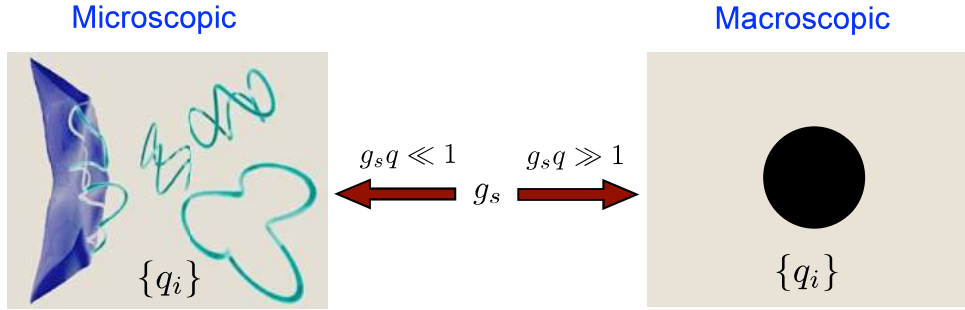
Phases of 4d SYM at large N and susy black holes in AdS5

Sameer Murthy
King's College London

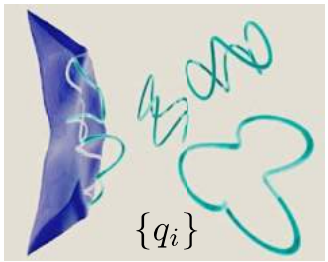
KIAS-YITP 2019
Seoul, Sep 23 2019

Explaining microscopic origin of BH entropy is a great success of string theory



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Microscopic



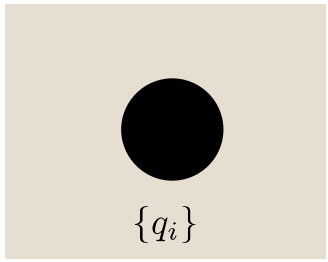
Strominger-Vafa '96

$$d_{\rm micro}(q) \;=\; e^{\pi\sqrt{q_1 q_2 q_3}}$$

$g_s q \ll 1$ $\quad$ $g_s q \gg 1$

g_s

Macroscopic



Bekenstein-Hawking '74

$$S^{\rm class}_{\rm BH}(q) \;=\; \frac{A_{\rm H}}{4\,\ell^2_{\rm Pl}} \;=\; \pi\sqrt{q_1 q_2 q_3}$$

$$\boxed{\log d_{\rm micro}(q) \;=\; S^{\rm class}_{\rm BH}(q) + \ldots}$$

The microstates of a susy BH are captured by a susy index in the microscopic theory

- Identify the states of the microscopic theory with the same charges as that of the BH.
- Calculate supersymmetric index $\mathrm{Tr}(-1)^F$ at weak coupling. Index protected, flow to strong coupling.

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- Does the susy BH contribute to the gravitational functional integral? Subtle question and needs a regulator.

[Sen '09; Dabholkar, Gomes, S.M., Sen '10]

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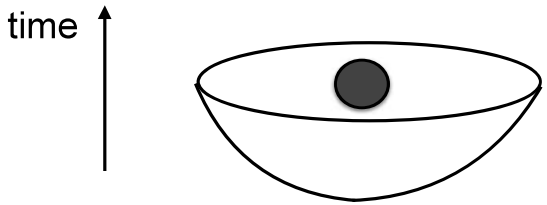
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[Sen '09; Dabholkar, Gomes, S.M., Sen '10]

- BHs in flat space, AdS_3, AdS_4

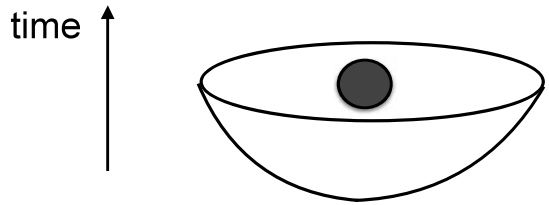
[Benini, Zaffaroni '15; + Hristov '16]

A puzzle: “The most general index in SCFT4 does not have exponential growth of states”



BH in $\mathrm{AdS}_5 \times S^5$

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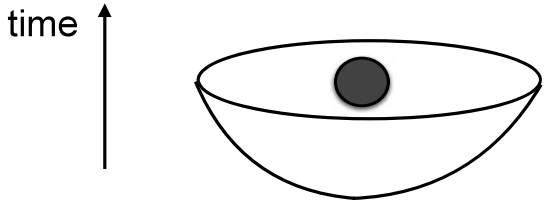
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$$S_{\mathrm{BH}} = \frac{A_H}{4G_N} = O(N^2)$$

