Brunn-Minkowski type inequalities and positivity of vector bundles.

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First version of Brunn-Minkowski, BM 1:

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Theorem

Let A_0 and A_1 be two convex bodies in $\mathbb{R}^n$ and let for $0 \leq t \leq 1$

$$A_t = tA_1 + (1-t)A_0 = \{ta_1 + (1-t)a_0; a_1 \in A_1, \quad a_0 \in A_0\}.$$

Then the volumes of A_t satisfy

$$|A_t|^{1/n} \geq t|A_1|^{1/n} + (1-t)|A_0|^{1/n}.$$

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In other words,

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We will get back to that in a while, but so far we stick to the convex case.

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Let $\mathcal{A}$ be a convex body in $\mathbb{R}^{n+1}$ and let

$$A_t = \{x \in \mathbb{R}^n; (t, x) \in \mathcal{A}\}$$

Then $t \to \log |A_t|$ is a concave function of t.

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$t \to |A_t|^{1/n}$ is also concave, but the two versions are 'equivalent' because of homogeneity of Lebesgue measure.

A version of Brunn-Minkowski for functions

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Prékopa's theorem Let $\phi(t,x)$ *be convex on* $\mathbb{R}^{n+m} = \mathbb{R}^m_t \times \mathbb{R}^n_x$. *Define* $\tilde{\phi}(t)$ *on* $\mathbb{R}^m$ *by*

$$e^{-\tilde{\phi}(t)} = \int_{\mathbb{R}^n} e^{-\phi(t,x)} dx.$$

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3. This implies the (second version of) Brunn-Minkowski.

Proof 1: Via the Brascamp-Lieb inequality.

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Theorem

Brascamp-Lieb, -72
Let ϕ be a smooth strictly convex function on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} e^{-\phi} dx < \infty.$$

Let f be a continuous function on $\mathbb{R}$. Then there is a solution to the equation

$$u' = f$$

with

$$\int_{\mathbb{R}} u^2 e^{-\phi} dx \leq \int_{\mathbb{R}} \frac{f^2}{\phi''} e^{-\phi} dx.$$

In other words

$$\int_{\mathbb{R}} (u - \hat{u})^2 e^{-\phi} dx \leq \int_{\mathbb{R}} \frac{(u')^2}{\phi''} e^{-\phi} dx,$$

if

$$\hat{u} = \int_{\mathbb{R}} u e^{-\phi} dx / \int_{\mathbb{R}} e^{-\phi} dx.$$

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This is a complex version of Hörmander's estimate

Theorem

Hörmander -65
There is a solution to $\bar{\partial} u = f$ in $\mathbb{C}$, such that

$$\int_{\mathbb{C}} |u|^2 e^{-\phi} d\lambda(z) \leq \int_{\mathbb{C}} \frac{|f|^2}{\Delta \phi} e^{-\phi} d\lambda(z).$$

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Hence

$$\int_{\mathbb{R}} (u')^2/\phi'' e^{-\phi} dx \geq \int_{\mathbb{R}} (2kk'\phi' + k^2\phi'')e^{-\phi} dx$$

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Now integrate by parts the first term in the right hand side:

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This was assuming $u(0) = 0$; in general

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Equality holds if and only if $k' = 0$, i. e. $u = c\phi'$.

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Notice how convexity with respect to all variables comes in!

Prékopa-Leindler theorem

Theorem

Let ϕ_0 and ϕ_1 be functions on $\mathbb{R}^n$, not necessarily convex. Take $0 < t < 1$ and let ϕ_t be a third function satisfying

$$\phi_t(tx_1 + (1-t)x_0) \leq t\phi_1(x_1) + (1-t)\phi_0(x_0),$$

for all x_0 and x_1.

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It implies Lyusternik's generalization of the Brunn-Minkowski theorem.

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Theorem

Let $e^{-\phi}$ be a smooth positive function on $\mathbb{R}^n$ of total integral 1. Let C be a convex body of volume 1. Then there is a convex function ψ such that

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1. This is very easy when $n = 1$, which is really all we need.
2. Note that the change of variables $y = \partial\psi(x)$ shows that

$$\int_{R(\partial\psi)} dy = \int_{\mathbb{R}^n} MA(\psi)(x)dx.$$

This explains why the volume of C has to be one.

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This means that

$$\psi(x) = \sup_{y \in R(\partial\psi)} (x \cdot y - \hat{\psi}(y).$$

The gradient map

So

$$\hat{\psi}(y) + \psi(x) \geq x \cdot y,$$

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Hence $MA(\psi)(x)MA(\hat{\psi})(y) = 1$ at corresponding points.

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We use 'the real Yau's theorem' on ϕ_0 and ϕ_1 and get two convex functions ψ_0, ψ_1, with $R(\partial\psi_j) = C$, such that

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$$\psi_s(x) = \sup_y (x \cdot y - \hat{\psi}_s).$$

It is convex in (s, x) since the RHS is affine in (s, x). (!)

Recall

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This is a line and

$$\psi_s(x_s(y)) = x_s(y) \cdot y - \hat{\psi}_s(y)$$

is linear along each such line.

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Moreover and more importantly

$$\log MA(\psi_s(x_s(y)) = -\log MA(\hat{\psi}_s(y))$$

is convex along those lines.

So, if

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$$\phi_t(x_t) \leq -\log MA(\psi_t)(x_t).$$

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Integrating we get

$$\int e^{-\phi_t} dx \geq \int MA(\psi_t) dx = \int_C dy = 1,$$

and we are done.

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Let ψ_s, $0 < s < 1$, be a curve of convex functions on $\mathbb{R}^n$, such that $\hat{\psi}_s$ is linear in s (a geodesic in the space of convex functions). Assume $R(\partial\psi_s) = C$ where C is a convex body of volume 1.

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$$\int_{\mathbb{R}^n} \log(MA(\psi_s))MA(\psi_s)dx$$

is a convex function of s.

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$$\int_{\mathbb{R}^n} \log(MA(\psi_s))MA(\psi_s)dx$$

is a convex function of s.

Given what we have done this follows easily from

$$\int_{\mathbb{R}^n} \log(MA(\psi_s))MA(\psi_s)dx = -\int_C \log(MA(\hat{\psi}_s))dy.$$

Convexity of entropy

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The complex version of this is the convexity of the *Mabuchi K-energy*.