\phi A} &= \frac{2g}{3}\int\prod_{r=1}^{3}dp^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)\ \mathrm{tr}\,\langle 0|\big(\phi(1)\phi(2)\mathbf{A}(3)+ \\
&\quad \phi(1)\mathbf{A}(2)\phi(3)+\mathbf{A}(1)\phi(2)\phi(3)\big)e^{\tau_0\sum_{r=1}^{3}\frac{1}{\alpha_r}\left(\frac{(p^{(r)})^2}{2}-1\right)} \\
&\quad \left(\sum_{r=1}^{3}\bar{N}_1^r a_1^{(r)\dagger}\cdot\mathbf{P}\right)|0\rangle \\
&= g\int\prod_{r=1}^{3}dp^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)\ \mathrm{tr}\,p_\mu^{(1)}\phi(p(1))\left[A^\mu(p^{(2)}),\phi(p^{(3)})\right] \\
&= -gi\int d^dx\,\mathrm{tr}\,\partial_\mu\phi\,[A^\mu,\phi]\,.
\end{aligned}
$$

Three-Open-String Scattering Amplitude

$$
\begin{aligned}
\mathcal{I}_{\phi AA} &= \frac{2g}{3}\int\prod_{r=1}^{3}dp^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)\,\mathrm{tr}\,\langle 0|\big(\phi(1)\mathbf{A}(2)\mathbf{A}(3)+ \\
&\quad \mathbf{A}(1)\phi(2)\mathbf{A}(3)+\mathbf{A}(1)\mathbf{A}(2)\phi(3)\big)e^{\tau_0\sum_{r=1}^{3}\frac{1}{\alpha_r}\left(\frac{(p^{(r)})^2}{2}-1\right)} \\
&\quad \left\{\frac{1}{2}\sum_{r,s=1}^{3}\bar{N}_{11}^{rs}a_1^{(r)\dagger}\cdot a_1^{(s)\dagger}+\frac{1}{2!}\left(\sum_{r=1}^{3}\bar{N}_1^{r}a_1^{(r)\dagger}\cdot\mathbf{P}\right)^2\right\}|0\rangle \\
&= g\int d^dx\,\mathrm{tr}\,\left(\phi A_\mu A^\mu+\phi\,\partial_\mu A^\nu\partial_\nu A^\mu\right).
\end{aligned}
$$

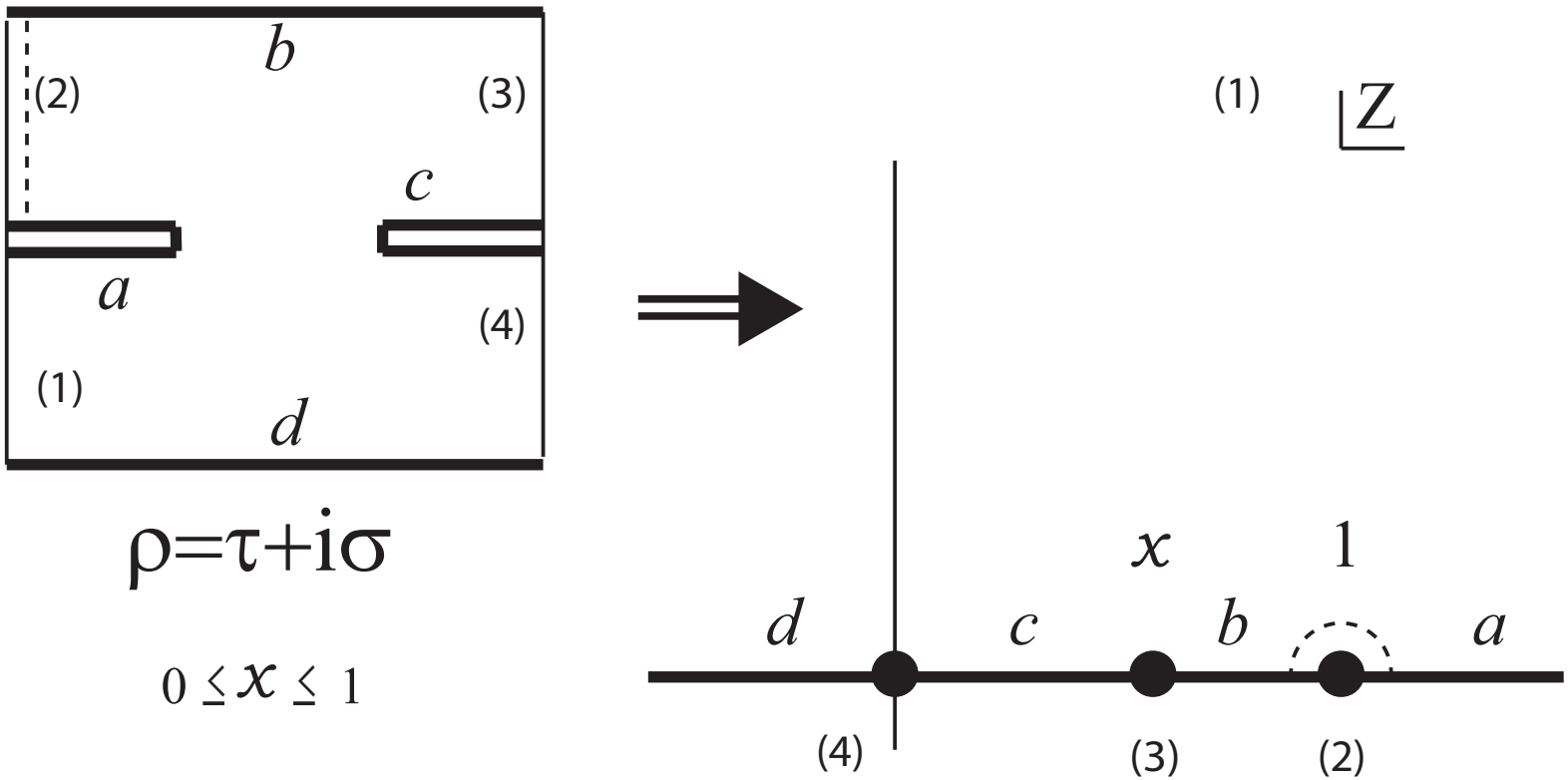
Three-Open-String Scattering Amplitude

$$
\begin{aligned}
\mathcal{I}_{AAA} &= \frac{2^4 g}{3} \int \prod_{r=1}^{3} dp^{(r)} \delta \left(\sum_{r=1}^{3} p^{(r)} \right) \operatorname{tr} \langle 0 | \mathbf{A}(1)\mathbf{A}(2)\mathbf{A}(3) \\
&\quad \left\{ \frac{1}{2} \sum_{r,s=1}^{3} \bar{N}_{11}^{rs} a_1^{(r)\dagger} \cdot a_1^{(s)\dagger} \left(\sum_{r=1}^{3} \bar{N}_1^r a_1^{(r)\dagger} \cdot \mathbf{P} \right) + \right. \\
&\quad \left. \frac{1}{3!} \left(\sum_{r=1}^{3} \bar{N}_1^r a_1^{(r)\dagger} \cdot \mathbf{P} \right)^3 \right\} |0\rangle \\
&= g \int d^d x \; i \operatorname{tr} \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) [A^\mu, A^\nu] + \\
&\quad \alpha' \frac{g}{3} i \int d^d x \operatorname{tr} \left(\partial_\mu A^\nu - \partial^\nu A_\mu \right) \left(\partial_\nu A^\lambda - \partial^\lambda A_\nu \right) \left(\partial_\lambda A^\mu - \partial^\mu A_\lambda \right) .
\end{aligned}
$$

Alpha prime correction.

Schwarz-Christoffel Mapping for Four-String Vertex

$$\begin{aligned} \rho &= \ln(z-1) - \ln(z-x) - \ln z. \\ Z_1 &= \infty, \quad Z_2 = 1, \quad Z_3 = x, \quad Z_4 = 0. \end{aligned}$$



