# Anomalies, inflow and holography

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## M-theory basics:

11D theory with 32 supercharges:

$$I_{11} = \frac{1}{2\kappa_{11}^2}\left[\int d^{11}x\sqrt{-G}R_G - \frac{1}{2}G\wedge *G - \frac{1}{6}C\wedge G\wedge G\right]$$

Supersymmetric flux backgrounds ($G\neq 0$ , $\mathrm{d}G=0$):

- FR: $AdS_7\times S^4$ max susy ($SO(5)$ symmetry)
- $AdS_5$ solutions
  - ◇ GM, BBBW:

$$\begin{cases} \text{(squashed) } S^4 \text{ fibered over } \Sigma_g \ - \text{ at least } U(1)^2 \text{ symmetry} \\ \text{8 or 16 supercharges} \end{cases}$$

  - ◇ GMSW:

$$\begin{cases} \text{Mink}_4\times_w \mathcal{C}(M_6) \quad \Rightarrow \quad AdS_7\times M_6 \\ M_6 - \text{compact complex} - S^2 \text{ bundle over Kähler } K_4 \end{cases}$$

## M-theory - strong coupling limit of type IIA strings

Higher derivative terms in string theory ($\alpha'$ and $g_s$ expansion) $\sim \mathcal{R}^{3l+1}$ lift to 11D

Special CP-odd couplings : $\Rightarrow$ $C \wedge X_8$

$$\begin{aligned} X_8 &= \frac{1}{48}\left(\frac{1}{4}p_1(TM)^2 - p_2(TM)\right) \\ &= \frac{1}{(2\pi)^4}\left(-\frac{1}{768}(\text{tr}\, R^2)^2 + \frac{1}{192}\text{tr}\, R^4\right) \end{aligned}$$

- 5pt function at one-loop
  - $\hat{\chi} = \frac{1}{4!(4\pi)^2} \cdot \frac{1}{2^4}$
    $\epsilon^{i_1\cdots i_8} R_{a_1a_2} \left(\Gamma^{a_1a_2}\right)^{i_1i_2} R_{a_3a_4} \left(\Gamma^{a_3a_4}\right)^{i_3i_4} R_{a_5a_6} \left(\Gamma^{a_5a_6}\right)^{i_5i_6} R_{a_7a_8} \left(\Gamma^{a_7a_8}\right)^{i_7i_8}$
  - $\hat{\chi} = \frac{1}{16}\left(8\chi + p_1(TM)^2 - 4p_2(TM)\right)$
- Needed for **string dualities**

$\Rightarrow$ solutions of M-theory $\neq$ solutions of D=11 sugra

## M5-branes

Classical soliton of 11D sugra

- the metric: $ds^2_{10} = e^{N_1 u(r)} \mathrm{d}s^2_6(W_{||}) + e^{N_2 u(r)} (\mathrm{d}s^2_5)_\perp$ ($r$ - distance away from M5)
- the four-form : $G_4 \sim \star_\perp du(r)$

$$\mathrm{d}G_4 = \delta_5(r)$$

- zero-mode expansion $G_4 \to G^{(0)} + h_3 \wedge \mathrm{d}u(r) + \cdots$

$$\mathrm{d} * G_4 \sim G_4 \wedge G_4 \qquad \Rightarrow \qquad h_3 = - *_{||} h_3$$

Theory on M5 $\Leftrightarrow$

- $(2,0)$ tensor multiplet
- $(\beta^-, \psi^\alpha, x^a) \quad \alpha = 1, ..., 4; \;\; a = 1, ..., 5$
- $SO(5)$ R-symmetry
- $ADE$ classification - non-Abelian M5

Symmetries of the theory without M5 $[C \wedge G \wedge G + C \wedge [\frac{1}{4}p_1^2(TM) - p_2(TM)]]$

- shift: $C_3 \to C_3 + \mathrm{d}\Lambda$
- diffeomorphisms

With M5 $i : W_6 \hookrightarrow M_{11}$

- $\delta(\int_{M_{11}} C \wedge G \wedge G) \;\to\; \int_{W_6} i^*(\Lambda \wedge G)$
  - ⋆ M5 coupling $\int_{W_6} h_3 \wedge i^*C$
  - ⋆ $\delta h_3 = i^*(\mathrm{d}\Lambda)$ .... relative cohomology ($\mathrm{d}h = i^*G$)

