Entanglement Wedges from Information Metric in CFTs

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Based on

arXiv: 1908.09939 with Yuki Suzuki (Kyoto U.)
and Koji Umemoto (YITP, Kyoto U.)

arXiv: 19XX.YYYYY with Yuya Kusuki (YITP, Kyoto),
Yuki Suzuki and Koji Umemoto

① Introduction

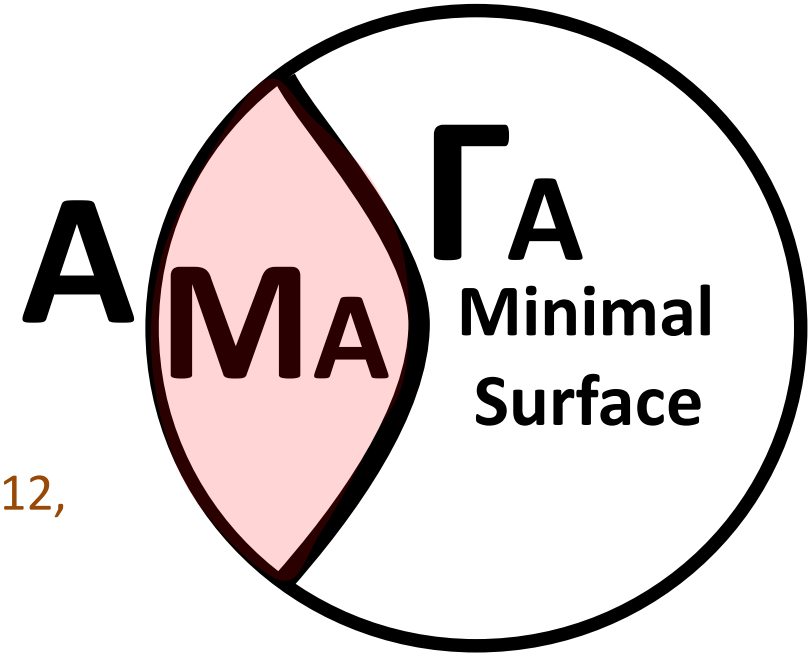
One of the basic mechanisms on how AdS/CFT works, is Entanglement Wedges (or Subregion-Subregion duality).

Which bulk region is dual to a given region A in CFT ?

⇒ Entanglement Wedge (EW): M_A

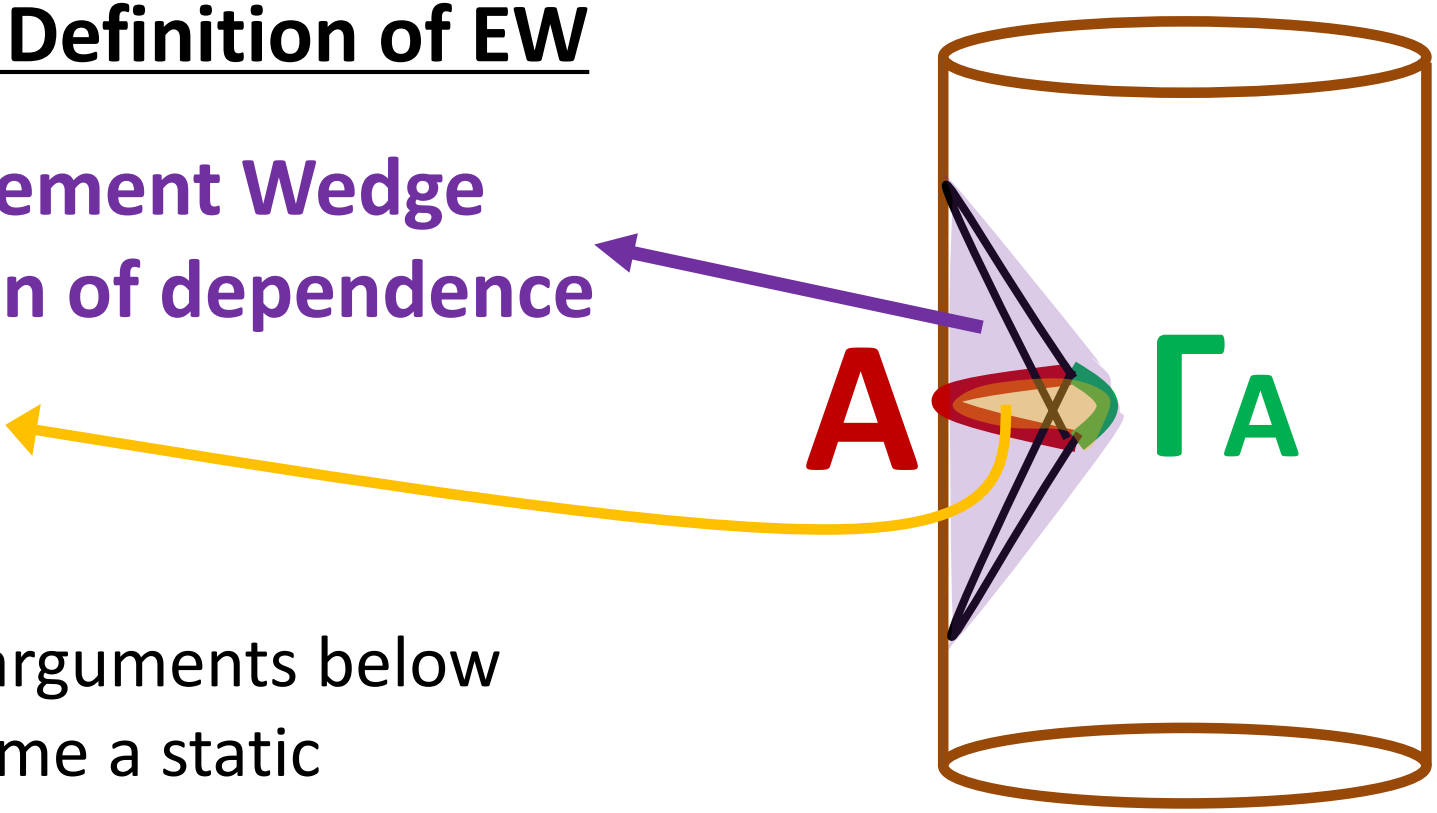
$$\rho_A \quad \text{in CFT}$$
$$\Leftrightarrow \quad \rho_{MA} \quad \text{in AdS gravity}$$

[Czech-Karczmarek-Nogueira-Raamsdonk 2012, Wall 2012, Headrick-Hubeny-Lawrence-Rangamani 2014]



Covariant Definition of EW

Entanglement Wedge
=Domain of dependence
of M_A

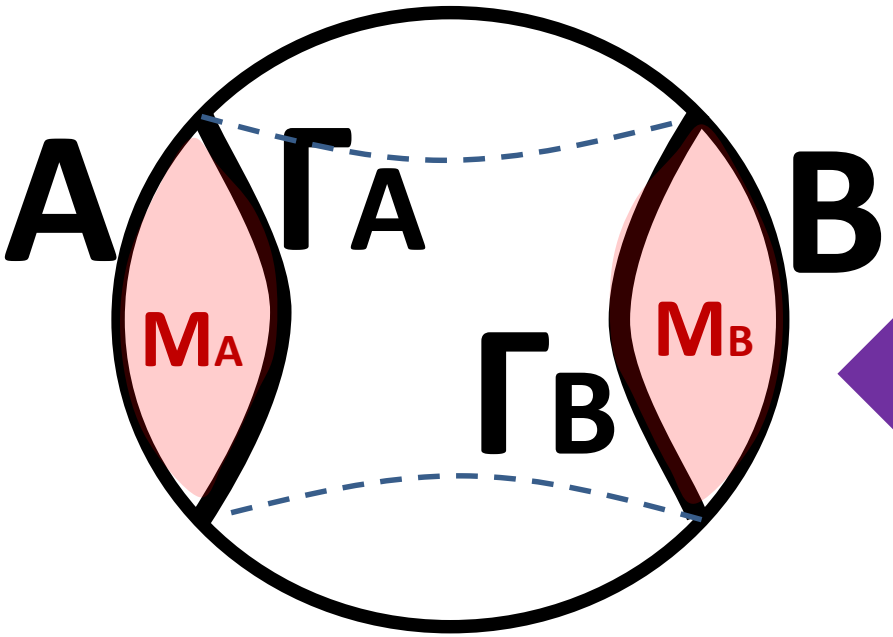


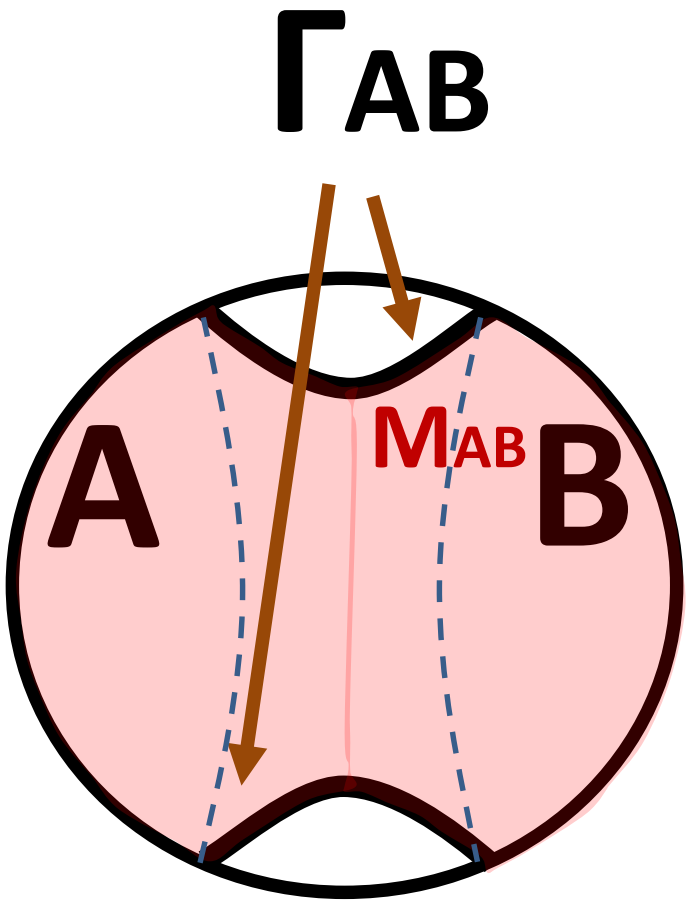
Note: Our arguments below mostly assume a static asymptotically AdS spacetime, which has a canonical time slice.