Four-Open-String Scattering Amplitude

$$\mathcal{A}_{[4]} = 2g^2 \int \left| \frac{\prod_{r=1}^{4} dZ_r}{dV_{abc}} \right| \prod_{r<s} |Z_r - Z_s|^{p_r \cdot p_s} \exp\left[- \sum_{r=1}^{4} \bar{N}_{00}^{[4]rr} \right]$$
$$\mathrm{tr} \left\langle \psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \psi^{(4)} \middle| \exp\left[E_{[4]} \right] \middle| 0 \right\rangle,$$

$$E_{[4]} = \frac{1}{2} \sum_{r,s=1}^{4} \sum_{m,n>0} \bar{N}_{mn}^{[4]rs} \alpha_{m\mu}^{(r)\dagger} \alpha_{n\nu}^{(s)\dagger} \eta^{\mu\nu} + \sum_{r,s=1}^{4} \sum_{n>0} \bar{N}_{n0}^{[4]rs} \alpha_{n\mu}^{(r)\dagger} p_{\nu}^{(s)\dagger} \eta^{\mu\nu}$$

where Z_r $r = 1, 2, 3, 4$ denote the Koba-Nelson variables,

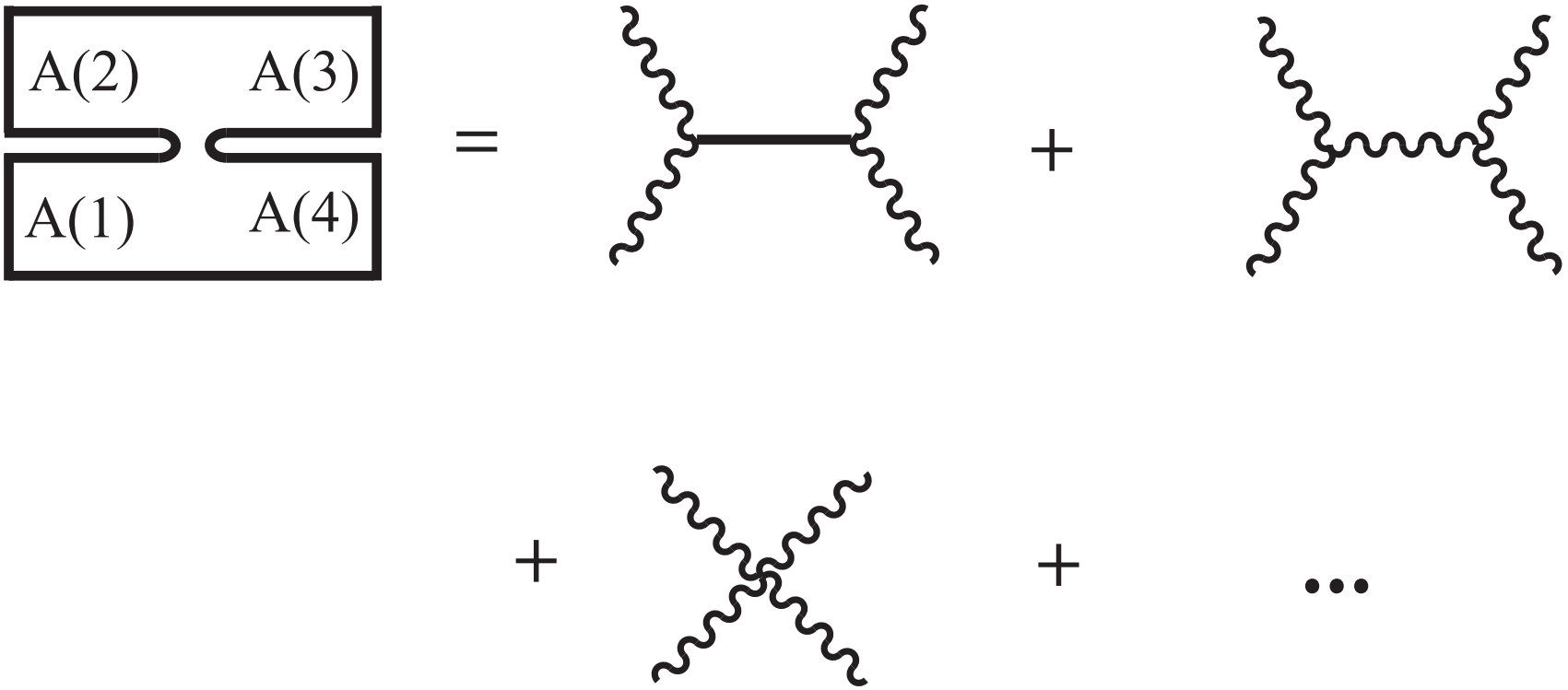
$$Z_1 = 0, \quad Z_2 = x, \quad Z_3 = 1, \quad Z_4 = \infty.$$

In the proper-time gauge, we fix the length parameters as

$$\alpha_1 = 1, \quad \alpha_2 = 1, \quad \alpha_3 = -1, \quad \alpha_4 = -1.$$

Four-Open-String Scattering Amplitude

If we choose the four-gauge-particle-state as the external string state,



Four-Open-String Scattering Amplitude

$$
\begin{aligned}
\mathcal{I}_{AAAA} \;=\; & 2g^2 \int \left| \frac{\prod_{r=1}^{4} dZ_r}{dV_{abc}} \right| \prod_{r<s} |Z_r - Z_s|^{p_r \cdot p_s} \exp\left[-\sum_{r=1}^{4} \bar{N}_{00}^{[4]rr} \right] \\
& \operatorname{tr} \langle 0 | \mathbf{A}(1)\mathbf{A}(2)\mathbf{A}(3)\mathbf{A}(4) \\
& \left\{ \frac{1}{2!} \left(\frac{1}{2} \sum_{r,s=1}^{4} \bar{N}_{11}^{[4]rs} a_1^{(r)\dagger} \cdot a_1^{(s)\dagger} \right)^2 + \right. \\
& \left(\frac{1}{2} \sum_{r,s=1}^{4} \bar{N}_{11}^{[4]rs} a_1^{(r)\dagger} \cdot a_1^{(s)\dagger} \right) \frac{1}{2!} \left(\sum_{r,s=1}^{4} \bar{N}_{10}^{[4]rs} a_1^{(r)\dagger} \cdot p^{(s)} \right)^2 \\
& \left. + \frac{1}{4!} \left(\sum_{r,s=1}^{4} \bar{N}_{10}^{[4]rs} a_1^{(r)\dagger} \cdot p^{(s)} \right)^4 \right\} |0\rangle .
\end{aligned}
$$

Four-Open-String Scattering Amplitude

$$\mathcal{I}^{(0)}_{AAAA} = \mathrm{tr}\Big\{A_{\mu_1}(1)A_{\mu_2}(2)A_{\mu_3}(3)A_{\mu_4}(4)\Big\}\frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)}{\Gamma\left(\frac{u}{2}+1\right)}$$
$$\left\{\frac{tu}{4}\frac{1}{s/2+1}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{4}\frac{1}{u/2+1}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\right.$$
$$\left. + \frac{us}{4}\frac{1}{t/2+1}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\right\}$$

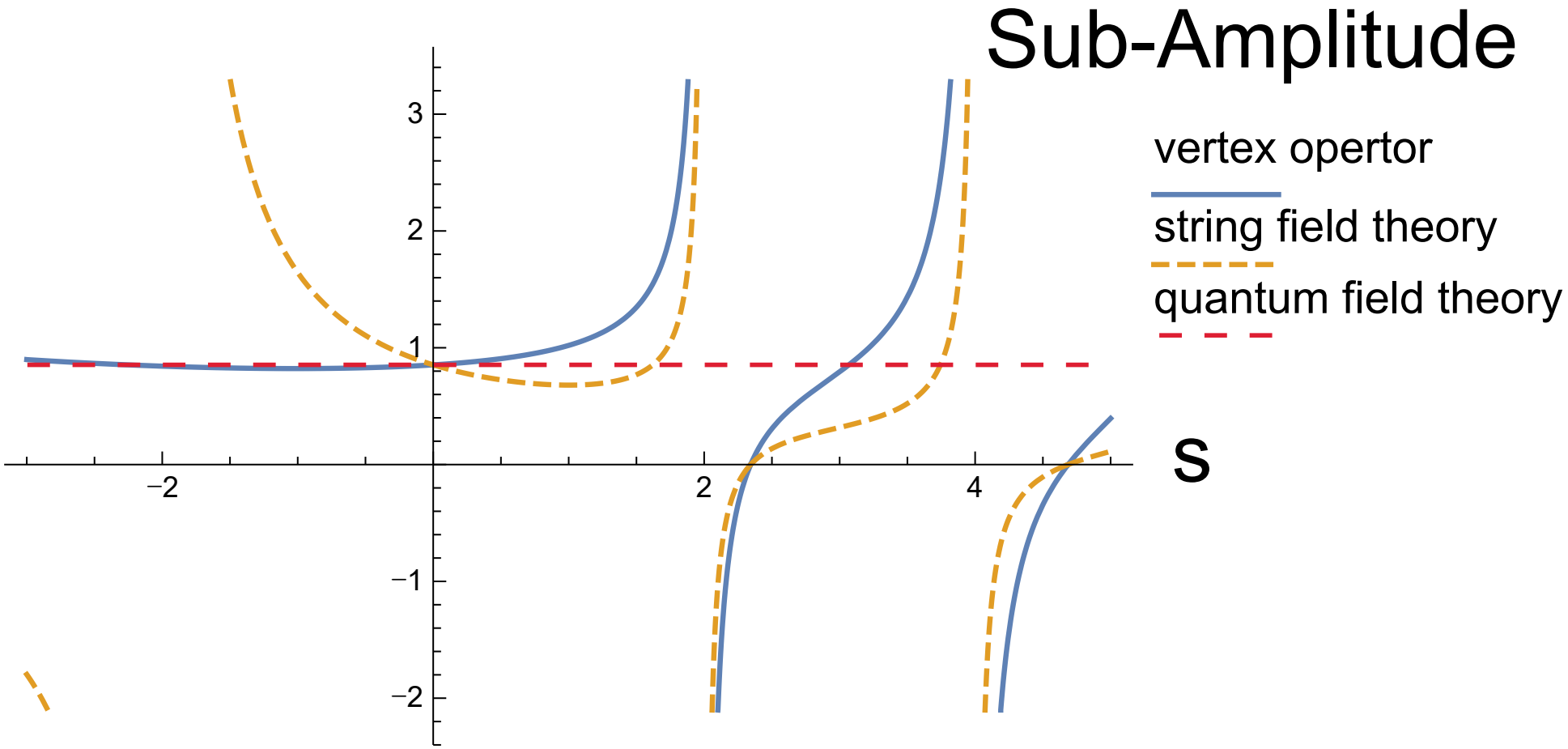
$\mathcal{I}^{(0)}_{AAAA}$ has a tachyon pole which has been missing in conventional calculation by using vertex operator.