- diffeomorphisms and $\delta \int C \wedge X_8$ ?
  - ⋆ $X_8$ - (made of) characteristic class(es)... $X_8 = \mathrm{d}X_7^{(0)}$ and $\delta X_7^{(0)} = \mathrm{d}X_6^{(1)}$
  - ⋆ assume trivial normal bundle ($p_i(TM)|_W = p_i(TW)$)
  - ⋆ $\delta \int C \wedge X_8 \;\to\; \int_{W_6} X_6^{(1)}$
  - ⋆ anomaly inflow

## M5 ANOMALY

(2,0) tensor multiplet:

- Woldvolume chiral 2-form
  - ◇ $I_\beta = \frac{1}{5760}\left(16p_1(TW)^2 - 112p_2(TW)\right) \sim L(TW)$
- Worldvolume fermions
  - ◇ $I_D = \frac{1}{2}\widehat{A}(TW)\,\mathrm{ch}S(N)$
  - ◇ for trivial normal bundle:

$$I_D = 4 \times \frac{1}{2}\widehat{A}(TW) = \frac{1}{5760}\left(14p_1(TW)^2 - 8p_2(TW)\right)$$

- Total anomaly: $I_{\text{M5}} = \frac{1}{48}\left(\frac{1}{4}p_1(TW)^2 - p_2(TW)\right)$
- Cancelled via inflow from a bulk coupling $\sim C_3 \wedge X_8(TM)$

$$G_4\delta X_7^{(0)} \to \eta(M5)X_6^{(1)}(TM) \quad \leftrightarrow \quad \mathrm{d}^{-1}\delta\mathrm{d}^{-1}I_{(2,0)}$$

- Nontrivial normal bundle... $C \wedge G \wedge G$ is NOT ... diff invariant!

## Non-trivial normal bundle

(single) M5 worldvolume :

- Chiral 2-form
  - $I_\beta = \frac{1}{5760}\left(16p_1(TW)^2 - 112p_2(TW)\right) \sim L(TW)$
- Fermions
  - $I_D = \frac{1}{2}\widehat{A}(TW)\,\mathrm{ch}S(N)$
  - $\mathrm{ch}S(N) = 4 + \frac{1}{2}p_1(N) + \frac{1}{96}p_1(N)^2 + \frac{1}{24}p_2(N) + ...$

$$I_D = 4 \times \frac{1}{2}\widehat{A}(TW) + ... = \frac{1}{5760}\left(14p_1(TW)^2 - 8p_2(TW)\right) + ...$$

- Total anomaly:
  $I_{\text{M5}} = \frac{1}{48}\left(\frac{1}{4}(p_1(TW)^2 + p_1(N)^2 - 2p_1(TW)p_1(N)) - p_2(TW) + p_2(N)\right)$
- Anomaly from the bulk (using $p_1(TM|_W) = p_1(TW) + p_1(N)$, ...)

$$I_{\text{bulk}} = -\frac{1}{48}\left(\frac{1}{4}(p_1(TW)^2 + p_1(N)^2 - 2p_1(TW)p_1(N)) - p_2(TW) - p_2(N)\right)$$

- The result: $I_{\text{M5}} + I_{\text{bulk}} = \frac{p_2(N)}{24}$ !!!

## Non-singular $p$-branes

Brane worldvolume $W_d$ ($d = p + 1$) located at $y^a = 0$ ($a = 1, ..., D - d$) in $M_D$

$S_\epsilon(W_d)$ - $S^{D-d-1}$ sphere bundle - boundary of tubular neighbourhood of rad $\epsilon$, $D_\epsilon(W_d)$

- Magnetic source:
  - $\mathrm{d}G_{D-d-1} = 2\pi\delta(y^1)...\delta(y^{D-d})\mathrm{d}y^1 \wedge ... \wedge \mathrm{d}y^{D-y} \quad \Rightarrow \quad 2\pi\Phi_{D-d}$

Thom class of $N$ $\Phi_{D-d}$ $=$

- $\mathrm{d}(\rho(r)e_{2n-1}/2) \qquad 2n - 1 = D - d - 1$
- $\mathrm{d}\rho(r)e_{2n}/2 \qquad 2n = D - d - 1$