However, it is possible to extend our arguments to general non-static setups in a covariant way.

EW for Disconnected Subregions

$$I(A:B) = S_A + S_B - S_{AB} = 0$$





$M_{AB}=M_A \cup M_B$, $\Gamma_{AB}=\Gamma_A \cup \Gamma_B$

$$I(A:B) > 0$$

Usually, EW is explained by combining several ideas:

(1) HKLL Reconstruction (CFT dual of bulk local fields)

[Hamilton-Kabat-Lifschytz-Lowe 2006]

(2) Quantum Corrections to HEE (or Hol. Relative entropy)

[Faulkner-Lewkowycz-Maldacena 2013, Jafferis-Lewkowycz-Maldacena-Suh 2015]

(3) Relation to Quantum Error Correcting Codes

[Almheri-Dong-Harlow 2014, Dong-Harlow-Wall 2016]

These arguments basically assume the AdS spacetime and its dynamics from the beginning, based on AdS/CFT.

In this talk, we would like to start from (holographic) CFTs and derive the entanglement wedge geometry in the CFT language. (i.e. postpone the use of AdS/CFT to the end.)

Our argument will clarify how entanglement wedges emerge from reduced density matrices in CFTs.

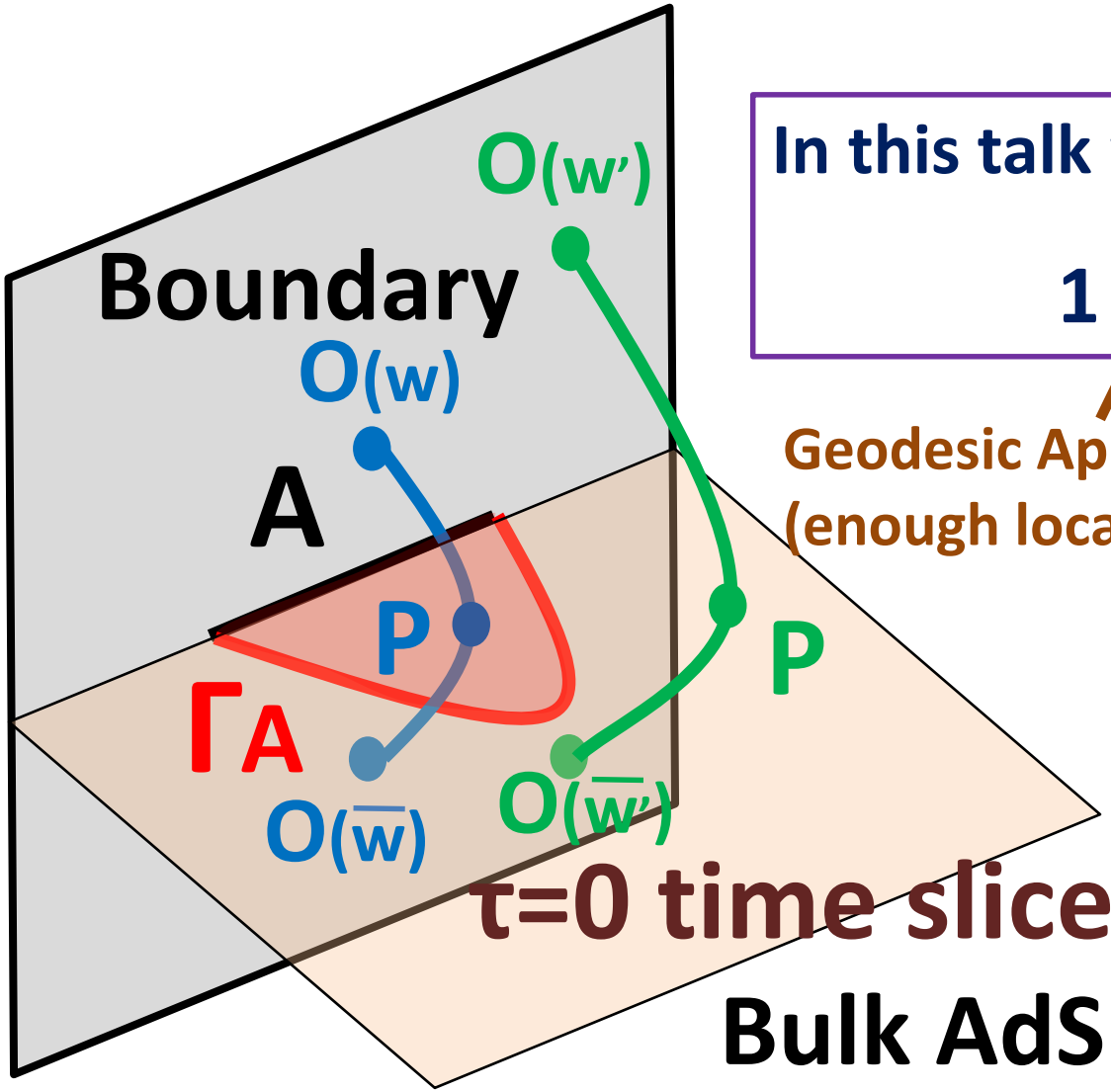
For this purpose we study locally excited states in 2d CFTs:

$$\boldsymbol{\rho_A(w,\bar{w}) = \mathrm{Tr}_B\left[O(w,\bar{w})|0\rangle\langle 0|O^\dagger(\bar{w},w)\right]}$$

$\mathrm{w} = x + i\tau$ is the coordinate of complex plane R^2.

$O(w,\bar{w})$: A primary in 2d CFT with (chiral) conformal dim. h

$\rho_A(w,\bar{w}) = \mathrm{Tr}_B\left[O(w,\bar{w})|0\rangle\langle 0|O^\dagger(\bar{w},w)\right]$ and its Dual



In this talk we always assume

$1 \ll h \ll c$.

Geodesic Approximation
(enough localization)

Negligible
Back-reactions

ρ_A may include the information of O(w) but not of O(w').

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① Introduction
② Bures Information Metric
③ Single Interval Case
④ Double Intervals Case
⑤ HKLL States
⑥ Conclusions

② Bures Information Metric

To study distinguishability of quantum states, consider a distance measure between density matrices ρ and ρ'.

Especially we focus on the Bures distance:

$$D_B(\rho,\rho')^2 = 2 - 2\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right]$$

When the states are pure, we have

$$D_B(|\Psi\rangle\langle\Psi|, |\Psi'\rangle\langle\Psi'|)^2 = 2(1 - |\langle\Psi|\Psi'\rangle|).$$

Assume density matrices depend on parameters λi, denoted by ρ(λ). **The Bures metric** is defined as follows:

$$ds^2 \equiv D_B(\rho(\lambda), \rho(\lambda + d\lambda))^2 \cong G_{ij} d\lambda^i d\lambda^j$$

Quantum Cramer-Rao Theorem

$$\left\langle \langle \delta\lambda_i \delta\lambda_j \rangle \right\rangle \geq G_{ij}^{-1}$$

[Heldstrom 1976]

Errors in estimation of values of λi
based on quantum (POVM) measurements of ρ(λ)

Warmup example: Pure States in CFTs

Consider the locally excited state $|\Psi(w)\rangle = O(w,\bar{w})|0\rangle$.