Four-Open-String Scattering Amplitude

$$\mathcal{I}^{(0)}_{AAAA} = \mathrm{tr}\Big\{A_{\mu_1}(1)A_{\mu_2}(2)A_{\mu_3}(3)A_{\mu_4}(4)\Big\}\frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)}{\Gamma\left(\frac{u}{2}+1\right)}$$
$$\left\{\frac{tu}{4}\frac{1}{s/2+1}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{4}\frac{1}{u/2+1}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4}\right.$$
$$\left. + \frac{us}{4}\frac{1}{t/2+1}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}\right\}$$

$\mathcal{I}^{(0)}_{AAAA}$ has a tachyon pole which has been missing in conventional calculation by using vertex operator.

Sub-amplitude of four-gauge-particle scattering amplitude



$$\mathcal{I}^{(1)}_{AAAA} =$$

$$\mathrm{tr}\Big\{A_{\mu_1}(1)A_{\mu_2}(2)A_{\mu_3}(3)A_{\mu_4}(4)\Big\}\frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)}{\Gamma\left(\frac{u}{2}+1\right)}$$

$$\Bigg\{-\frac{1}{4}\eta^{\mu_1\mu_2}\left(-\frac{s}{2}p^{(1)\mu_3}p^{(2)\mu_4}+\frac{u}{2}p^{(1)\mu_3}p^{(3)\mu_4}+\frac{u}{2}p^{(4)\mu_3}p^{(2)\mu_4}-\frac{u(u-2)}{4}\frac{1}{\frac{s}{2}+1}p^{(4)\mu_3}p^{(3)\mu_4}\right)$$

$$+\frac{1}{4}\eta^{\mu_1\mu_3}\left(-\frac{s}{2}p^{(1)\mu_2}p^{(2)\mu_4}+\frac{u}{2}p^{(1)\mu_2}p^{(3)\mu_4}+\frac{s(s-2)}{4}\frac{1}{\frac{u}{2}+1}p^{(4)\mu_2}p^{(2)\mu_4}-\frac{s}{2}p^{(4)\mu_2}p^{(3)\mu_4}\right)$$

$$-\frac{1}{4}\eta^{\mu_1\mu_4}\left(-\frac{t}{2}p^{(4)\mu_2}p^{(1)\mu_3}+\frac{u}{2}p^{(4)\mu_2}p^{(2)\mu_3}+\frac{u}{2}p^{(3)\mu_2}p^{(1)\mu_3}-\frac{u(u-2)}{4}\frac{1}{\frac{t}{2}+1}p^{(3)\mu_2}p^{(2)\mu_3}\right)$$

$$-\frac{1}{4}\eta^{\mu_2\mu_3}\left(-\frac{t}{2}p^{(3)\mu_1}p^{(2)\mu_4}+\frac{u}{2}p^{(3)\mu_1}p^{(1)\mu_4}+\frac{u}{2}p^{(4)\mu_1}p^{(2)\mu_4}-\frac{u(u-2)}{4}\frac{1}{\frac{t}{2}+1}p^{(4)\mu_1}p^{(1)\mu_4}\right)$$

$$+\frac{1}{4}\eta^{\mu_2\mu_4}\left(\frac{s(s-2)}{4}\frac{1}{\frac{u}{2}+1}p^{(3)\mu_1}p^{(1)\mu_3}-\frac{s}{2}p^{(3)\mu_1}p^{(4)\mu_3}-\frac{s}{2}p^{(2)\mu_1}p^{(1)\mu_3}+\frac{u}{2}p^{(2)\mu_1}p^{(4)\mu_3}\right)$$

$$-\frac{1}{4}\eta^{\mu_3\mu_4}\left(-\frac{s}{2}p^{(3)\mu_1}p^{(4)\mu_2}+\frac{u}{2}p^{(3)\mu_1}p^{(1)\mu_2}+\frac{u}{2}p^{(2)\mu_1}p^{(4)\mu_2}-\frac{u(u-2)}{4}\frac{1}{\frac{s}{2}+1}p^{(2)\mu_1}p^{(1)\mu_2}\right)\Bigg\}.$$

Scattering amplitude $\mathcal{I}^{(1)}_{AAAA}$ contains the tachyon poles in all three channels.

Comparison with Conventional Vertex Operator Calculation

$$
\begin{aligned}
\mathcal{I}_{AAAA}^{\text{vertex}(1)} &= \mathrm{tr}\Big\{A_{\mu_1}(1)A_{\mu_2}(2)A_{\mu_3}(3)A_{\mu_4}(4)\Big\}\frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)}{\Gamma\left(\frac{u}{2}+1\right)} \\
&\quad \Bigg\{\frac{\eta^{\mu_1\mu_2}}{2}\left(tp^{(1)\mu_3}p^{(2)\mu_4}+up^{(2)\mu_3}p^{(1)\mu_4}\right) \\
&\quad +\frac{\eta^{\mu_1\mu_3}}{2}\left(tp^{(1)\mu_2}p^{(3)\mu_4}+sp^{(3)\mu_2}p^{(1)\mu_4}\right) \\
&\quad +\frac{\eta^{\mu_1\mu_4}}{2}\left(up^{(1)\mu_2}p^{(4)\mu_3}+sp^{(4)\mu_2}p^{(1)\mu_3}\right) \\
&\quad +\frac{\eta^{\mu_2\mu_3}}{2}\left(up^{(2)\mu_1}p^{(3)\mu_4}+sp^{(3)\mu_1}p^{(2)\mu_4}\right) \\
&\quad +\frac{\eta^{\mu_2\mu_4}}{2}\left(tp^{(2)\mu_1}p^{(4)\mu_3}+sp^{(4)\mu_1}p^{(2)\mu_3}\right) \\
&\quad +\frac{\eta^{\mu_3\mu_4}}{2}\left(tp^{(3)\mu_1}p^{(4)\mu_2}+up^{(4)\mu_1}p^{(3)\mu_2}\right)\Bigg\}.
\end{aligned}
$$

Kinematic Factor: Proper-Time Gauge

$$
\begin{aligned}
K^{\mu_1\mu_2\mu_3\mu_4}_{\text{proper}} = & -\frac{ut}{4\left(1+\frac{s}{8}\right)}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \frac{st}{4\left(1+\frac{u}{8}\right)}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \frac{us}{4\left(1+\frac{t}{8}\right)}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\
& + \frac{1}{2}\eta^{\mu_1\mu_2}\left(-sp^{(1)\mu_3}p^{(2)\mu_4} + up^{(1)\mu_3}p^{(3)\mu_4} + up^{(4)\mu_3}p^{(2)\mu_4} + \frac{u\left(1-\frac{u}{8}\right)}{\left(1+\frac{s}{8}\right)}p^{(4)\mu_3}p^{(3)\mu_4}\right) \\
& - \frac{1}{2}\eta^{\mu_1\mu_3}\left(-sp^{(1)\mu_2}p^{(2)\mu_4} + up^{(1)\mu_2}p^{(3)\mu_4} - \frac{s\left(1-\frac{s}{8}\right)}{\left(1+\frac{u}{8}\right)}p^{(4)\mu_2}p^{(2)\mu_4} - sp^{(4)\mu_2}p^{(3)\mu_4}\right) \\
& + \frac{1}{2}\eta^{\mu_1\mu_4}\left(-tp^{(4)\mu_2}p^{(1)\mu_3} + up^{(4)\mu_2}p^{(2)\mu_3} + up^{(3)\mu_2}p^{(1)\mu_3} + \frac{u\left(1-\frac{u}{8}\right)}{\left(1+\frac{t}{8}\right)}p^{(3)\mu_2}p^{(2)\mu_3}\right) \\
& + \frac{1}{2}\eta^{\mu_2\mu_3}\left(-tp^{(3)\mu_1}p^{(2)\mu_4} + up^{(3)\mu_1}p^{(1)\mu_4} + up^{(4)\mu_1}p^{(2)\mu_4} + \frac{u\left(1-\frac{u}{8}\right)}{\left(1+\frac{t}{8}\right)}p^{(4)\mu_1}p^{(1)\mu_4}\right) \\
& - \frac{1}{2}\eta^{\mu_2\mu_4}\left(-\frac{s\left(1-\frac{s}{8}\right)}{\left(1+\frac{u}{8}\right)}p^{(3)\mu_1}p^{(1)\mu_3} - sp^{(3)\mu_1}p^{(4)\mu_3} - sp^{(2)\mu_1}p^{(1)\mu_3} + up^{(2)\mu_1}p^{(4)\mu_3}\right) \\
& + \frac{1}{2}\eta^{\mu_3\mu_4}\left(-sp^{(3)\mu_1}p^{(4)\mu_2} + up^{(3)\mu_1}p^{(1)\mu_2} + up^{(2)\mu_1}p^{(4)\mu_2} + \frac{u\left(1-\frac{u}{8}\right)}{\left(1+\frac{s}{8}\right)}p^{(2)\mu_1}p^{(1)\mu_2}\right) .
\end{aligned}
$$