[Gutowski, Reall ’04]

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$\mathcal{N} = 4$ $SU(N)$ SYM
on $S^3 \times$ time

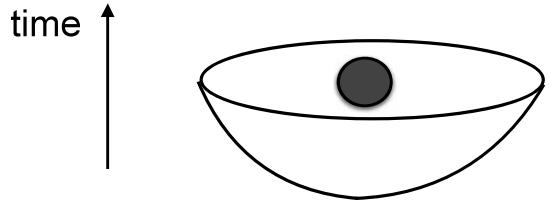
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BH in $\mathrm{AdS}_5 \times S^5$

$$\mathcal{I} = \mathrm{Tr}(-1)^F e^{2\pi \mathrm{i} \tau_i q_i}$$
$$\log \mathcal{I} = O(1)$$

[Kinney, Maldacena, Minwalla, Raju '05]

$$S_{\mathrm{BH}} = \frac{A_H}{4G_N} = O(N^2)$$

[Gutowski, Reall '04]

The N=4 superconformal index is a large-N matrix integral

[Kinney, Maldacena, Minwalla, Raju '05; Romelsberger '05]

$$\mathcal{I}(\vec{\tau}) \;=\; \mathrm{Tr}_{\mathcal{H}_{\rm phys}}\,(-1)^F\,\mathrm{e}^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+2\pi\mathrm{i}\tau_i q_i}$$

$\vec{\tau} = (\tau_1, \tau_2, \dots)$: Chemical potentials/Coupling constants

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- Purely two-particle interaction among eigenvalues.

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- Bethe-ansatz-type approach at large N.
  (Different contour integral representation of $\mathcal{I}$)
  [Benini, Milan 1811, 1812.]

Aim of talk: direct analysis of matrix model for finite charges at large N

[Cabo-Bizet, S.M. 1909.]

Relate to recent progress along three threads:

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Recent related work

Cabo-Bizet, Cassani, Martelli, S.M. 1810
Choi, Kim, Kim, Nahmgoong 1810
Choi, Hwang, Kim, Nahmgoong 1811
Benini, Milan 1811, 1812
Suh 1812
Honda 1901
Arabi Ardehali 1902
Kim, Kim, Song 1902
Cabo-Bizet, Cassani, Martelli, S.M. 1904
Amariti, Garozzo, Lo Monaco 1904
Cassani, Papini 1906
Larsen, Nian, Zeng 1907
Kantor, Papageorgakis, Richmond 1907
Nahmgoong 1907
Lezcano, Pando-Zayas 1907
Lanir, Nedelin, Sela 1908

Susy BHs in AdS5 are labelled by charge and two angular momenta

$$\text{IIB on } \mathrm{AdS}_5 \times S^5 \longleftrightarrow \mathcal{N} = 4 \text{ SYM}$$

Charges of BPS states/BH

$SO(4) \quad (J_1, J_2)$

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Charges

(ω_1, ω_2)

φ

Chemical potentials

The entropy of these BHs can be recast as an extremization (of SCFT Casimir energy)

[Hosseini, Hristov, Zaffaroni 1705.]

$$S_{\rm BH}(J_1,J_2,Q) \;=\; -{\rm ext}_{\varphi,\omega_i}\big(I(\omega_1,\omega_2,\varphi)-\omega_1 J_1-\omega_2 J_2-\varphi\, Q\big)$$

where $\quad I \;=\; \dfrac{2\pi R^2_{\rm AdS}}{27G}\,\dfrac{\varphi^3}{\omega_1\omega_2}$

under the constraint $\quad \omega_1+\omega_2-2\varphi \;=\; 2\pi{\rm i}$

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(For $\mathcal{N}=4$ SYM $\;\varphi^3 \to \varphi_1\,\varphi_2\,\varphi_3$)

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Is there a physical principle that leads to this extremization problem?