$e_{D-d-1}$ - global angular form

- rank$(N) = 2n$ - sphere bundle has fibers $S^{2n-1}$
  - $\mathrm{d}e_{2n-1} = -\pi^*(\chi(N)) \qquad \chi(N) \in H^{2n}(M, \mathbb{Z})$
- rank$(N) = 2n + 1$ - sphere bundle has fibers $S^{2n}$
  - $\mathrm{d}e_{2n} = 0 \qquad (e_{2n} = \mathrm{d}e^{(0)}_{2n-1}, \ \delta e^{(0)}_{2n-1} = e^{(1)}_{2n-2})$
  - cohomology class $e_{2n}$ : $\quad [e^2_{2n}] = \pi^*(p_n(N))$
  - at the level of differential forms : $\quad \pi_*(e^3_{2n}) = \pi_*(e_{2n}\pi^* p_n(N)) = 2p_n(N)$

## M5 ($W_6 \hookrightarrow M_{11}$ ) - $SO(5)$ $N$ bundle

$\mathbf{so(5)} \cong \Lambda^2\mathbb{R}^5$ - connection on $N$: $\quad \Theta^{ab} = -\Theta^{ba}$

Define $\hat{y}^a = y^a/r\,; \quad (D\hat{y})^a = \mathrm{d}\hat{y}^a - \Theta^{ab}\hat{y}^b\,; \quad F^{ab} = \mathrm{d}\Theta^{ab} - \Theta^{ac}\wedge\Theta^{ca}$

- $e_4(\Theta) = \frac{1}{64\pi^2}\epsilon_{a_1\ldots a_5}\left((D\hat{y})^{a_1}\ldots(D\hat{y})^{a_4} - 2F^{a_1a_2}(D\hat{y})^{a_3}(D\hat{y})^{a_4} + F^{a_1a_2}F^{a_3a_4}\right)\hat{y}^{a_5}$
- $e_3^{(0)}(\Theta) = \frac{1}{32\pi^2}\epsilon_{a_1\ldots a_5}\left(\Theta^{a_1a_2}\mathrm{d}\Theta^{a_3a_4}\hat{y}^{a_5} - \frac{1}{2}\Theta^{a_1a_2}\Theta^{a_3a_4}\mathrm{d}\hat{y}^{a_5} - 2\Theta^{a_1a_2}\mathrm{d}\hat{y}^{a_3}\mathrm{d}\hat{y}^{a_4}\hat{y}^{a_5}\right)$
- $e_2^{(1)}(\varepsilon,\,\Theta) = \frac{1}{16\pi^2}\epsilon_{a_1\ldots a_5}\varepsilon^{a_1a_2}\left(\mathrm{d}\hat{y}^{a_3}\mathrm{d}\hat{y}^{a_4}\hat{y}^{a_5} - \Theta^{a_3a_4}\mathrm{d}\hat{y}^{a_5}\right)$

Under gauge transformations: $\delta\Theta^{ab} = (D\varepsilon)^{ab}, \quad \delta\hat{y}^a = \varepsilon^{ab}\hat{y}^b$

In the presence of M5:

- $G = \mathrm{d}C \quad \longrightarrow \quad \mathrm{d}C - 2\pi\mathrm{d}\rho\wedge e_3^{(0)}/2$
  - $\delta C = -2\pi\mathrm{d}\rho\wedge e_2^{(1)}/2$
- Introduce $\sigma_3$: $\quad G_4 - 2\pi\rho e_4/2 = \mathrm{d}(C_3 - 2\pi\rho e_3^{(0)}/2) \equiv \mathrm{d}(C_3 - 2\pi\sigma_3)$
- CS couplings in presence of M5:

$$S_{\mathsf{cs}} = \lim_{\epsilon\to 0} -\frac{1}{6(2\pi)^2}\int_{M_{11}-D_\epsilon(W_6)}(C_3 - 2\pi\sigma_3)\wedge\mathrm{d}(C_3 - 2\pi\sigma_3)\wedge\mathrm{d}(C_3 - 2\pi\sigma_3)$$

Remember $\pi_*(e_4^3) = \pi^* p_n(N)) = 2p_2(N)$

Under diffs ($SO(5)$ transformations), $S_{CS}$ varies....