Primary with conformal dim. h

The Bures distance is computed as

$$D_B(|\Psi(w)\rangle\langle\Psi(w)|, |\Psi(w')\rangle\langle\Psi(w')|)^2 = 2(1 - |\langle\Psi(w)|\Psi(w')\rangle|).$$

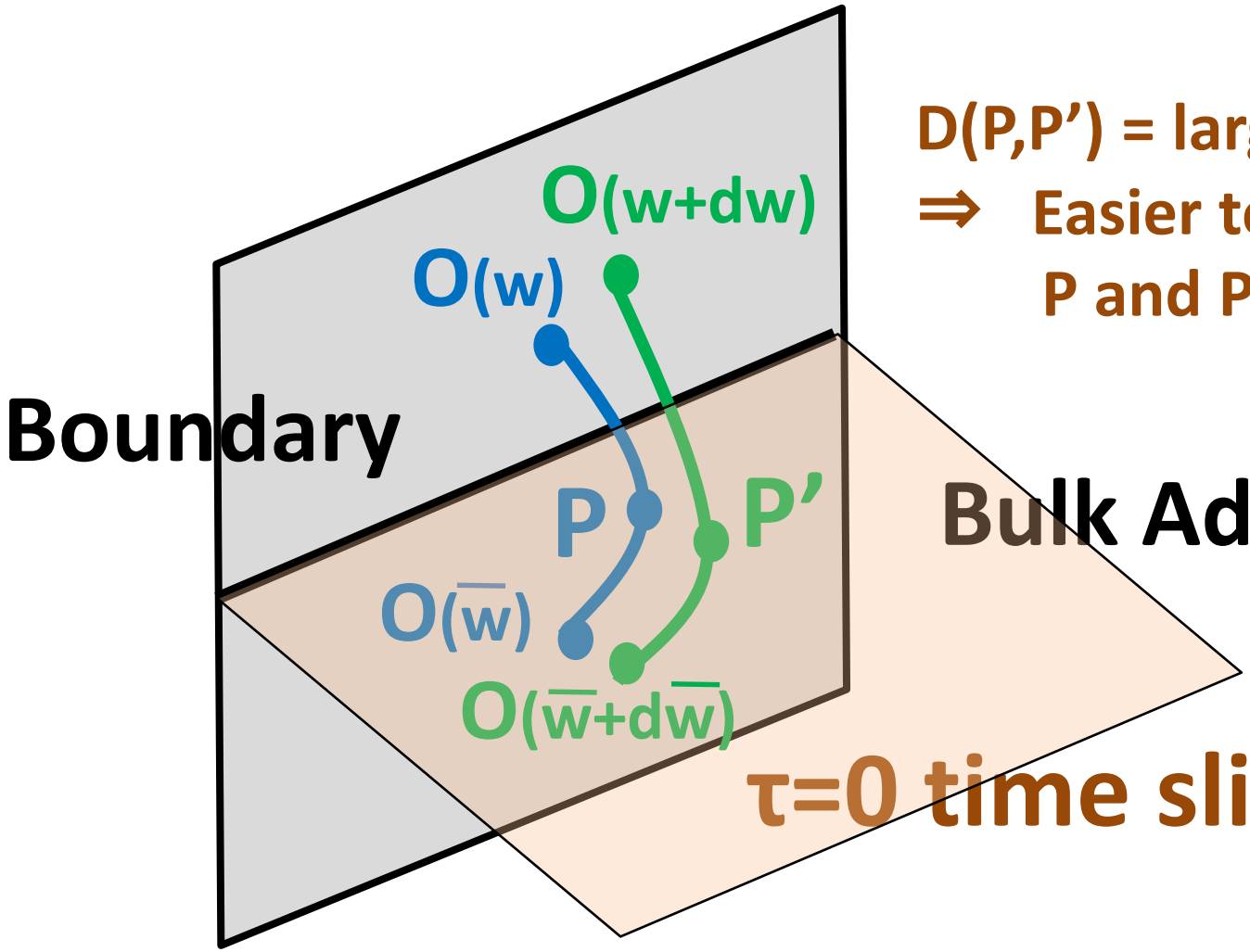
$$|\langle\Psi(w)|\Psi(w')\rangle| = |w-\bar{w}|^{2h}|w'-\overline{w'}|^{2h}|w-\overline{w'}|^{-4h}.$$

The Bures metric reads (w=x+iτ)

$$ds_B^2 = \frac{h}{\tau^2}(dx^2 + d\tau^2)$$

Coincides with the metric on the time slice of AdS up to normalization

Information Metric ∝ Geometric Metric in Gravity Dual



D(P,P') = larger

⇒ Easier to distinguish P and P'

However, the info. metric is universal for any CFTs.

⇒ The situation largely changes for ρ_A as we will see.

③ Single Interval Case

Primary with conformal dim. h

Consider the reduced density matrix

$$\boldsymbol{\rho_A(w,\bar{w}) = \mathrm{Tr}_B\left[O(w,\bar{w})|0\rangle\langle 0|O^{\dagger}(\bar{w},w)\right].}$$

We choose A to be an interval A=[0,L] on R^2 .

We would like to calculate its Bures metric.

As a first step, let us start with the computation of `Renyi version' of Bures distance:

[Introduced by Cardy 2014]

$$\boldsymbol{I(\rho,\rho') = \frac{\mathrm{Tr}[\rho\rho']}{\sqrt{\mathrm{Tr}[\rho^2]\mathrm{Tr}[\rho'^2]}} \leq 1.}$$

$$\boldsymbol{I(\rho,\rho') = 1 \;\leftrightarrow\; \rho = \rho'\,.}$$

Cf. $2 - D_B(\rho,\rho')^2$

$$= 2\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right]$$

We can calculate **Tr[ρρ']** by pasting the w-plane for ρ and that for ρ' via the conformal map: $z^2 = \frac{w}{w-L}$.

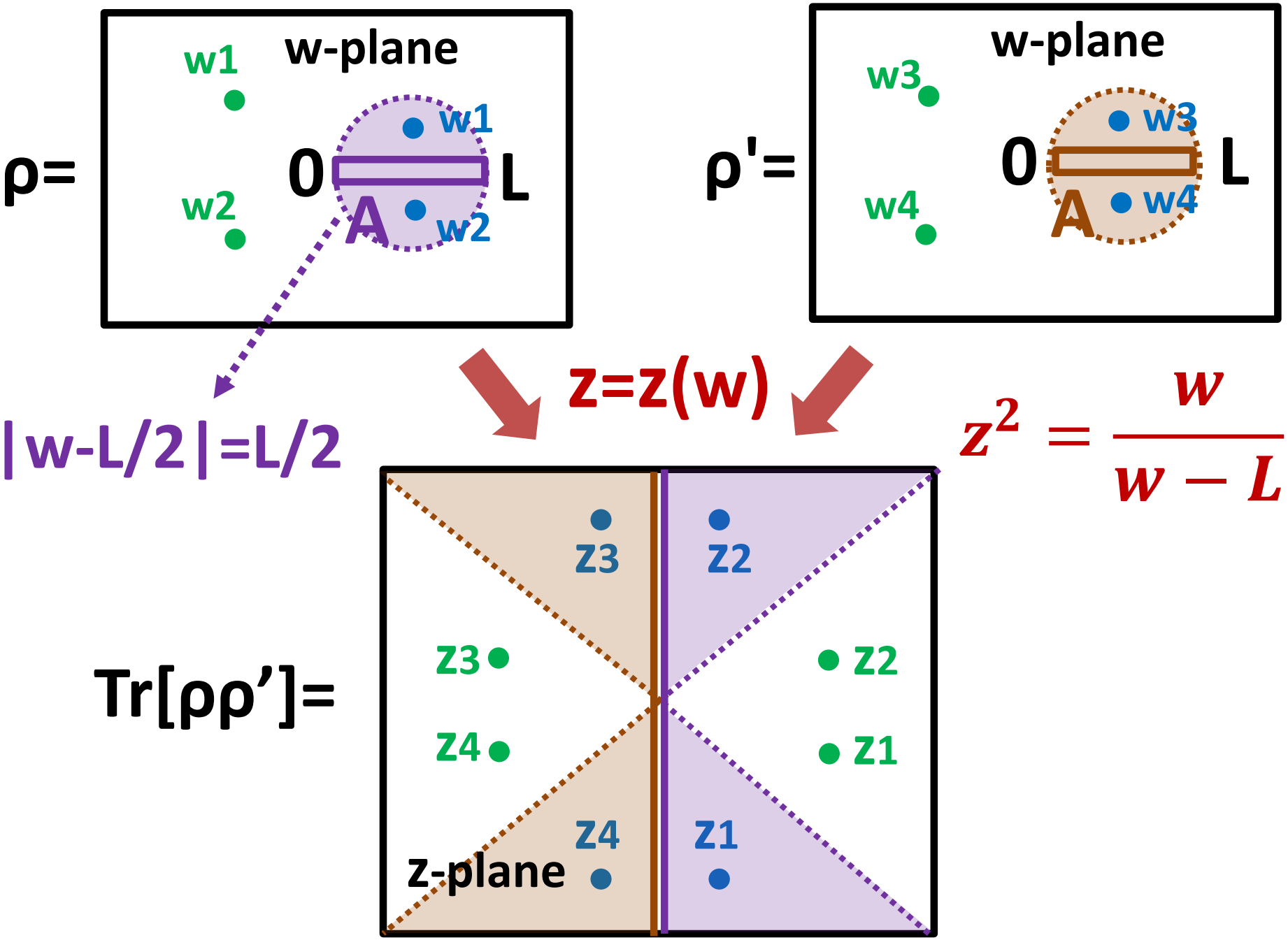
$$\mathrm{Tr}[\rho\rho'] = \left[\left|\frac{dz_1}{dw_1}\right|\left|\frac{dz_2}{dw_2}\right|\left|\frac{dz_3'}{dw_3'}\right|\left|\frac{dz_4'}{dw_4'}\right|\right]^{2h} \cdot \frac{H(z_1,z_2,z_3',z_4')\cdot Z^{(2)}}{(Z^{(1)})^2},$$

$$H(z_1,z_2,z_3',z_4') \equiv \frac{\langle O^\dagger(z_1,\bar{z}_1)O(z_2,\bar{z}_2)O^\dagger(z_3',\bar{z}_3')O(z_4',\bar{z}_4')\rangle}{\langle O^\dagger(w_1,\bar{w}_1)O(w_2,\bar{w}_2)\rangle\langle O^\dagger(w_3',\bar{w}_3')O(w_4',\bar{w}_4')\rangle},$$

where $\langle\cdots\rangle$ denotes the normalized correlation function such that $\langle 1\rangle = 1$ and we also write the vacuum partition function on a n-sheeted complex plane by $Z^{(n)}$. Finally we obtain

$$I(\rho,\rho') = \frac{F(z_1,z_2,z_3',z_4')}{\sqrt{F(z_1,z_2,z_3,z_4)F(z_1',z_2',z_3',z_4')}},$$

$$F(z_1,z_2,z_3',z_4') \equiv \langle O^\dagger(z_1,\bar{z}_1)O(z_2,\bar{z}_2)O^\dagger(z_3',\bar{z}_3')O(z_4',\bar{z}_4')\rangle.$$