BH thermodynamics

Temperature	=	surface gravity β
Chemical potential for angular momentum	=	angular velocity $\Omega_{1,2}$
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- Wrong relation! $1 + \Omega^*_1 + \Omega^*_2 - 2\Phi^* = 0$

Four-parameter family of (non-susy) BH solutions of minimal sugra

$$\mathcal{L} = (R + 12g^2) *1 - \tfrac{2}{3} F \wedge *F + \tfrac{8}{27} F \wedge F \wedge A$$

- BH parameters: (a, b, m, q)

[Gutowski, Reall, '04] [Chong, Cvetic, Lu, Pope, '05; Kunduri, Lucietti, Reall, '06]

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- Measure charges (E, J_1, J_2, Q) at infinity

- Measure chemical potentials $\Omega_{1,2}(a, b, m, q)$, $\Phi(a, b, m, q)$

Charges, potentials, and on-shell action obey *Quantum Statistical Relation (QSR)*

[Gibbons, Hawking '77]

- Calculate regularized on-shell action $I(a, b, m, q)$

  On-shell action diverges because of the infinite Vol(AdS)
  Holographic renormalization / Background subtraction

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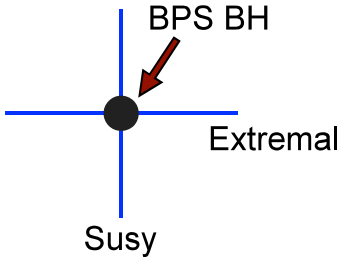
$$I = \beta E - S_{\rm BH} - \beta\Omega_1 J_1 - \beta\Omega_2 J_2 - \beta\Phi Q$$

$$E = \frac{\partial I}{\partial \beta}\,, \qquad J_1 = -\frac{1}{\beta}\frac{\partial I}{\partial \Omega_1}\,, \qquad J_2 = -\frac{1}{\beta}\frac{\partial I}{\partial \Omega_2}\,, \qquad Q = -\frac{1}{\beta}\frac{\partial I}{\partial \Phi}$$

There is a three-parameter supersymmetric family of solutions

- Supersymmetry $\neq$ extremality

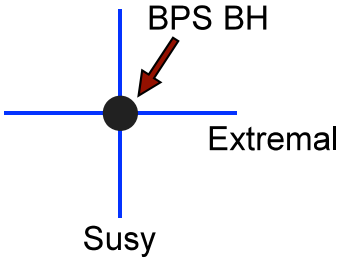
$$q = \frac{m}{1+a+b} \qquad r_+ \to r_* \quad (\beta \to \infty)$$



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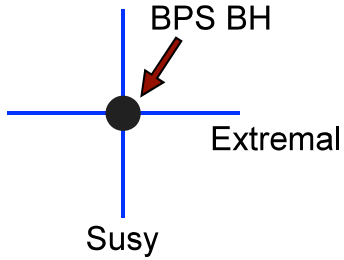


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- The supersymmetric Euclidean solutions are complex and the Lorentzian solutions are not physical (CTCs) away from the extremal limit.

- Supersymmetric chemical potentials obey the constraint

$$\beta\,(1+\Omega_1+\Omega_2-2\Phi) \;=\; 2\pi \mathrm{i}$$

The sub-leading fluctuations of chemical potentials are the interesting variables

Extremal limit of susy potentials:

$$\Omega_{1,2}(a,b,m) \to \Omega^*_{1,2} = 1 \qquad \Phi(a,b,m) \to \Phi^* = \frac{3}{2} \qquad \text{as } \beta \to \infty$$

Subleading fluctuations:

$$\omega_{1,2} := \beta(\Omega_{1,2} - \Omega^*_{1,2}) \qquad \varphi := \beta(\Phi - \Phi^*)$$

They obey the constraint $\omega_1 + \omega_2 - 2\varphi = 2\pi \mathrm{i}$

On-shell action:

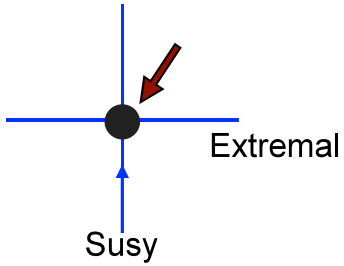
$$I = \frac{2\pi}{27 g^3 G_5} \frac{\varphi^3}{\omega_1 \omega_2}$$

QSR:

$$I = -S_{\rm BH} - \omega_1 J_1 - \omega_2 J_2 - \varphi Q$$

We thus reach a new way to describe AdS5 susy BH thermodynamics

- Go away from extremality, but preserve manifest susy. Keep first-order fluctuations.

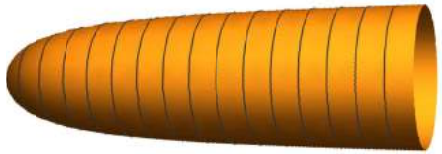


[cf. Silva '06]

- Infinite number of limits. Only one is supersymmetric through the process.