- $\delta((C_3 - 2\pi\sigma_3) = -2\pi \mathrm{d}(\rho e_2^{(1)}/2)$

$$\begin{aligned}
\delta S_{\mathsf{cs}} &= \lim_{\epsilon\to 0} \frac{1}{12\pi} \int_{M_{11}-D_\epsilon(W_6)} \mathrm{d}(\rho e_2^{(1)}/2) \wedge \mathrm{d}(C_3 - 2\pi\sigma_3) \wedge \mathrm{d}(C_3 - 2\pi\sigma_3) \\
&= -\frac{2\pi}{6} \int_{S_\epsilon(W_6)} \frac{e_4}{2} \wedge \frac{e_4}{2} \wedge \frac{e_2^{(1)}}{2}
\end{aligned}$$

Anomaly cancellation!... and a key to non-Abelian $(2,0)$ theories

- $I_{\text{M5}} + I_{\text{bulk}} + \delta S_{\mathsf{CS}} = 0$
- $Q$ coincident M5 - symmetry enhancement to $SU(Q)$
  - $\mathrm{d}G_4 = 2\pi Q \mathrm{d}\rho e_4/2$
  - no new ingredients in the anomaly cancellation

$\star$ $I_{\text{M5}}(Q) = Q I_{\text{M5}}(Q=1) + \frac{Q^3-Q}{24} p_2(N)$

## $(2,0)$ theories have ADE symmetry enhancement

- $A_{Q-1}$
  - ◇ remove a centre of mass (one free (2,0) multiplet) anomaly
  - ◇ $I^{(2,0)}_{A_{Q-1}} = (Q-1) I_{\text{M5}}(Q=1) + \frac{Q^3-Q}{24} p_2(N)$
- $D_Q$
  - ◇ $\mathbb{R}^5/\mathbb{Z}_2$ fixed points
  - ◇ $I^{(2,0)}_{D_Q} = Q I_{\text{M5}}(Q=1) + \frac{Q(2Q-1)(2Q-2)}{24} p_2(N)$
- $E$
  - ◇ no direct calculation (brane picture), but ... using 5D gauge CS confirm
  - ◇ $I^{(2,0)}_{A_{Q-1}} = r_G I_{\text{M5}}(Q=1) + \frac{d_G \cdot h^\vee_G}{24} p_2(N)$
  - ◇ $r_G, d_G, h^\vee_G$ - rank, dimension, dual Coxeter

## $E$-strings

- ◇ M5 inflow + HW
- ◇ HW: $G|_{x^{11}=0} = I_4(1) \sim \alpha'/4(\text{tr}\, R^2/2 - \text{tr}\, F_1^2)$
- ◇ $I_{12}(i) = I_4(i) \wedge X_8 + \frac{1}{6} I_4(i) \wedge I_4(i) \wedge I_4(i) \quad (i=1,2); \quad I_{12} = I_{12}(1) + I_{12}(2)$

## CLASS $\mathcal{I}_{12}$

$$\mathcal{I}_{12} = -\frac{1}{6}\, e_4 \wedge e_4 \wedge e_4 - e_4 \wedge X_8$$

M-theory on a spacetime with non-trivial boundary:

- $S_\epsilon(W_6) \sim M_{11} - D_\epsilon(W_6)$ : $\qquad S^4 \hookrightarrow S_\epsilon(W_6) \to W_6$

Non-invariance of the action under diffeomorphisms:

- $\delta S_{\rm M} = 2\pi \int_{S_\epsilon(W_6)} \mathcal{I}_{10}^{(1)}$
  - $\mathrm{d}\mathcal{I}_{10}^{(1)} = \delta \mathcal{I}_{11}^{(0)}$ , $\qquad \mathrm{d}\mathcal{I}_{11}^{(0)} = \mathcal{I}_{12}$

$$I_8^{\rm inflow} = \int_{S^4} \mathcal{I}_{12}$$

$$I_8^{\rm inflow} + I_8^{\rm CFT} + I_8^{\rm decoup} = 0$$

## Four-dimensional anomalies

M5 on a Riemann surface $\Sigma_g$ : $\qquad W_6 = W_4 \times \Sigma_g$

- Space $M_6$ (fixed by supersymmetry)
  - $\diamond$ $S^4 \hookrightarrow M_6 \to \Sigma_{g,0}$
  - $\diamond$ $M_6^{n=0} \hookrightarrow S_\epsilon(W_6) \to W_4$
  - $\diamond$ $SO(5) \longrightarrow SO(2) \times SO(3)$ or $SO(2) \times SO(2)$
- $I_6^{\text{inflow}} = \int_{M_6^{n=0}} \mathcal{I}_{12} \qquad \& \qquad I_6^{\text{inflow}} + I_6^{\text{CFT}} + I_6^{\text{decoup}} = 0$