ρ=
w-plane
w1
w2
0
L
A
w1
w2
ρ'=
w-plane
w3
w4
0
L
A
w3
w4
|w-L/2|=L/2
z=z(w)
$z^2 = \frac{w}{w-L}$
Tr[ρρ']=
z-plane
z3
z2
z4
z1
z3
z2
z4
z1

I(ρ,ρ') in 2d Holographic CFTs

The 4-pt function $F(z_1, z_2, z_3, z_4) = \langle O^\dagger O O^\dagger O \rangle$ is given by applying the generalized free field prescription:

$$F \approx |z_1 - z_2|^{-4h} |z_3 - z_4|^{-4h} + |z_2 - z_3|^{-4h} |z_1 - z_4|^{-4h}.$$

Trivial Wick Contraction

$\Rightarrow$ Always $I(\rho, \rho') = 1$

$\Rightarrow$ $\rho = \rho'$.

(i.e. Indistinguishable)

Non-Trivial Wick Contraction

$\Rightarrow$ In general, $I(\rho, \rho') < 1$

(i.e. distinguishable)

Having in mind the information metric, we take w ≈ w'.

[i] Case 1: the trivial Wick contraction is dominant:

$$|z_2 - z_3| \; < |z_1 - z_4| \quad \Rightarrow |w - L/2| > L/2$$

w ∉ Entanglement Wedge

$\rho_A(w)$ and $\rho_A(w')$ are indistinguishable !
Always we find I(ρ,ρ')=1.

[ii] Case 2: the non-trivial Wick contraction is domiant:

$$|z_2 - z_3| \; > |z_1 - z_4| \quad \Rightarrow |w - L/2| < L/2$$

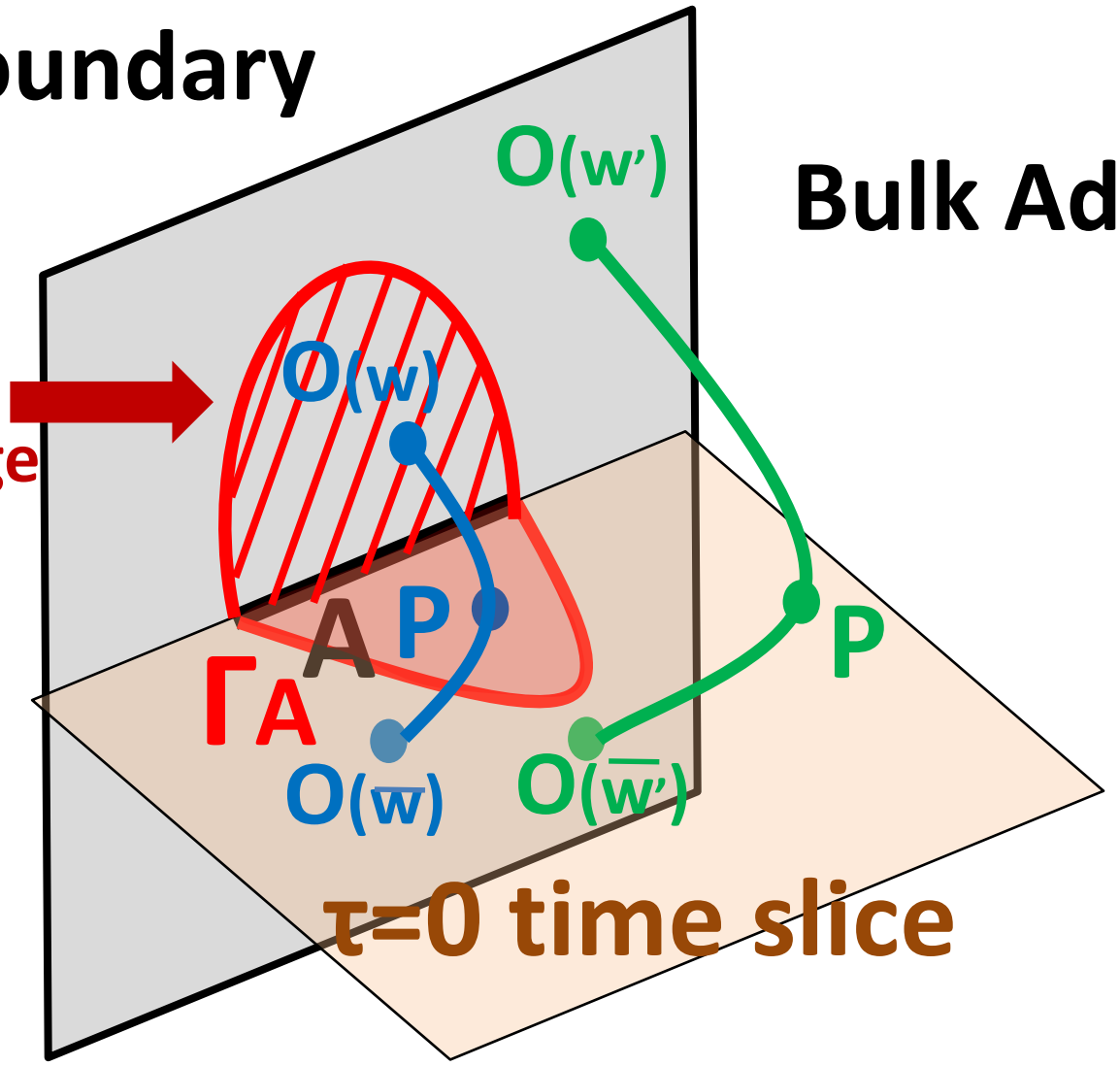
w∈ Entanglement Wedge

$\rho_A(w)$ and $\rho_A(w')$ are distinguishable !
I(ρ,ρ')=1 only when w=w'.

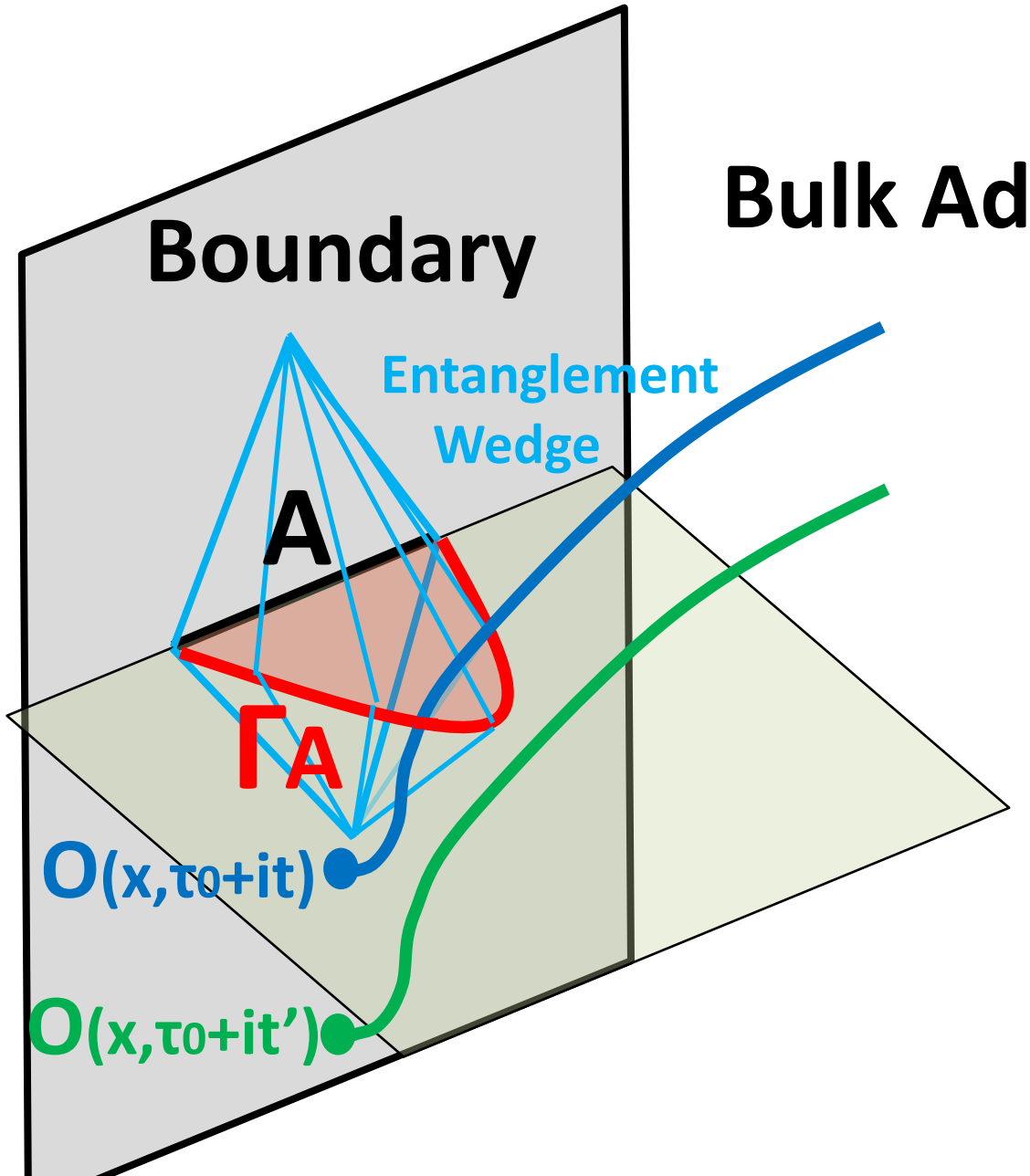
A Sketch of Entanglement Wedge from CFTs