Kinematic Factor: Vertex Opertor

$$
\begin{aligned}
K_{\text{vertex}}^{\mu_1\mu_2\mu_3\mu_4} \quad = \quad & -\frac{tu}{4}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \frac{st}{4}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} - \frac{us}{4}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\
& + \frac{\eta^{\mu_1\mu_2}}{2}\left(tp^{(1)\mu_3}p^{(2)\mu_4} + up^{(2)\mu_3}p^{(1)\mu_4}\right) \\
& + \frac{\eta^{\mu_1\mu_3}}{2}\left(tp^{(1)\mu_2}p^{(3)\mu_4} + sp^{(3)\mu_2}p^{(1)\mu_4}\right) \\
& + \frac{\eta^{\mu_1\mu_4}}{2}\left(up^{(1)\mu_2}p^{(4)\mu_3} + sp^{(4)\mu_2}p^{(1)\mu_3}\right) \\
& + \frac{\eta^{\mu_2\mu_3}}{2}\left(up^{(2)\mu_1}p^{(3)\mu_4} + sp^{(3)\mu_1}p^{(2)\mu_4}\right) \\
& + \frac{\eta^{\mu_2\mu_4}}{2}\left(tp^{(2)\mu_1}p^{(4)\mu_3} + sp^{(4)\mu_1}p^{(2)\mu_3}\right) \\
& + \frac{\eta^{\mu_3\mu_4}}{2}\left(tp^{(3)\mu_1}p^{(4)\mu_2} + up^{(4)\mu_1}p^{(3)\mu_2}\right).
\end{aligned}
$$

$$
K_{\text{proper}}^{\mu_1\mu_2\mu_3\mu_4} \xrightarrow{\text{zero-slope limit}} K_{\text{vertex}}^{\mu_1\mu_2\mu_3\mu_4}.
$$

Alpha prime corrections

$$
\begin{aligned}
&\mathcal{I}^{(2)}_{AAAA} \\
= \quad & (2g^2)\mathrm{tr}\left\{ A(1)^{\mu_1} A(2)^{\mu_2} A(3)^{\mu_3} A(4)^{\mu_4} \right\} \frac{\Gamma\left(-\frac{s}{2}\right)\Gamma\left(-\frac{t}{2}\right)}{\Gamma\left(\frac{u}{2}+1\right)} \\
& \frac{1}{\left(\frac{u}{2}+1\right)} \left\{ \frac{\frac{t}{2}\left(\frac{t}{2}-1\right)\left(\frac{t}{2}-2\right)}{\left(\frac{s}{2}+1\right)} p^{(2)}_{\mu_1} p^{(1)}_{\mu_2} p^{(4)}_{\mu_3} p^{(3)}_{\mu_4} \right. \\
& -\frac{t}{2}\left(\frac{t}{2}-1\right)\left(p^{(2)}_{\mu_1} p^{(1)}_{\mu_2} p^{(4)}_{\mu_3} p^{(1)}_{\mu_4} + p^{(2)}_{\mu_1} p^{(1)}_{\mu_2} p^{(2)}_{\mu_3} p^{(3)}_{\mu_4} + p^{(2)}_{\mu_1} p^{(3)}_{\mu_2} p^{(4)}_{\mu_3} p^{(3)}_{\mu_4} + p^{(4)}_{\mu_1} p^{(1)}_{\mu_2} p^{(4)}_{\mu_3} p^{(3)}_{\mu_4} \right) \\
& +\frac{st}{4}\left(p^{(2)}_{\mu_1} p^{(1)}_{\mu_2} p^{(2)}_{\mu_3} p^{(1)}_{\mu_4} + p^{(2)}_{\mu_1} p^{(3)}_{\mu_2} p^{(4)}_{\mu_3} p^{(1)}_{\mu_4} + p^{(2)}_{\mu_1} p^{(3)}_{\mu_2} p^{(2)}_{\mu_3} p^{(3)}_{\mu_4} \right. \\
& \left. + p^{(4)}_{\mu_1} p^{(1)}_{\mu_2} p^{(4)}_{\mu_3} p^{(1)}_{\mu_4} + p^{(4)}_{\mu_1} p^{(1)}_{\mu_2} p^{(2)}_{\mu_3} p^{(3)}_{\mu_4} + p^{(4)}_{\mu_1} p^{(3)}_{\mu_2} p^{(4)}_{\mu_3} p^{(3)}_{\mu_4} + \right) \\
& -\frac{s}{2}\left(\frac{s}{2}-1\right)\left(p^{(2)}_{\mu_1} p^{(3)}_{\mu_2} p^{(2)}_{\mu_3} p^{(1)}_{\mu_4} + p^{(4)}_{\mu_1} p^{(1)}_{\mu_2} p^{(2)}_{\mu_3} p^{(1)}_{\mu_4} + p^{(4)}_{\mu_1} p^{(3)}_{\mu_2} p^{(4)}_{\mu_3} p^{(1)}_{\mu_4} + p^{(4)}_{\mu_1} p^{(3)}_{\mu_2} p^{(2)}_{\mu_3} p^{(3)}_{\mu_4} \right) \\
& \left. + \frac{\frac{s}{2}\left(\frac{s}{2}-1\right)\left(\frac{s}{2}-2\right)}{\left(\frac{t}{2}+1\right)} p^{(4)}_{\mu_1} p^{(3)}_{\mu_2} p^{(2)}_{\mu_3} p^{(1)}_{\mu_4} \right\} \sim \mathcal{O}(F^4).
\end{aligned}
$$

Closed String Scattering

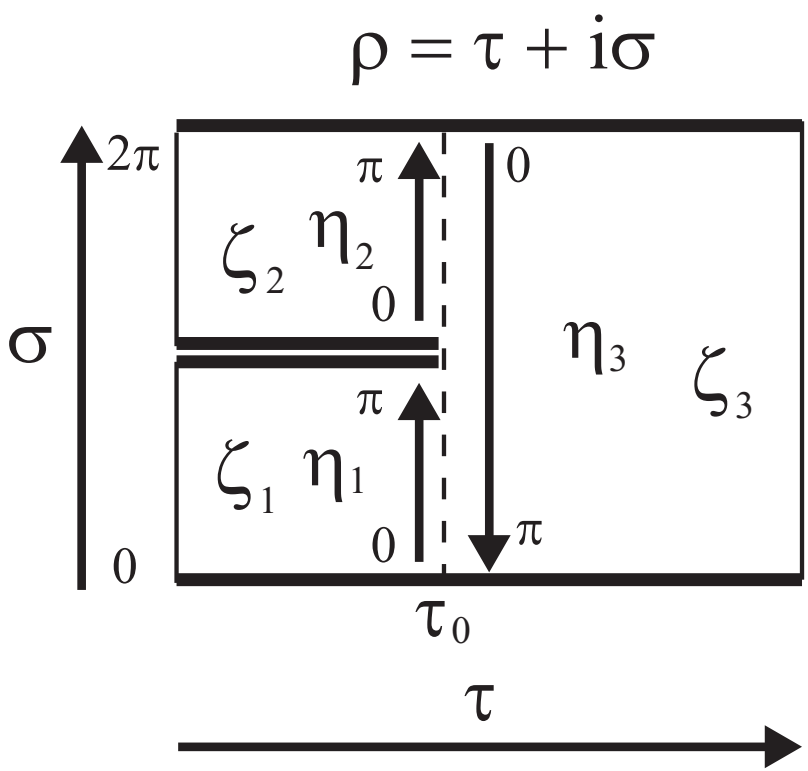
$$\mathcal{A}_{[N]} = g^2 \int D[X]D[h] \exp\left(iS + i \int_{\partial M} \sum_{r=1}^{3} P^{(r)} \cdot X^{(r)} d\sigma \right),$$