Riemann surface with **punctures** $\Sigma_{g,n}$:

- $M_6 = M_6^{\text{bulk}} \cup \bigcup_{\alpha=1}^n X_6^\alpha$
  - $\diamond$ $S^4 \hookrightarrow M_6^{\text{bulk}} \to \Sigma_{g,n}$
  - $\diamond$ $X_6^\alpha$ – geometry encoding puncture data
- $I_6^{\text{inflow}} = \int_{M_6} \mathcal{I}_{12} = I_6^{\text{inflow}}(\Sigma_{g,n}) + \sum_{\alpha=1}^n I_6^{\text{inflow}}(P_\alpha)$
  - $\diamond$ $I_6^{\text{inflow}}(\Sigma_{g,n}) = \int_{M_6^{\text{bulk}}} \mathcal{I}_{12} = \int_{\Sigma_{g,n}} I_8^{\text{inflow}} \qquad \& \qquad I_6^{\text{inflow}}(P_\alpha) = \int_{X_6^\alpha} \mathcal{I}_{12}$

## $\mathcal{N}=2$ class $\mathcal{S}$ SCFTs

$$I_6^{\rm CFT} = (n_v - n_h)\left[\frac{1}{3}\,(c_1^r)^3 - \frac{1}{12}\,c_1^r\,p_1(TW_4)\right] - n_v\,c_1^r\,c_2^R - k_G\,c_1^r\,{\rm ch}_2(G_F)$$

- ◇ Flavor symmetry $\quad G_F$
- ◇ R-symmetry $\quad U(1)_r \times SU(2)_R$

M5 geometry: $\qquad TM_{11}|_W = TW_4 \oplus \underbrace{T\Sigma_{g,n} \oplus SO(2)}_{T^*\Sigma_{g,n}\text{ (hyper-Kähler)}} \oplus SO(3)$

- ◇ Chern root of $SO(2)$ : $\; n = n^{\text{4d}} + \int_{\Sigma_{g,n}} \chi(\Sigma_{g,n}) = 2c_1^r + (2(1-g) - n)$

$$\begin{cases} \bullet & n_v^{\rm inflow} + n_v^{\rm CFT} = n_v^{\text{free tensor}}(\Sigma_{g,0}) = \frac{1}{2}\chi(\Sigma_{g,0}) \\ \bullet & n_h^{\rm inflow} + n_h^{\rm CFT} = n_h^{\text{free tensor}}(\Sigma_{g,0}) = 0 \\ \bullet & k_{SU(k_a)}^{\rm inflow} + k_{SU(k_a)}^{\rm CFT} = 0 \end{cases}$$

## $S^4$ reduction to $AdS_7$

- ◇ vacuum configuration: $G = 2\pi Q\,\text{Vol}(S^4)$
- ◇ expand to include fluctuations - flux needs to be
  - ▷ invariant under $SO(5)$ gauge transformations
  - ▷ $\mathrm{d}G = 0$
  - ▷ quantisation $\int_{S^4} G/2 = 2\pi\, Q$
- ◇ $G = 2\pi Q e_4/2 +$ fluctuations in $C_3$ works!
  - ◇ $\int_{M_{11}} C \wedge G \wedge G \quad \longrightarrow \quad \sim Q^3 \int_{AdS_7} p_2^{(0)}(A)$

- "topological sector" - extension to $AdS_5$ reductions
- Scalars can be included

$$64\pi^2 e_4 \longrightarrow \epsilon_{a_1\ldots a_5} \quad \Big(-\tfrac{1}{3}(D\hat{y})^{a_1}\ldots(D\hat{y})^{a_4}\,\tfrac{(T\cdot\hat{y})^{a_5}}{\hat{y}\cdot T\cdot\hat{y}} + \tfrac{4}{3}(D\hat{y})^{a_1}\ldots D\left(\tfrac{(T\cdot\hat{y})^{a_4}}{\hat{y}\cdot T\cdot\hat{y}}\right)\hat{y}^{a_5}$$
$$-2F^{a_1a_2}(D\hat{y})^{a_3}(D\hat{y})^{a_4}\,\tfrac{(T\cdot\hat{y})^{a_5}}{\hat{y}\cdot T\cdot\hat{y}} + F^{a_1a_2}F^{a_3a_4}\hat{y}^{a_5}\Big)$$