CFT counterpart of Entanglement Wedge

$|w - L/2| < L/2$



Lorentzian Version



I(ρ,ρ’) in 2d Free Scalar CFTs

As a comparison, consider the massless free scalar CFT in two dimensions. We simply choose O to be

$$\boldsymbol{O(w,\bar{w}) = e^{i\alpha\varphi(w,\bar{w})}.} \qquad \boldsymbol{h = \alpha^2/2}$$

In this case we find (z1=z, z’3=-z’)

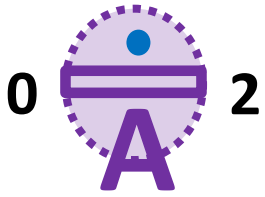
$$\boldsymbol{I(\rho,\rho') = \left[\frac{|z+z'|^2|z+\bar{z}||z'+\bar{z}'|}{4|z||z'||z+\bar{z}'|^2}\right]^{4h}}$$

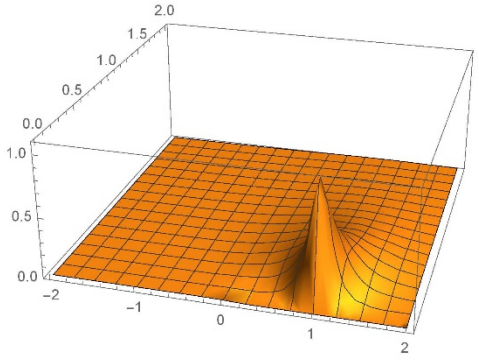
Plot of I(ρ,ρ')

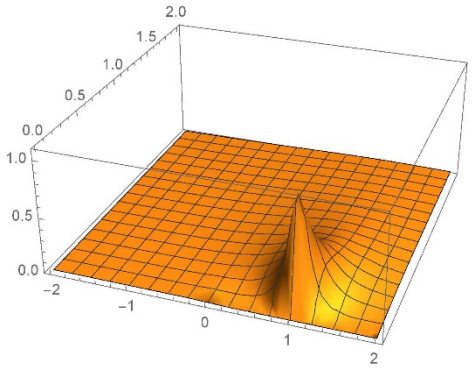
Holographic CFT

Free Scalar CFT

w=1+0.1i

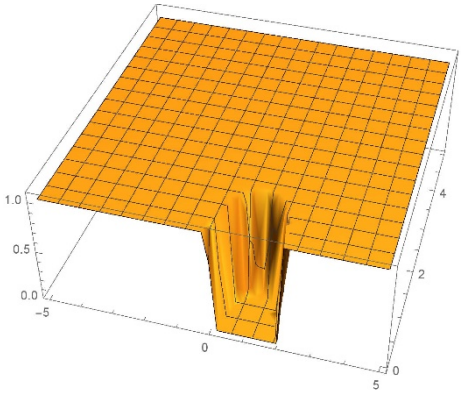


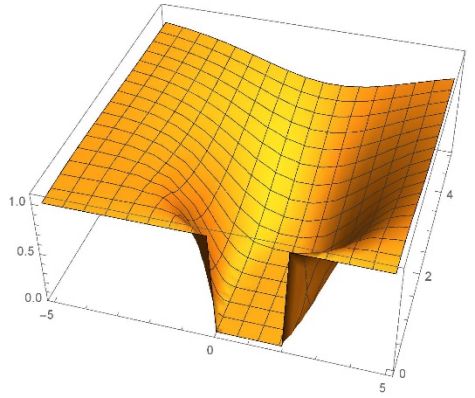




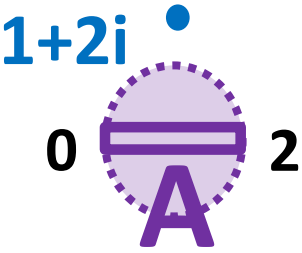
w=-1+0.1i

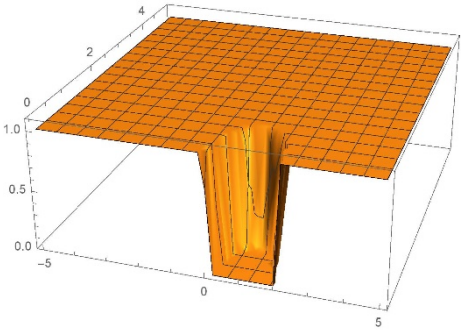


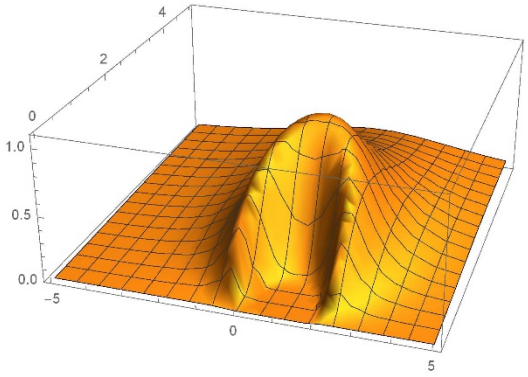




w=1+2i







Bures Metric in Holographic CFTs

To be exact, we need to calculate the Bures metric.

We introduce $A_{n,m} = \mathrm{Tr}[(\rho^m \rho' \rho^m)^n]$.

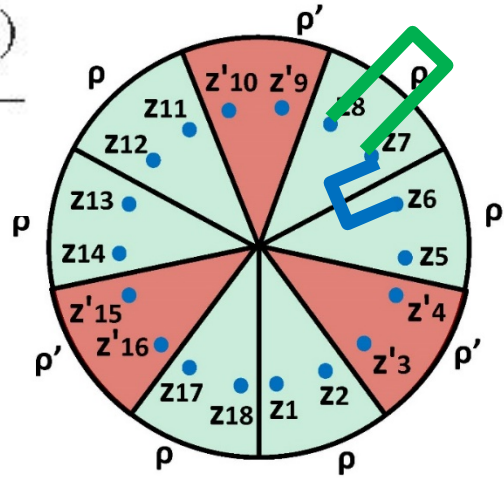
We compute the Bures metric via an analytical continuation:

$$A_{n=\frac{1}{2},m=\frac{1}{2}} = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right].$$

To evaluate Anm, consider the map: $z^k = \frac{w}{w-L}$, $k=(2m+1)n$

$$A_{n,m} = \frac{\langle O^\dagger(w_1)O(w_2)\cdots O^\dagger(w_{2k-1})O(w_{2k})\rangle \cdot Z^{(k)}}{\prod_{i=1}^{k}\langle O^\dagger(w_{2i-1})O(w_{2i})\rangle \cdot (Z^{(1)})^k}$$

$$A_{n=3,m=1} =$$



[See also Lashkari 2014, 2015, Ugajin 2016 for relative entropy]

Trivial Wick Contraction (Outside EW: |w-L/2|>L/2)

$$\langle O^\dagger(z_1)O(z_2)\cdots O^\dagger(z_{2k-1})O(z_{2k})\rangle$$
$$\simeq \prod_{j=1}^{k}\langle O^\dagger(z_{2j-1})O(z_{2j})\rangle \simeq \prod_{j=1}^{k}|z_{2j-1}-z_{2j}|^{-4h}$$

➡ $A_{1/2,1/2}=1$, $D_B(\rho,\rho')=0$ Trivial metric

Non-Trivial Wick Contraction (Inside EW: |w-L/2|<L/2)

$$\langle O^\dagger(z_1)O(z_2)\cdots O^\dagger(z_{2k-1})O(z_{2k})\rangle$$
$$\simeq \prod_{j=1}^{k}\langle O^\dagger(z_{2j-2})O(z_{2j-1})\rangle \simeq \prod_{j=1}^{k}|z_{2j-2}-z_{2j-1}|^{-4h}.$$