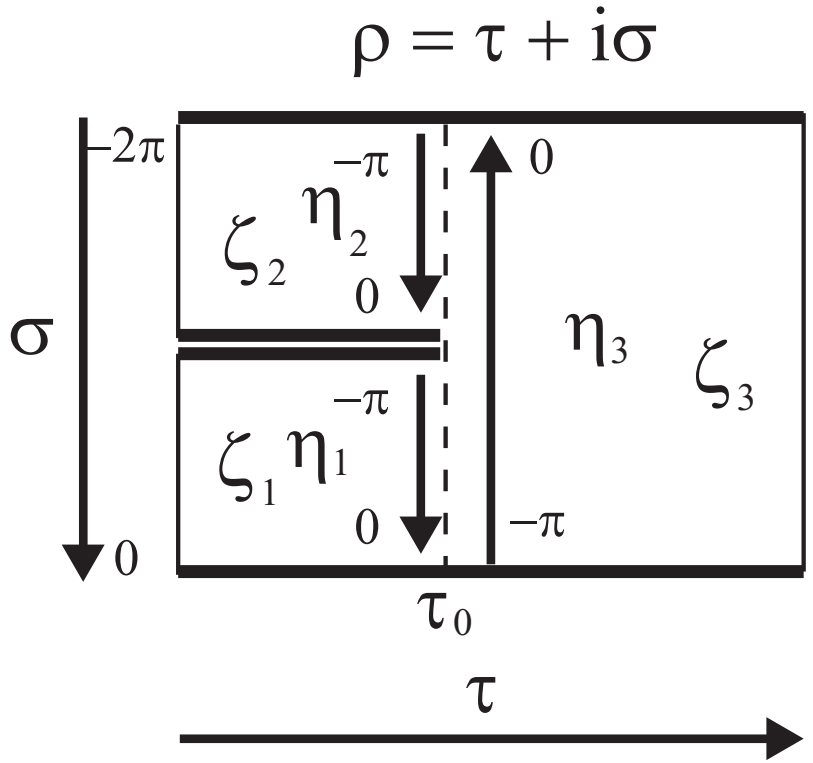
$$S = -\frac{1}{4\pi} \int_M d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}, \ \ \mu, \nu = 0, \ldots, d-1.$$

$$X_L(\tau, \sigma) = x_L + \sqrt{\frac{1}{2}} p_L(\tau + \sigma) + i\sqrt{\frac{1}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in(\tau+\sigma)},$$

$$X_R(\tau, \sigma) = x_R + \sqrt{\frac{1}{2}} p_R(\tau - \sigma) + i\sqrt{\frac{1}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n e^{-in(\tau-\sigma)}.$$

Local Coordinates



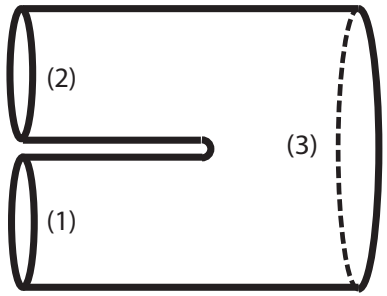


Neumann Functions of the Closed String Vertices

Fourier components of the Green's function on complex plane

$$
\begin{aligned}
G_C(\rho_r, \rho'_s) &= \ln|z_r - z'_s| \\
&= -\delta_{rs}\left\{\sum_{n=1}\frac{e^{-n\Delta}}{2n}\left(e^{in(\eta'_s-\eta_r)}+e^{-in(\eta'_s-\eta_r)}\right)-\max(\xi,\xi')\right\} \\
&\quad +\sum_{n,m}\bar{C}^{rs}_{nm}e^{|n|\xi_r+|m|\xi'_s}e^{in\eta_r}e^{im\eta'_s}.
\end{aligned}
$$

Three-Closed-String Scattering



$$
\begin{aligned}
|V[3]_{\text{closed}}\rangle \;=\; & \exp\left\{\sum_{r,s}\Big(\sum_{n,m\geq 1}\frac{1}{2}\bar{N}^{rs}_{nm}\,\alpha^{(r)\dagger}_{n}\cdot\alpha^{(r)\dagger}_{m}+\sum_{n\geq 1}\bar{N}^{rs}_{n0}\,\alpha^{(r)\dagger}_{n}\cdot p^{(s)}\Big)\right\} \\
& \exp\left\{\tau_0\sum_{r}\frac{1}{\alpha_r}\left(\frac{1}{2}\left(p^{(r)}\right)^2-1\right)\right\} \\
& \exp\left\{\sum_{r,s}\Big(\sum_{n,m\geq 1}\frac{1}{2}\bar{N}^{rs}_{nm}\,\tilde{\alpha}^{(r)\dagger}_{n}\cdot\tilde{\alpha}^{(r)\dagger}_{m}+\sum_{n\geq 1}\bar{N}^{rs}_{n0}\,\tilde{\alpha}^{(r)\dagger}_{n}\cdot p^{(s)}\Big)\right\} \\
& \exp\left\{\tau_0\sum_{r}\frac{1}{\alpha_r}\left(\frac{1}{2}\left(p^{(r)}\right)^2-1\right)\right\}|0;p\rangle.
\end{aligned}
$$

Factorization of Three-Closed-String Scattering Amplitude

Factorization of three-closed-string scattering amplitude

$$\mathcal{A}_{\text{closed}}[1,2,3] = \mathcal{A}_{\text{open}}[1,2,3]\,\mathcal{A}_{\text{open}}[1,2,3].$$

Scattering amplitude of three closed strings can be completely factorized into those of three open strings except for the zero modes.

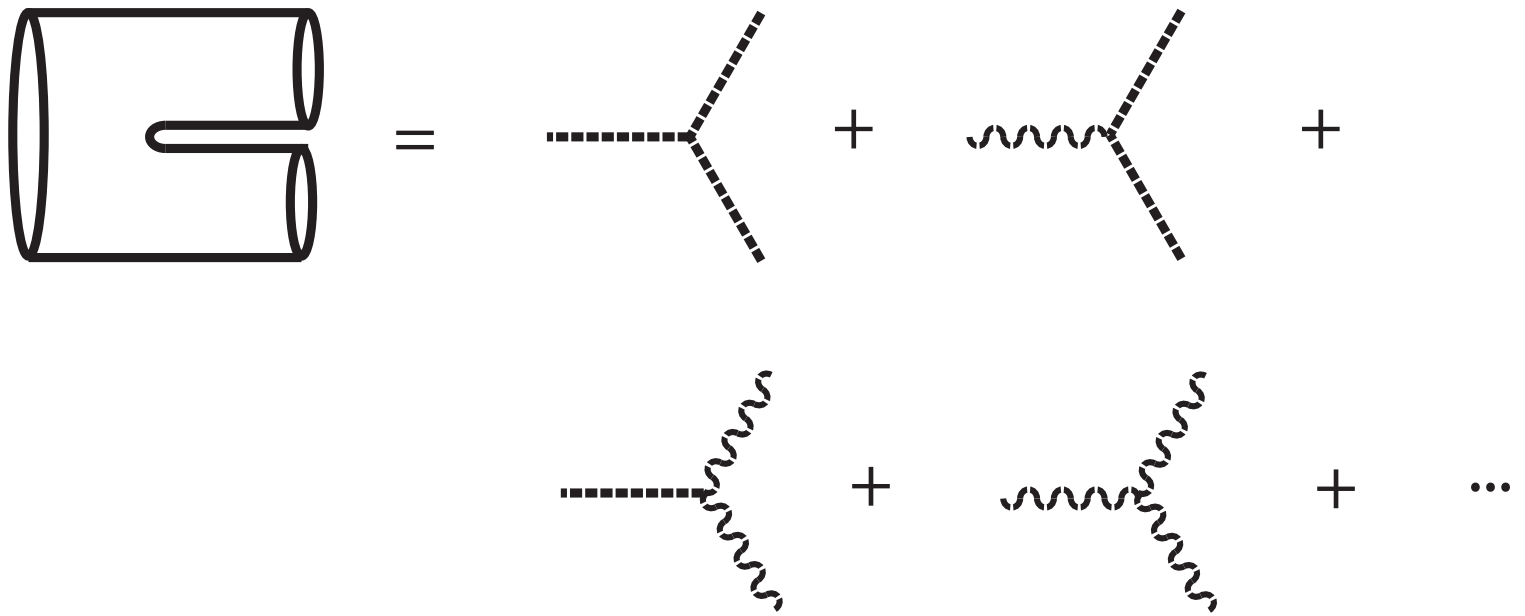
Question: Can we factorize general closed string scattering amplitudes into those of open string theory?

Realization of the Kawai-Lewellen-Tye (KLT) relations in the framework of the second quantized string theory .