  - ◇ consistent truncation of D=11 supergravity to D=7 fields
  - ◇ $T^{ab} = (\Pi^{-1}\cdot\Pi^{-1})^{ab}$ - coset elements of $SL(5)/SO(5)$
  - ◇ Deformations of GAF $\quad\Rightarrow\quad$ Consistent truncations (!?)
  - ◇ Note $\quad e_5 \mapsto SL(6)\times SL(2) \subset E_6 \qquad \& \qquad e_7 \mapsto SL(8) -$ max subgroup of $E_7$

$AdS_{d+1} \times_w M^{\text{hol}}_{10-d}$

(think of M5 with $W_6 = W_d \times \Sigma_{6-d}$ & $S^4 \hookrightarrow M_{10-d} \to \Sigma_{6-d}$ )

- Flux $G|_{r=\epsilon} = 2\pi\, E_4$
  - $E_4$ invariant under symmetries of $M_{10-d}$
  - $\left.\begin{array}{l} \mathrm{d}E_4 = 0 \\ E_4 \ - \ \text{globally defined} \end{array}\right\} \quad \leftrightarrow \quad \int_{S^4} E_4 = N$
  - $E_4$ not uniquely fixed (yet)

- $M_{10-d}$ has isometries (killing vectors $k_I$) and harmonic 2-forms $\omega_\alpha$:
  - Isometries
    - ⋆ Lie algebra: $\pounds_I k_J \equiv \pounds_{k_I} k_J = [k_I, k_J] = f_{IJ}{}^K\, k_K$
    - ⋆ Gaugings: $\mathrm{d}\xi^m \ \to \ D\xi^m = \mathrm{d}\xi^m + k_I^m\, A^I$ (forms: $\omega \to \omega^{\mathrm{g}}$)
    - ⋆ Connections: $F^I = \mathrm{d}A^I - \frac{1}{2}\, f_{JK}{}^I\, A^J\, A^K$
  - $H^2(M_{10-d})$
    - ⋆ $C_3 = A^\alpha \omega_\alpha\,, \qquad \alpha = 1, ..., b^2(M_{10-d})$
  - ... but no NWVV fields (cf $S^7$) - generally $\chi(M_{10-d}) \neq 0$

General parametrisation of $E_4$:

$$E_4 = V_4^{\rm eq} + \frac{F^\alpha}{2\pi}\,\omega_\alpha^{\rm eq} + \gamma_4$$

where $\begin{cases} V_4^{\rm eq} = V_4^{\rm g} + F^I\,\omega_I^{\rm g} + F^I\,F^J\,\sigma_{IJ} \\ \omega_\alpha^{\rm eq} = \omega_\alpha^{\rm g} + 2\,F^I\,\sigma_{I\alpha} \end{cases}$ $\Leftarrow$ closed & gauge-inv.

$$E_4 = V_4^{\rm g} + F^X\,\omega_X^{\rm g} + F^X\,F^Y\,\sigma_{XY} + \gamma_4 \qquad \text{for} \quad X = (I,\alpha)$$

$$F^X = (F^I, \tfrac{1}{2\pi}\,F^\alpha)\,, \qquad \omega_X = (\omega_I, \omega_\alpha)\,, \qquad \sigma_{XY} = \begin{pmatrix} \sigma_{IJ} & \sigma_{I\beta} \\ \sigma_{J\alpha} & 0 \end{pmatrix}$$

$$\begin{cases} \pounds_X V_4 = 0\,, \qquad \pounds_X \omega_Y = f_{XY}{}^Z\,\omega_Z\,, \qquad \pounds_X \sigma_{Y_1Y_2} = f_{XY_1}{}^Z\,\sigma_{ZY_2} + f_{XY_2}{}^Z\,\sigma_{Y_1Z} \\ dV_4 = 0\,, \qquad \iota_X V_4 + d\omega_X = 0\,, \qquad \iota_{(X}\omega_{Y)} + d\sigma_{XY} = 0 \\ + \qquad \pounds_\alpha \equiv 0 \quad \& \quad \iota_\alpha \equiv 0\,, \qquad f_{XY}{}^Z = \{f_{IJ}{}^K, 0\} \end{cases}$$