In the limit $n \to 1/2$ and $m \to 1/2$, this leads to

$$A_{1/2,1/2} \simeq |w-\bar{w}|^{2h}|w'-\bar{w}'|^{2h}|w'-\bar{w}|^{-4h},$$

$$dD_B^2 \simeq \frac{h}{\tau^2}(dx^2+d\tau^2).$$

➡ Reproduce the time slice of AdS

Other examples of Bure Metrics in Hol. CFTs

[1] Hol. CFT on Cylinder

$$ds_B^2 = \frac{h}{(\sinh\tau)^2}\left(d\tau^2 + dx^2\right)$$ ➡ **Agrees with Global AdS**

[2] Hol. CFT at Finite temp.

$$ds_B^2 = \frac{h(2\pi/\beta)^2}{\left(\sin\frac{2\pi}{\beta}\tau\right)^2}\left(d\tau^2 + dx^2\right)$$ ➡ **Agrees with Global BTZ**

Bures Distance/Metric in free scalar CFT

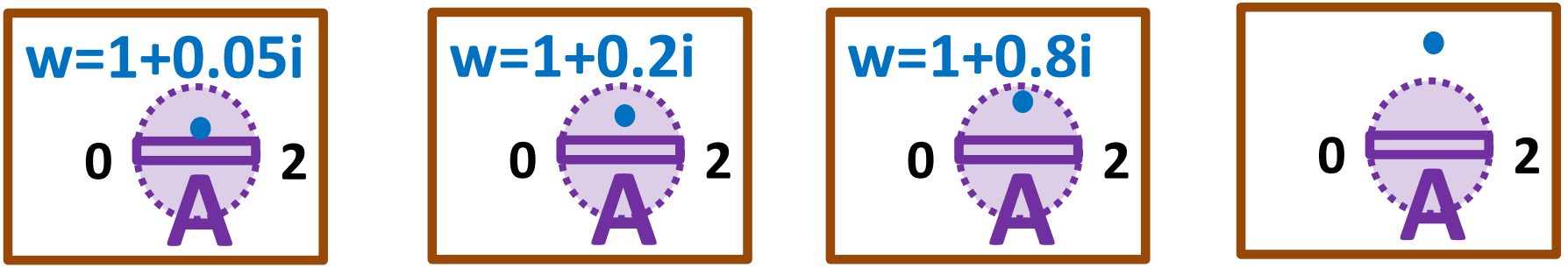
$$A_{\frac{1}{2},\frac{1}{2}}=\frac{(\sqrt{z}+\sqrt{z'})(\sqrt{\bar{z}}+\sqrt{\bar{z}'})(\sqrt{z}+\sqrt{\bar{z}})(\sqrt{z'}+\sqrt{\bar{z}'})}{4\sqrt{|z||z'|}(\sqrt{z}+\sqrt{\bar{z}'})(\sqrt{\bar{z}}+\sqrt{z'})},$$

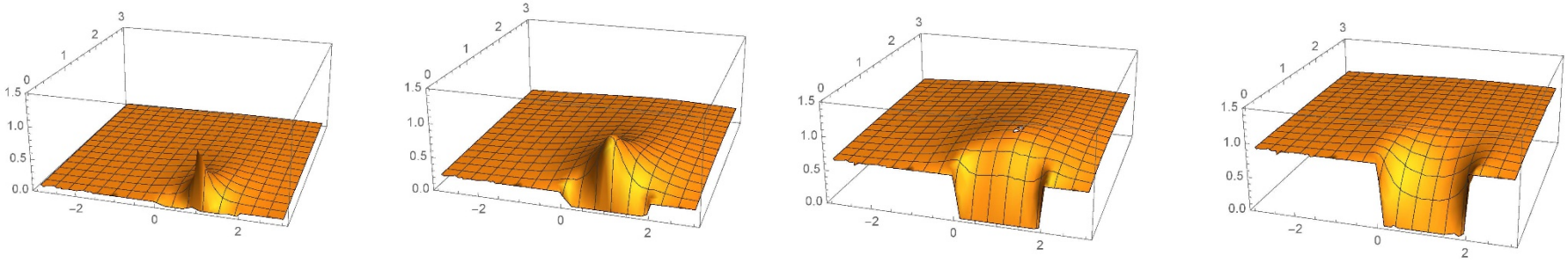
$$dD_B^2=-\frac{L^2(dw)^2}{16w^2(L-w)^2}-\frac{L^2(d\bar{w})^2}{16\bar{w}^2(L-\bar{w})^2}$$
$$+\frac{L^2(dw)(d\bar{w})}{2|w||w-L|\left(\sqrt{\bar{w}(w-L)}+\sqrt{w(\bar{w}-L)}\right)^2}.$$

We choose

$\boldsymbol{O=e^{i\varphi}}$

$\boldsymbol{(\alpha=1)}$





④ Double Intervals Case

Let us move on to a more complicated case where the subsystem A consists of double intervals in 2d CFTs.

We choose A as A=A1 ∪ A2, A1=[0,s] and A2=[l+s,l+2s].

Comformal Map: from a cpx plane with two slits to a cylinder

$$z = f(w) = \frac{J(\kappa^2)}{2} - \frac{J(\kappa^2)}{2K(\kappa^2)} \mathrm{sn}^{-1}(\widetilde{w}, \kappa^2),$$

where we introduced

$$\tilde{w} = \frac{2}{l}\left(w - s - \frac{l}{2}\right), \quad \mathbf{J}\,(\kappa^2) = 2\pi \frac{K(\kappa^2)}{K(1-\kappa^2)},$$

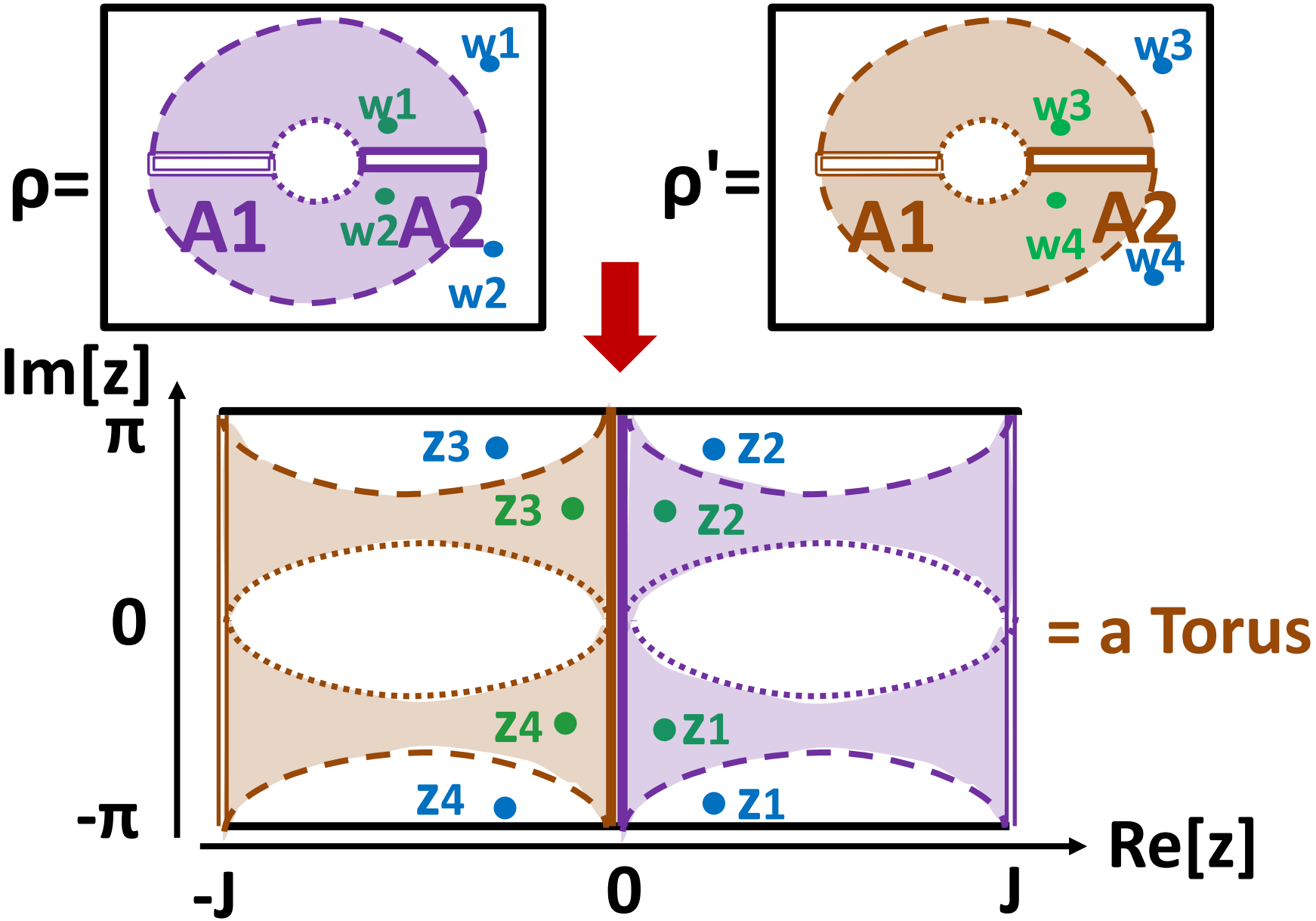
$$K(\kappa^2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}}, \quad \kappa = \frac{l}{l+2s}.$$

The function $\mathrm{sn}^{-1}(\tilde{w}, \kappa^2)$ is the Jacobi elliptic function:

$$\mathrm{sn}^{-1}(\tilde{w}, \kappa^2) = \int_0^{\tilde{w}} \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}}.$$

[e.g. Rajabpour 2015]

Computation of Tr[ρρ']



There are two phases depending on the torus moduli.