Three-Closed-String Scattering

Three-closed-string vertex may be completely factorized into those of three-open-string vertices. If we expand the external three closed string states in terms of the mass eigenstates of free closed string,

$$|\Psi^{(r)}\rangle = \phi(p^{(r)})|0;p\rangle + h_{\mu\nu}(p^{(r)})a_1^\mu \tilde{a}_1^\nu|0;p\rangle + \cdots, \quad r = 1,2,3,$$



Three-Closed-String Scattering

Choose the three external closed string states as follows

$$\langle \Psi^{(1)}, \Psi^{(2)}, \Psi^{(3)}| = \langle 0| \prod_r (\phi(r) + \mathbf{h}(r))$$

where $\phi(r) = \phi(p^{(r)}),\ \mathbf{h}(r) = h_{\mu\nu}(p^{(r)}) a_1^{(r)\mu} \tilde{a}_1^{(r)\nu},\quad r = 1, 2, 3$

$$\begin{aligned}
\mathcal{I}_{\phi\phi\phi} &= \frac{2g}{3} \int \prod_{r=1}^{3} dp^{(r)} \delta \left(\sum_{r=1}^{3} p^{(r)} \right) \phi(p^{(1)}) \phi(p^{(2)}) \phi(p^{(3)}), \\
\mathcal{I}_{\phi\phi h} &= g \int \prod_{r=1}^{3} dp^{(r)} \delta \left(\sum_{r=1}^{3} p^{(r)} \right) h(1)_{(\mu\nu)} \\
&\quad \left(p^{(2)\mu} p^{(2)\nu} + p^{(3)\mu} p^{(3)\nu} - 2 p^{(2)\mu} p^{(3)\nu} \right) \phi(2) \phi(3) \\
&= 4g \int d^d x\, h^{\mu\nu} \partial_\mu \phi \partial_\nu \phi.
\end{aligned}$$

Two gravitons and one tachyon couplings

$$
\begin{aligned}
\mathcal{I}_{\phi hh} &= \mathcal{I}^{(0,0)}_{\phi hh} + \mathcal{I}^{(0,1)}_{\phi hh} + \mathcal{I}^{(1,0)}_{\phi hh} + \mathcal{I}^{(1,1)}_{\phi hh}, \\
\mathcal{I}^{(0,0)}_{\phi hh} &= 2g \int d^d x \Big(\phi \, h^{\mu\nu} h_{\mu\nu} \Big), \\
\mathcal{I}^{(0,1)}_{\phi hh} &= 2g \int d^d x \Big(\partial_\nu \partial_\rho \phi \, h^{\mu\nu} h_\mu{}^\rho \Big), \\
\mathcal{I}^{(1,0)}_{\phi hh} &= 2g \int d^d x \Big(\partial_\nu \partial_\rho \phi \, h^{\nu\mu} h^\rho{}_\mu \Big), \\
\mathcal{I}^{(1,1)}_{\phi hh} &= 2g \int d^d x \Big(\partial_\mu \partial_\nu \partial_\rho \partial_\sigma \phi \, h^{\mu\nu} h^{\rho\sigma} \Big).
\end{aligned}
$$

Three-graviton coupling

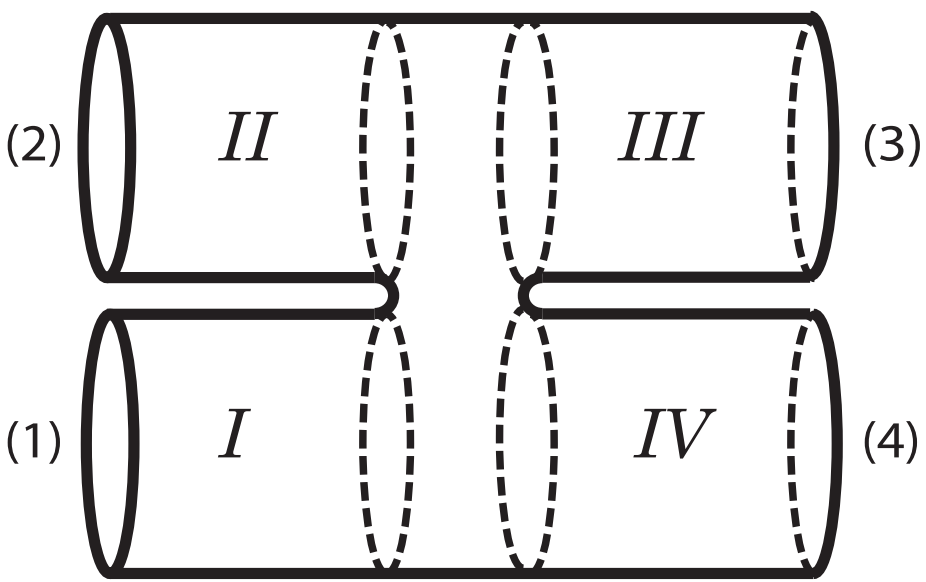
$$
\begin{aligned}
\mathcal{I}_{hhh} &= \mathcal{I}^{(0,0)}_{hhh} + \mathcal{I}^{(1,0)}_{hhh} + \mathcal{I}^{(0,1)}_{hhh} + \mathcal{I}^{(1,1)}_{hhh}, \\
\mathcal{I}^{(0,0)}_{hhh} &= \kappa \int \prod_{i=1}^{3} dp^{(i)} \delta \left(\sum_{i=1}^{3} p^{(i)} \right) h_{\mu_1\nu_1}(p^{(1)}) h_{\mu_2\nu_2}(p^{(2)}) h_{\mu_3\nu_3}(p^{(3)}) \\
&\quad \left\{ \eta^{\mu_1\mu_2} p^{(1)\mu_3} + \eta^{\mu_2\mu_3} p^{(2)\mu_1} + \eta^{\mu_3\mu_1} p^{(3)\mu_2} \right\} \\
&\quad \left\{ \eta^{\nu_1\nu_2} p^{(1)\nu_3} + \eta^{\nu_2\nu_3} p^{(2)\nu_1} + \eta^{\nu_3\nu_1} p^{(3)\nu_2} \right\}.
\end{aligned}
$$

This is precisely the three-graviton interaction term of the conventional perturbative Einstein gravity.

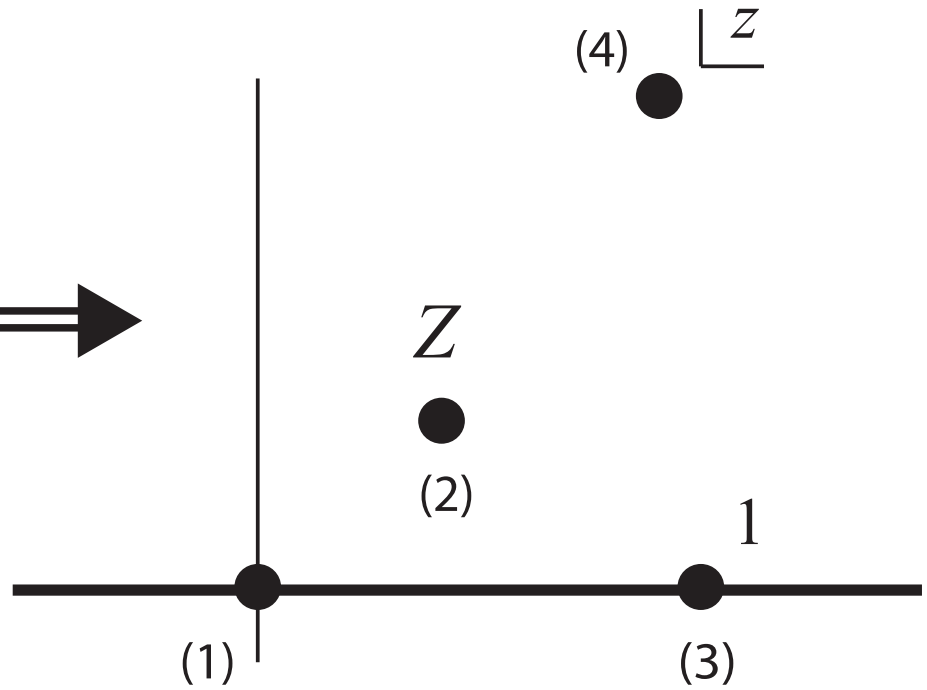
Alpha-Prime corrections to three-graviton-scattering

$$
\begin{aligned}
\mathcal{I}^{(0,1)}_{hhh} &= \frac{g}{3}\int\prod_{i=1}^{3}dp^{(i)}\delta\left(\sum_{i=1}^{3}p^{(i)}\right)h_{\mu_1\nu_1}(p^{(1)})h_{\mu_2\nu_2}(p^{(2)})h_{\mu_3\nu_3}(p^{(3)}) \\
&\quad \left(\eta^{\mu_1\mu_2}p^{(1)\mu_3}+\eta^{\mu_2\mu_3}p^{(2)\mu_1}+\eta^{\mu_3\mu_1}p^{(3)\mu_2}\right)p^{(1)\nu_3}p^{(2)\nu_1}p^{(3)\nu_2}, \\
\mathcal{I}^{(1,0)}_{hhh} &= \frac{g}{3}\int\prod_{i=1}^{3}dp^{(i)}\delta\left(\sum_{i=1}^{3}p^{(i)}\right)h_{\mu_1\nu_1}(p^{(1)})h_{\mu_2\nu_2}(p^{(2)})h_{\mu_3\nu_3}(p^{(3)}) \\
&\quad p^{(1)\mu_3}p^{(2)\mu_1}p^{(3)\mu_2}\left(\eta^{\nu_1\nu_2}p^{(1)\nu_3}+\eta^{\nu_2\nu_3}p^{(2)\nu_1}+\eta^{\nu_3\nu_1}p^{(3)\nu_2}\right), \\
\mathcal{I}^{(1,1)}_{hhh} &= \frac{g}{3!}\int\prod_{r=1}^{3}dp^{(r)}\delta\left(\sum_{r=1}^{3}p^{(r)}\right)h_{\mu_1\nu_1}(p^{(1)})h_{\mu_2\nu_2}(p^{(2)})h_{\mu_3\nu_3}(p^{(3)}) \\
&\quad p^{(1)\mu_3}p^{(2)\mu_1}p^{(3)\mu_2}p^{(1)\nu_3}p^{(2)\nu_1}p^{(3)\nu_2} \\
&= \frac{g}{3!}\int d^dx\;\partial_\mu\partial_\nu h^{\alpha\beta}\partial_\alpha\partial_\beta h^{\gamma\delta}\partial_\gamma\partial_\delta h^{\mu\nu}.
\end{aligned}
$$