- $\gamma_4 \longrightarrow 0$
- D=4: $N_\alpha\,F^\alpha = 0\,, \quad (N_\alpha = \int_{M_6} V_4\,\omega_\alpha) \quad \Leftarrow \quad \int_{M_6} V_4\,\omega_I = 0$

$E_4 \quad \Rightarrow \quad$ equivariant class

- Action of $G$ on $\Omega^*(M_{10-d})$

$$\begin{array}{ccccc} \alpha & : & \mathfrak{g} & \to & \Omega^*(M_{10-d}) \\ & & \mathcal{X} & \mapsto & \alpha(\mathcal{X}) \end{array}$$

  - ⋆ $\mathfrak{g} \ni \mathcal{X} = \mathcal{X}^I\, t_I$ ( $\{t_I\}$ - a basis of $\mathfrak{g}$)
    - ◇ $\alpha(\mathcal{X}) = V_4 + \omega_I\, \mathcal{X}^I + \sigma_{IJ}\, \mathcal{X}^I\, \mathcal{X}^J$
    - ◇ $\alpha(\mathcal{X})$ - homogeneous of degree $4$ ([(dif. form) $+ 2 \times$ (polynomial)])
  - ⋆ can extend the Lie algebra $\mathfrak{g}$ by adding new Abelian generators $t_\alpha$
    - ◇ $\mathcal{X} = \mathcal{X}^X\, t_X$
      - ∗ $\iota_\alpha = 0\,, \qquad \pounds_\alpha = 0$ (trivial action on forms)
  - ⋆ Invariance & closure of $E_4 \quad \Leftrightarrow \quad d_{\rm eq}\alpha = 0$
    - ◇ $(d_{\rm eq}\alpha)(\mathcal{X}) = d\big(\alpha(\mathcal{X})\big) + \iota_{\mathcal{X}}\alpha(\mathcal{X})$
  - ⋆ Deformations: mod out $\alpha(\mathcal{X}) \to \alpha(\mathcal{X}) + (d_{\rm eq}\beta)(\mathcal{X})$

- Fixing leftover ambiguities

$$E_4^2 + 2\, X_8 = 0 \qquad \text{in cohomology of } M_{10-d}$$

(cf. $\frac{2\pi}{(2\pi\ell_p)^3}\, d * G_4 = \frac{1}{2}\, \frac{G_4}{2\pi}\, \frac{G_4}{2\pi} + X_8$)

## EX: $M_6 -$ GMSW

- $S^2$ bundle over a product of 2 constant-curvature Riemann surfaces

$$ds^2(M_6) = h_{S^2}^2\, ds^2(S^2) + h_\Sigma^2\, ds^2(\Sigma_g) + h_y^2\, dy^2 + h_\psi^2\, D\psi^2$$

- ⋆ $y \in [y_{\rm min}, y_{\rm max}]$, $h_{S^2}$, $h_\Sigma$, $h_y$, $h_\psi$ - functions of $y$ only
- ⋆ topology:
  - ◇ $\Sigma_g$ $g = 0$ or $g \geq 2$
  - ◇ $\{y, \psi\}$ - topologically $S^2_{y\psi}$
  - ◇ $dD\psi = -2\, V_{S^2} - \chi\, V_\Sigma\,,$ $(V_{S^2}, V_\Sigma) -$ normalised volume forms
- ⋆ Fluxes $(N_{S^2}, N_\Sigma, M)$
  - ◇ $N_\Sigma := \int_{\Sigma_g \times S^2_{y\psi}} V_4$
  - ◇ $N_{S^2} := \int_{S^2 \times S^2_{y\psi}} V_4$
  - ◇ $M = \frac{1}{2}\,(N_{\rm N} + N_{\rm S}) = \frac{1}{2}\left(\int_{y=y_{\rm max}} V_4 + \int_{y=y_{\rm min}} V_4\right)$ ($\Sigma_g \times S^2$ integrals)
  - ◇ constraint: $N_{\rm N} - N_{\rm S} + 2\, N_\Sigma + \chi\, N_{S^2} = 0$
- ⋆ 3 independent harmonic 2-forms $\omega_\alpha$ & constraint:

$$N_\alpha\, F^\alpha = N_{S^2}\, F^1 + N_\Sigma\, F^2 + M\, F^3 = 0$$

The result in terms of $F^2$, $F^3$ and $F^\psi = dA^\psi$

- $A^\psi$ 4d connection associated to the $U(1)_\psi$ isometry

$$
\begin{aligned}
(2\pi)^3\, I_6^{\text{inflow}} = & -\frac{1}{24}\left(\chi N_{S^2} + 2N_\Sigma\right) p_1(TW_4)\, F^\psi + \frac{1}{24}\,\chi\, p_1(TW_4)\, F^3 \\
& + \left[\frac{N_{S^2}^2\left(12M^2 + 4\chi N_{S^2} N_\Sigma + \chi^2 N_{S^2}^2 - 12N_\Sigma^2\right)}{24\left(\chi N_{S^2} - 2N_\Sigma\right)} + \frac{1}{12}\,\chi\, N_{S^2}\right] p_1(SO(3))\, F^\psi \\
& - \frac{1}{4}\, M\, N_{S^2}\, p_1(SO(3))\, F^2 - \frac{1}{8} N_{S^2}\left(\chi N_{S^2} + 2N_\Sigma\right)\, p_1(SO(3))\, F^3 \\
& + \frac{1}{6}\left(\chi N_{S^2} + 2N_\Sigma\right)\, (F^\psi)^3 \\
& + \left[-\frac{N_{S^2} N_\Sigma\left(-2M - \chi N_{S^2} + 2N_\Sigma\right)\left(2M - \chi N_{S^2} + 2N_\Sigma\right)}{\left(2N_\Sigma - \chi N_{S^2}\right)^2} - \frac{1}{2}\,\chi\right] (F^\psi)^2\, F^3 \\
& + \frac{4M N_\Sigma}{2N_\Sigma - \chi N_{S^2}}\, F^\psi\, F^2\, F^3 + \frac{4M^2 - \chi^2 N_{S^2}^2 + 4N_\Sigma^2}{2\left(2N_\Sigma - \chi N_{S^2}\right)}\, F^\psi\, (F^3)^2 \\
& + \frac{N_\Sigma}{N_{S^2}}\, (F^2)^2\, F^3 + \frac{M}{N_{S^2}}\, F^2\, (F^3)^2 - \frac{1}{6}\,\chi\, (F^3)^3
\end{aligned}
$$

## Central charge ar large $N$

- $F^{\psi} \to F^{\mathrm{R}}\,, \qquad F^2 \to s^2\, F^{\mathrm{R}}\,, \qquad F^3 \to s^3\, F^{\mathrm{R}}$
  - ◇ turn off the flavour $SO(3)$
- Large $N$ : $N_{S^2} \sim N_\Sigma \sim M \sim F^2 \sim F^3 \sim \mathcal{O}(N)$
- maximise the coefficient of $(F^{\mathrm{R}})^3$:

$$(2\pi)^3\, I_6^{\mathrm{CFT}} = (2\pi)^3\, (-I_6^{\mathrm{inflow}}) = \frac{1}{6}\, \mathrm{tr}\, \mathrm{R}^3\, (F^{\mathrm{R}})^3$$

- $a = c = \frac{9}{32}\, \mathrm{tr}\, \mathrm{R}^3$

$$c = \frac{9\, N_{S^2}^2\, N_\Sigma^2\, (4\, N_\Sigma^2 + 2\, \chi\, N_{S^2}\, N_\Sigma + \chi^2\, N_{S^2}^2 - 3\, M^2)^{3/2}}{4\, (2\, \chi\, N_{S^2}\, N_\Sigma + 3\, M^2)^2}$$
$$- \frac{9\, N_{S^2}^2\, N_\Sigma^2\, (2\, N_\Sigma + \chi\, N_{S^2})\, (8\, N_\Sigma^2 + 2\, \chi\, N_{S^2}\, N_\Sigma + 2\, \chi^2\, N_{S^2}^2 - 9\, M^2)}{8\, (2\, \chi\, N_{S^2}\, N_\Sigma + 3\, M^2)^2}$$

- agreement with holographic calculation for $M = 0$ and $\Sigma_g = S^2$ ($\chi = 2$)

$* * *$

- Chern-Simons couplings in $AdS_{d+1}$ $\Leftrightarrow$ Holographic computation of anomalies
- Topology of $M_{10-d}$ and $E_4$ determinant!

$\star$ Extension to IIB ($E_5$ is very different from $E_4$!)

$\star$ Beyond "topological deformations" ...

$* * *$