[As in mutual information for double intervals: Headrick 2010, Hartman 2013,…]

(i) Connected Phase

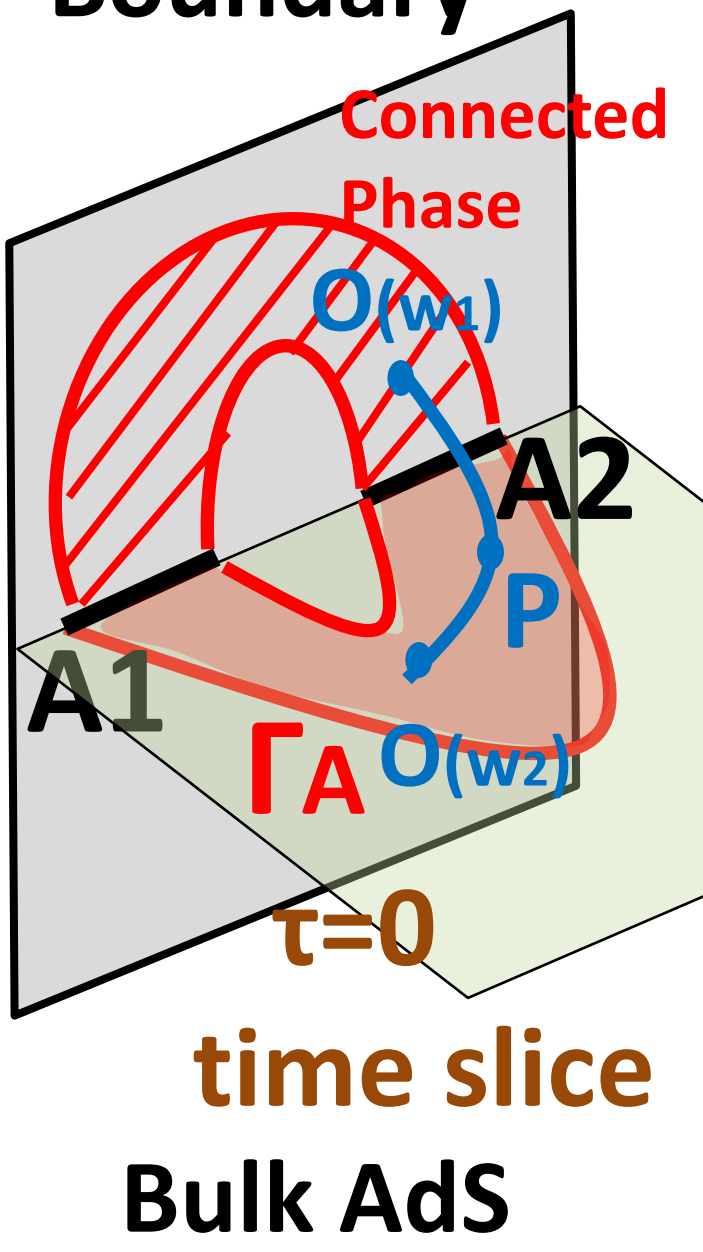
$J < \pi \quad (or\ \kappa < 3-2\sqrt{2})$

$$\langle O^\dagger(z) O(z') \rangle \approx \left| \mathrm{Sin}\left(\frac{\pi(z-z')}{2J} \right) \right|^{-4h}$$

(ii) Disconnected Phase

$J > \pi \quad (or\ \kappa > 3-2\sqrt{2})$

$$\langle O^\dagger(z) O(z') \rangle \approx \left| \mathrm{Sinh}\left(\frac{z-z'}{2} \right) \right|^{-4h}$$



Again, there are two possibilities:
Trivial Wick contraction and Non-trivial Wick contraction

(i) Connected Phase

$J < \pi \ \ (or\ \kappa < 3-2\sqrt{2})$

$$\left|\mathrm{Sin}\left(\frac{\pi(z2-z1)}{2J}\right)\right| > \left|\mathrm{Sin}\left(\frac{\pi(z3-z2)}{2J}\right)\right|$$

(ii) Disconnected Phase

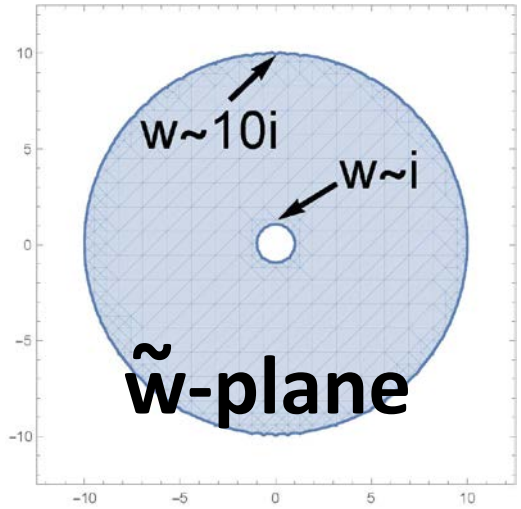
$J > \pi \ \ (or\ \kappa > 3-2\sqrt{2})$

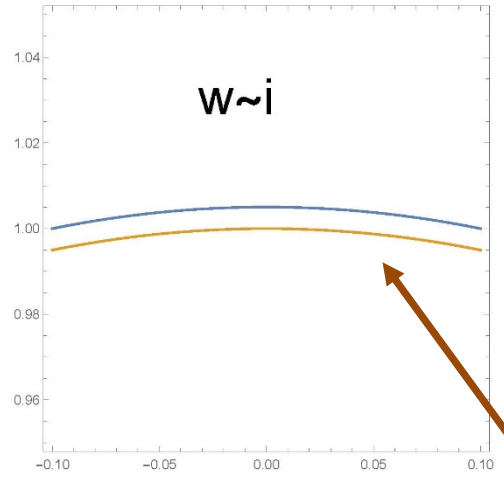
$$\left|\mathrm{Sinh}\left(\frac{z2-z1}{2}\right)\right| > \left|\mathrm{Sinh}\left(\frac{\pi(z3-z2)}{2J}\right)\right|$$

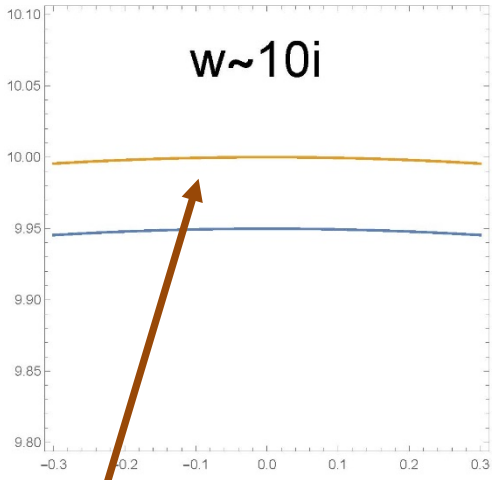
Non-trivial Contraction is favored

Numerical Plots of Regions of Non-trivial Contraction

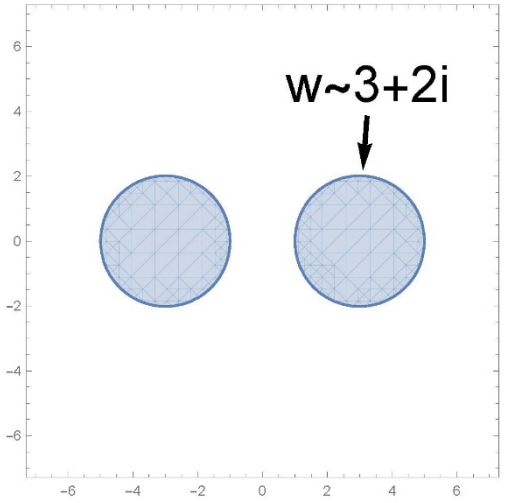
(i) Connected Phase (κ=0.1)