Four-Closed-String Scattering



$\rho = \tau + i\sigma$



Scattering Amplitude of Four Strings

Scattering amplitude of four closed strings:

$$\mathcal{A}[1,2,3,4] = g^2 \int \prod_r dZ_r^2 \frac{|Z_a - Z_b|^2 |Z_b - Z_c|^2 |Z_c - Z_a|^2}{d^2 Z_a d^2 Z_b d^2 Z_c}$$

$$\prod_{r<s} |Z_r - Z_s|^{2\,\frac{p^{(r)}}{2}\cdot\frac{p^{(s)}}{2}} \prod_{r<s} |Z_r - Z_s|^{2\left\{\frac{\alpha_s}{\alpha_r}\left(1-\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2\right)+\frac{\alpha_r}{\alpha_s}\left(1-\frac{1}{2}\left(\frac{p^{(s)}}{2}\right)^2\right)\right\}}$$

$$\langle\{\mathbf{k}^{(r)}\}| \exp\left\{\frac{1}{4}\sum_{r,s}\sum_{n,m\geq 1}\left(\bar{C}^{rs}_{nm}\,\tilde{\alpha}^{(r)\dagger}_n\cdot\tilde{\alpha}^{(s)\dagger}_m + \bar{C}^{rs*}_{nm}\,\alpha^{(r)\dagger}_n\cdot\alpha^{(r)\dagger}_m\right)\right.$$

$$+\sum_{r,s}\left(\sum_{n\geq 1}\left(\bar{C}^{rs}_{n0}\tilde{\alpha}^{(r)\dagger}_n\cdot\frac{p^{(s)}}{2} + \bar{C}^{rs*}_{n0}\alpha^{(r)\dagger}_n\cdot\frac{p^{(s)}}{2}\right)\right)$$

$$\left.+2\tau^{(r)}\sum_r\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2-1\right)\right\}|0\rangle.$$

Shapiro-Virasoro Amplitude

Scattering of four closed string tachyons

$$\begin{aligned}
\mathcal{A}_{\text{Tachyon}}[1,2,3,4] &= g^2 \int d^2Z\, |Z|^{2\,\frac{p^{(1)}}{2}\cdot\frac{p^{(2)}}{2}} |1-Z|^{2\,\frac{p^{(2)}}{2}\cdot\frac{p^{(3)}}{2}} \\
&= g^2 \int d^2Z\, |Z|^{2\left(-\frac{s}{8}-2\right)} |1-Z|^{2\left(-\frac{u}{8}-2\right)} \\
&= 2\pi g^2 \frac{\Gamma\left(-1-\frac{s}{8}\right)\Gamma\left(-1-\frac{t}{8}\right)\Gamma\left(-1-\frac{u}{8}\right)}{\Gamma\left(2+\frac{s}{8}\right)\Gamma\left(2+\frac{t}{8}\right)\Gamma\left(2+\frac{u}{8}\right)}.
\end{aligned}$$

Mandelstam variables:

$$s=-(p_1+p_2)^2, \quad t=-(p_1+p_3)^2, \quad u=-(p_1+p_4)^2.$$

The Koba-Nielsen variables:

$$Z_1=0, \quad Z_2=Z, \quad Z_3=1, \quad Z_4=\infty.$$

Scattering amplitude for four gravitons

$$
\begin{aligned}
\mathcal{A}_{[4]} \;=\; & g^2 \int |Z_4|^4 d^2Z \prod_{r<s} |Z_r - Z_s|^{2\,\frac{p^{(r)}}{2}\cdot\frac{p^{(s)}}{2}} e^{-2\sum_{r=1}^{4} \bar{C}_{00}^{[4]rr}} \\
& \langle 0| \left\{ \prod_{i=1}^{4} h_{\mu\nu}(p^{(i)}) a_1^{(i)\mu} \tilde{a}_1^{(i)\nu} \right\} \frac{1}{2!}\cdot\frac{1}{2^4} \left\{ \left(\sum_{r,s} \bar{C}_{11}^{rs*} a_1^{(r)\dagger}\cdot a_1^{(s)\dagger} \right)^2 \right. \\
& \left. + \left(\sum_{r,s} \bar{C}_{11}^{rs*} a_1^{(r)\dagger}\cdot a_1^{(s)\dagger} \right) \left(\sum_{r,s} \bar{C}_{10}^{rs*} a_1^{(r)\dagger}\cdot p^{(s)} \right)^2 \right\} \\
& \frac{1}{2!}\cdot\frac{1}{2^4} \left\{ \left(\sum_{r,s} \bar{C}_{11}^{rs}\, \tilde{a}_1^{(r)\dagger}\cdot \tilde{a}_1^{(s)\dagger} \right)^2 + \left(\sum_{r,s} \bar{C}_{11}^{rs}\, \tilde{a}_1^{(r)\dagger}\cdot \tilde{a}_1^{(s)\dagger} \right) \right. \\
& \left. \left(\sum_{r,s} \bar{C}_{10}^{rs} \tilde{a}_1^{(r)\dagger}\cdot p^{(s)} \right)^2 \right\} |0\rangle .
\end{aligned}
$$

$$\mathcal{A}_{[4]} = \frac{g^2}{2^{10}} \int d^2Z\, |Z|^{-\frac{s}{4}} |1-Z|^{-\frac{t}{4}} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4}$$

$$\left\{ \frac{4}{Z^{*2}} \eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + 4\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{4}{(1-Z^*)^2}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \right.$$

$$-\eta^{\mu_1\mu_2} \frac{1}{(1-Z^*)Z^{*2}} \left(Z^* p^{(1)\mu_3} + p^{(4)\mu_3}\right)\left(Z^* p^{(2)\mu_4} + p^{(3)\mu_4}\right)$$

$$+\eta^{\mu_1\mu_3} \frac{1}{(1-Z^*)Z^{*}} \left(p^{(1)\mu_2} + Z^* p^{(4)\mu_2}\right)\left(Z^* p^{(2)\mu_4} + p^{(3)\mu_4}\right)$$

$$-\eta^{\mu_1\mu_4} \frac{1}{(1-Z^*)^2Z^{*}} \left((1-Z^*) p^{(4)\mu_2} + p^{(3)\mu_2}\right)\left((1-Z^*) p^{(1)\mu_3} + p^{(2)\mu_3}\right)$$

$$-\eta^{\mu_2\mu_3} \frac{1}{(1-Z^*)^2Z^{*}} \left((1-Z^*) p^{(3)\mu_1} + p^{(4)\mu_1}\right)\left(p^{(1)\mu_4} + (1-Z^*) p^{(2)\mu_4}\right)$$

$$+\eta^{\mu_2\mu_4} \frac{1}{(1-Z^*)Z^{*}} \left(Z^* p^{(3)\mu_1} + p^{(2)\mu_1}\right)\left(Z^* p^{(1)\mu_3} + p^{(4)\mu_3}\right)$$

$$\left. -\eta^{\mu_3\mu_4} \frac{1}{(1-Z^*)Z^{*2}} \left(Z^* p^{(3)\mu_1} + p^{(2)\mu_1}\right)\left(p^{(1)\mu_2} + Z^* p^{(4)\mu_2}\right) \right\} \left\{ Z^* \Rightarrow Z,\ \mu \Rightarrow \nu \right\}.$$

Kawai-Lewellen-Tye (KLT) Relations

$$
\begin{aligned}
I &= \int d^2Z \prod_{r<s} |Z_r - Z_s|^{2\,\frac{p^{(r)}}{2}\cdot\frac{p^{(s)}}{2}} (Z^*)^{-2n} Z^{-2m} (1-Z^*)^{-2p} (1-Z)^{-2q} \\
&= \int d^2Z\, Z^{-2m-\frac{s}{8}} (1-Z)^{-2q-\frac{t}{8}} (Z^*)^{-2n-\frac{s}{8}} (1-Z^*)^{-2p-\frac{t}{8}}.
\end{aligned}
$$

If we write $Z = x + iy$, then we treat x and y as independent two complex variables. The integrand is an analytic function of y with four branch points, $\pm x$, $\pm i(1-x)$. We may deform the contour of y which is along the real line to the contour along the imaginary axis.

$$
\begin{aligned}
I &= \sin\left(\frac{\pi}{8}t\right) \int_0^1 d\xi\, |\xi|^{-\frac{s}{8}} |1-\xi|^{-\frac{t}{8}} \xi^{-2m} (1-\xi)^{-2q} \\
&\quad \int_1^\infty d\eta\, |\eta|^{-\frac{s}{8}} |1-\eta|^{-\frac{t}{8}} \eta^{-2n} (1-\eta)^{-2p}.
\end{aligned}
$$

Kawai-Lewellen-Tye (KLT) Relations

Factorization

$$
\begin{aligned}
I &= \sin\left(\frac{\pi t}{8}\right) I_1(n,p) I_2(m,q), \\
I_1(n,p) &= (-1)^{2p} \frac{\Gamma\left(-\frac{u}{8}+2n+2p-1\right)\Gamma\left(-\frac{t}{8}-2p+1\right)}{\Gamma\left(\frac{s}{8}+2n\right)}, \\
I_2(m,q) &= \frac{\Gamma\left(-\frac{s}{8}-2m+1\right)\Gamma\left(-\frac{t}{8}-2q+1\right)}{\Gamma\left(\frac{u}{8}-2m-2q+2\right)}
\end{aligned}
$$

where n, p, m, q are integers or half-integers.

Four-graviton scattering

$$\mathcal{A}_{[4G]} = g^2 c_{[4]} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \sin\left(\frac{\pi t}{8}\right)$$

$$\Big\{ I_1(2,0)\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + I_1(0,0)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + I_1(0,2)\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}$$

$$-\frac{1}{4}\eta^{\mu_1\mu_2}\left(I_1(0,1)p^{(1)\mu_3}p^{(2)\mu_4} + I_1(1,1)p^{(1)\mu_3}p^{(3)\mu_4} + I_1(1,1)p^{(4)\mu_3}p^{(2)\mu_4} + I_1(2,1)p^{(4)\mu_3}p^{(3)\mu_4}\right)$$

$$+\frac{1}{4}\eta^{\mu_1\mu_3}\left(I_1(0,1)p^{(1)\mu_2}p^{(2)\mu_4} + I_1(1,1)p^{(1)\mu_2}p^{(3)\mu_4} + I_1(-1,1)p^{(4)\mu_2}p^{(2)\mu_4} + I_1(0,1)p^{(4)\mu_2}p^{(3)\mu_4}\right)$$

$$-\frac{1}{4}\eta^{\mu_1\mu_4}\left(I_1(1,0)p^{(4)\mu_2}p^{(1)\mu_3} + I_1(1,1)p^{(4)\mu_2}p^{(2)\mu_3} + I_1(1,1)p^{(3)\mu_2}p^{(1)\mu_3} + I_1(1,2)p^{(3)\mu_2}p^{(2)\mu_3}\right)$$

$$-\frac{1}{4}\eta^{\mu_2\mu_3}\left(I_1(1,0)p^{(3)\mu_1}p^{(2)\mu_4} + I_1(1,1)p^{(3)\mu_1}p^{(1)\mu_4} + I_1(1,1)p^{(4)\mu_1}p^{(2)\mu_4} + I_1(1,2)p^{(4)\mu_1}p^{(1)\mu_4}\right)$$

$$+\frac{1}{4}\eta^{\mu_2\mu_4}\left(I_1(-1,1)p^{(3)\mu_1}p^{(1)\mu_3} + I_1(0,1)p^{(3)\mu_1}p^{(4)\mu_3} + I_1(0,1)p^{(2)\mu_1}p^{(1)\mu_3} + I_1(1,1)p^{(2)\mu_1}p^{(4)\mu_3}\right)$$

$$-\frac{1}{4}\eta^{\mu_3\mu_4}\left(I_1(0,1)p^{(3)\mu_1}p^{(4)\mu_2} + I_1(1,1)p^{(3)\mu_1}p^{(1)\mu_2} + I_1(1,1)p^{(2)\mu_1}p^{(4)\mu_2} + I_1(2,1)p^{(2)\mu_1}p^{(1)\mu_2}\right)\Big\}$$

$$\Big\{ I_1(n,p) \Rightarrow I_2(n,p),\ \mu \Rightarrow \nu \Big\}.$$

In the zero-slope limit, it reduces to the four-graviton-scattering amplitude of Einstein gravity:

$$
\begin{aligned}
\mathcal{A}_{[4]} \;=\; & \frac{g^2\pi}{2^{10}} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \frac{1}{stu} \\
& \left\{ \frac{ut}{2}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{su}{2}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \right. \\
& -\eta^{\mu_1\mu_2}\left(t p^{(1)\mu_3} p^{(2)\mu_4} + u p^{(2)\mu_3} p^{(1)\mu_4} \right) \\
& -\eta^{\mu_1\mu_3}\left(t p^{(1)\mu_2} p^{(3)\mu_4} + s p^{(3)\mu_2} p^{(1)\mu_4} \right) \\
& -\eta^{\mu_1\mu_4}\left(u p^{(1)\mu_2} p^{(4)\mu_3} + s p^{(4)\mu_2} p^{(1)\mu_3} \right) \\
& -\eta^{\mu_2\mu_3}\left(u p^{(2)\mu_1} p^{(3)\mu_4} + s p^{(3)\mu_1} p^{(2)\mu_4} \right) \\
& -\eta^{\mu_2\mu_4}\left(s p^{(4)\mu_1} p^{(2)\mu_3} + t p^{(2)\mu_1} p^{(4)\mu_3} \right) \\
& \left. -\eta^{\mu_3\mu_4}\left(t p^{(3)\mu_1} p^{(4)\mu_2} + u p^{(4)\mu_1} p^{(3)\mu_2} \right) \right\} \left\{ \mu \Rightarrow \nu \right\}.
\end{aligned}
$$

Four-Graviton Scattering Amp by using vertex operators

$$\mathcal{A}_{[4]\text{vertex}} = \frac{\kappa^2}{128}\frac{\Gamma\left(-\frac{s}{8}\right)\Gamma\left(-\frac{t}{8}\right)\Gamma\left(-\frac{u}{8}\right)}{\Gamma\left(1+\frac{s}{8}\right)\Gamma\left(1+\frac{t}{8}\right)\Gamma\left(1+\frac{u}{8}\right)} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4}$$

$$\left\{ \frac{ut}{2}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{su}{2}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \right.$$

$$-\eta^{\mu_1\mu_2}\left(tp^{(1)\mu_3}p^{(2)\mu_4} + up^{(2)\mu_3}p^{(1)\mu_4}\right) - \eta^{\mu_1\mu_3}\left(tp^{(1)\mu_2}p^{(3)\mu_4}\right.$$

$$\left.+sp^{(3)\mu_2}p^{(1)\mu_4}\right) - \eta^{\mu_1\mu_4}\left(up^{(1)\mu_2}p^{(4)\mu_3} + sp^{(4)\mu_2}p^{(1)\mu_3}\right)$$

$$-\eta^{\mu_2\mu_3}\left(up^{(2)\mu_1}p^{(3)\mu_4} + sp^{(3)\mu_1}p^{(2)\mu_4}\right)$$

$$-\eta^{\mu_2\mu_4}\left(sp^{(4)\mu_1}p^{(2)\mu_3} + tp^{(2)\mu_1}p^{(4)\mu_3}\right)$$

$$\left.-\eta^{\mu_3\mu_4}\left(tp^{(3)\mu_1}p^{(4)\mu_2} + up^{(4)\mu_1}p^{(3)\mu_2}\right)\right\}\left\{\mu \Rightarrow \nu\right\}.$$

Comparison of Four-Graviton Scattering Amplitudes

$$\mathcal{A}_{[4]\text{vertex}} = \frac{\kappa^2}{128}\frac{\Gamma\left(-\frac{s}{8}\right)\Gamma\left(-\frac{t}{8}\right)\Gamma\left(-\frac{u}{8}\right)}{\Gamma\left(1+\frac{s}{8}\right)\Gamma\left(1+\frac{t}{8}\right)\Gamma\left(1+\frac{u}{8}\right)} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} K_{\text{vertex}}^{\mu_1\mu_2\mu_3\mu_4} K_{\text{vertex}}^{\nu_1\nu_2\nu_3\nu_4},$$

$$\Longrightarrow$$

$$\mathcal{A}_{[4]\text{proper}} = \frac{\kappa^2}{128}\frac{\Gamma\left(-\frac{s}{8}\right)\Gamma\left(-\frac{t}{8}\right)\Gamma\left(-\frac{u}{8}\right)}{\Gamma\left(1+\frac{s}{8}\right)\Gamma\left(1+\frac{t}{8}\right)\Gamma\left(1+\frac{u}{8}\right)} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} K_{\text{proper}}^{\mu_1\mu_2\mu_3\mu_4} K_{\text{proper}}^{\nu_1\nu_2\nu_3\nu_4}.$$

Conclusions and Discussions

1. The four-particle-scattering amplitudes in the proper-time gauge contains the tachyon poles in a manner consistent with the three-particle-scattering amplitudes of two gauge/graviton particles and one tachyon.
2. The scattering amplitudes evaluated by using the Polyakov string paht integral in the proper-time gauge may be also important to understand low energy dynamics of string theory.
3. Systematic evaluation of α'-corrections: The scattering amplitudes evaluated by using the Polyakov string paht integral in the proper-time gauge may be more suitable than the conventional ones for exploring the string corrections to the quantum field theories.