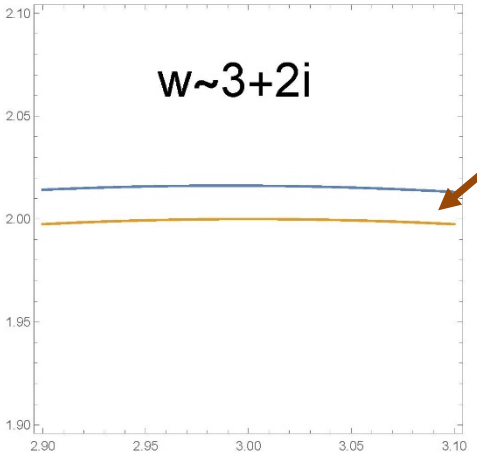






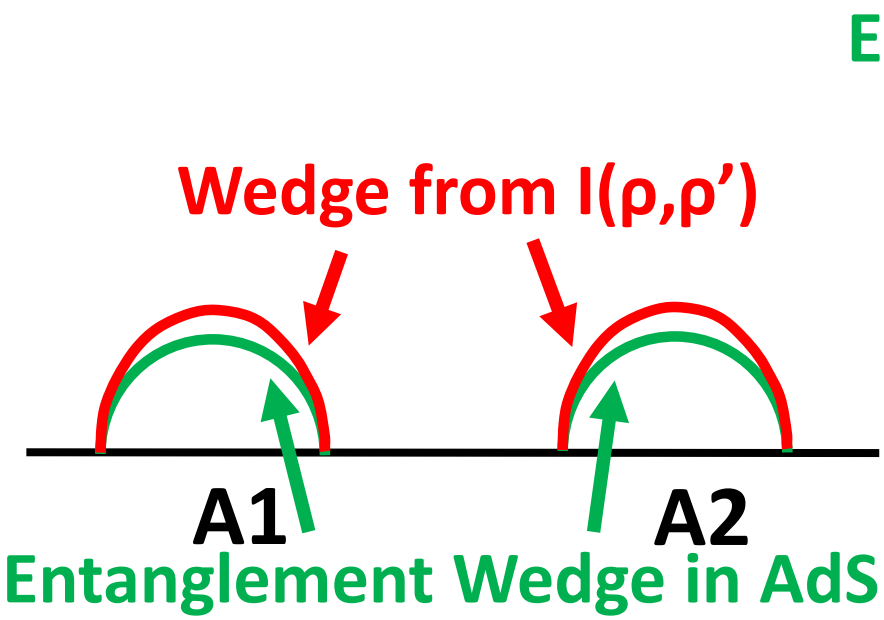
(ii) Disconnected Phase (κ=0.2)

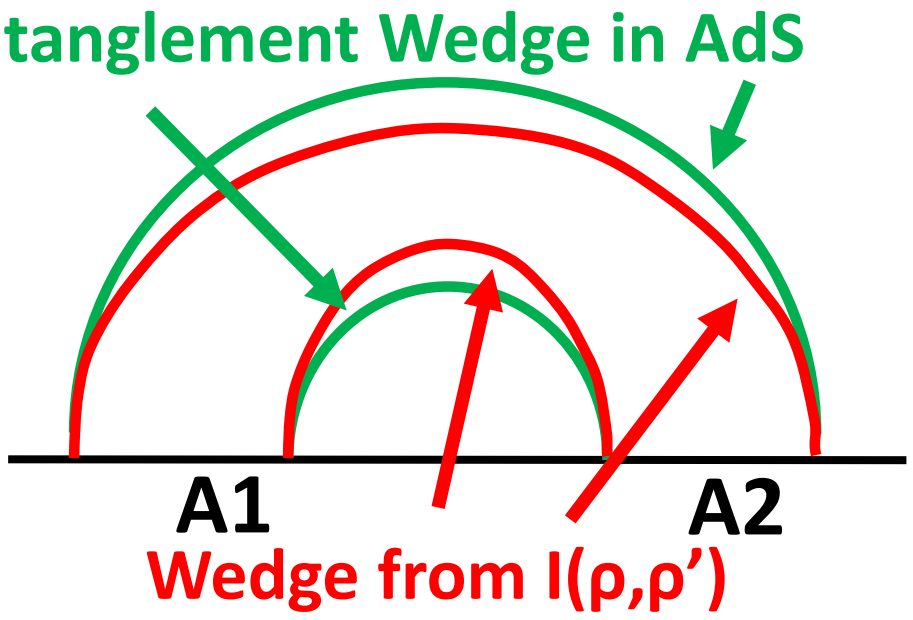




Actual EW

The distinguishability analysis for I(ρ,ρ') reproduces correct entanglement wedges up to a few percent errors.





The wedge from I(ρ,ρ') satisfies

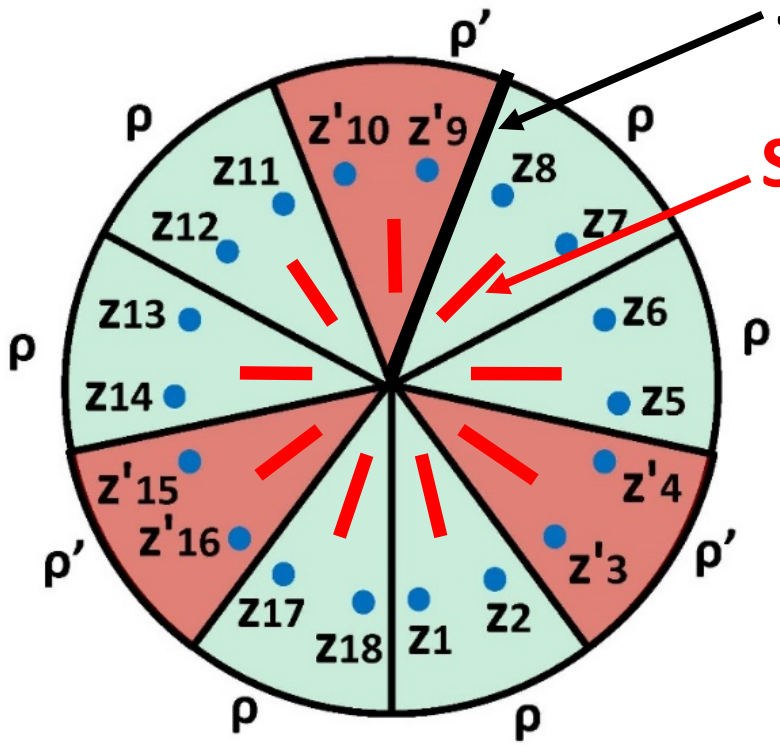
(a) the wedge for A1 ∪ A2 ⊃ the wedge for A1 (or A2)
(b) the wedge for A is the complement of that for A^c

We expect these small deviations are because we employ $I(\rho,\rho')$ instead of the Bures distance $D_B(\rho,\rho')$.

Indeed, the following heuristic argument shows the Bures metric reproduces the genuine Entanglement Wedge:

[Kusuki-Suzuki-Umemoto-TT, in preparation]

$$A_{n=3,m=1} =$$



Slit of A1

Slit of A2

can be negligible in n=m=1/2 limit !

⑤ HKLL States (Speculative Argument)

The quantum Cramer-Rao predicts

$$\left\langle\left\langle \delta x_i \delta x_j \right\rangle\right\rangle \geq G_{ij}^{-1} \sim O\left(\frac{1}{h}\right).$$

However, in the actual AdS/CFT predicts
a sub-AdS scale resolution of coordinate values:

$$\left\langle\left\langle \delta x_i \delta x_j \right\rangle\right\rangle \sim O\left(\frac{1}{c^2}\right).$$

This is because the operator $O(w, \bar{w})$ is not precisely dual to a fully localized bulk excitation.

The actual bulk local excitation is described by the HKLL states (2d CFT on a cylinder):

$$|\Psi_\alpha(\rho,\theta)\rangle = U(\rho,\theta)\cdot e^{-\delta\cdot H_{CFT}}\cdot e^{\frac{\pi i}{2}\cdot H_{CFT}}|J_\alpha\rangle$$

Unitary by SL(2,R) which changes localized points

UV regularization (for descendants)

Global Ishibashi State [SL(2,R) boundary state]

[Miyaji-Numasawa-Shiba-Watanbe-TT 2015]

The Bures metric reads

$$\boldsymbol{ds_B^2 = \frac{1}{4\delta^2}\left(d\rho^2 + \sinh^2\rho d\theta^2\right).}$$

(i) From our previous analysis, we expect Bures metric for the reduced density matrix ρ_A is the same as that for the pure state $|\Psi_\alpha(\rho,\theta)\rangle$ if the point is inside the wedge.

(ii) Moreover, since we consider large c CFTs (Hol. CFTs), it is natural to take $\delta \sim 1/c$.

If we assume these speculations, we find the Bures metric for reduced matrices of HKLL state:

$$\boldsymbol{ds_B^2 \approx c^2\left(d\rho^2 + \sinh^2\rho d\theta^2\right).}$$

This agrees with the correct metric of global AdS and the expected resolution: $\left\langle\left\langle \delta x_i \delta x_j \right\rangle\right\rangle \sim O\left(c^{-2}\right).$

⑥ Conclusions

- In this talk, we presented an approach using the Bures metric $D_B(\rho,\rho')$ to detect entanglement wedges directly from CFTs.

- We found that the large c property of holographic CFTs plays a crucial role for the emergence of EW.

- We can perfectly reproduce the EW for a single interval. For double intervals, we had to look at an analogous quantity $I(\rho,\rho')$. This "approximation" reproduces the correct EW up to a few percent errors. A heuristic argument shows that the Bures metric in CFTs correctly reproduces the actual EW.

Future Directions

- Other Information Distances ?
  [Trace distance, Quantum Fisher Metric,…]
- EW for Higher Dim. AdS/CFT ?
- Quantum Corrections to EW ?
- Connection to Entanglement Edge Cross Section ?
  EoP [Umemoto-TT 2017, Nguyen-Devakul-Halasch-Zaletel-Swingle 2017]
  Negativity [Kudler-Flam-Ryu 2018]
  Odd Entropy [Tamaoka 2018]
  Reflected Entropy [Dutta-Faulkner 2019]
- Connection to Path-integral Optimizations ?
  [Caputa-Kundu-Miyaji-Takayanagi-Watanabe 2017]

  ⇒ Sub AdS bulk locality in AdS/Tensor Network ?

Thank you very much !