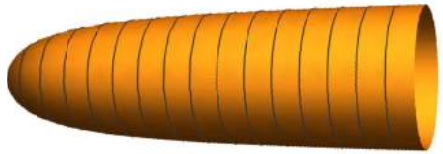
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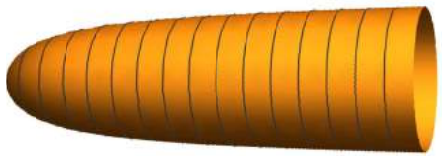


R-symmetry background Wilson line

- Global regularity of Killing spinor → $\omega_1 + \omega_2 - 2\varphi = 2\pi \mathrm{i}$

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R-symmetry background Wilson line

- Global regularity of Killing spinor → $\omega_1 + \omega_2 - 2\varphi = 2\pi \mathrm{i}$

- On-shell action reproduces BH entropy functional, topology explains constraint.

How does the dual matrix model capture the growth of states of the black hole?

Today: $\omega_1 = \omega_2 = 2\pi \mathrm{i} \tau$ $\quad \tau \in \mathbb{H}$

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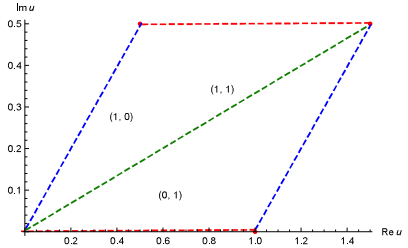
Eigenvalues $\{e^{2\pi i u_i}\}_{i=1,\ldots,N}$

$$\underline{u} \equiv \{u_i\}_{i=1,\ldots,N} \qquad [D\underline{u}] = \frac{1}{N!} \prod_{i=1}^{N} du_i$$

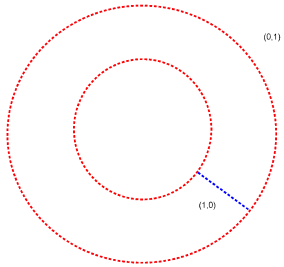
$$\mathcal{I}(\tau) = \int [D\underline{u}] \exp\left(-S(\underline{u})\right) \qquad S(\underline{u}) = \sum_{i,j=1}^{N} V(u_{ij})$$

Soln: matrix integral has complex saddles

[Cabo-Bizet, 사.무. 1909.]

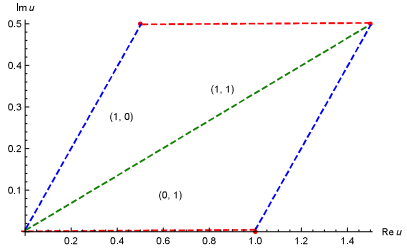


$u \mapsto e^{2\pi i u}$

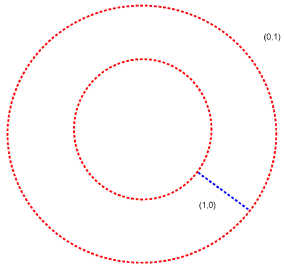


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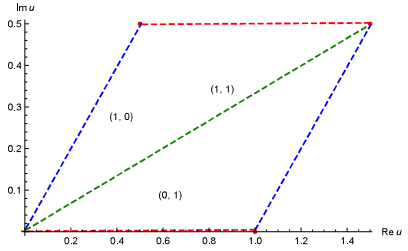


Family of saddles labelled by lattice points (m, n)

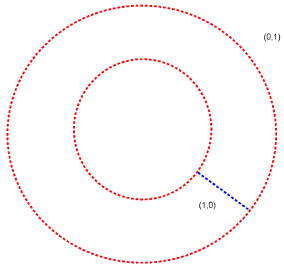
$m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \qquad \gcd(m, n) = 1$

Soln: matrix integral has complex saddles with macroscopic entropy

[Cabo-Bizet, 사.무. 1909.]



$u \mapsto e^{2\pi i u}$



Family of saddles labelled by lattice points (m, n)

$m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \qquad \gcd(m, n) = 1$

$$\mathcal{S}_{(0,1)} = \text{entropy of pure } \mathrm{AdS}_5 = O(1)$$

$$\mathcal{S}_{(1,0)} = \text{entropy of } \mathrm{AdS}_5 \text{ BH} = O(N^2)$$

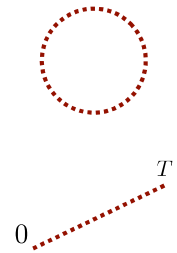
Lemma: Periodic eigenvalue distributions are saddle points of periodic potentials

$$\partial_{u_i} S(\underline{u}) = \partial_{\overline{u}_i} S(\underline{u}) = 0 \ , \qquad S(\underline{u}) = \sum_{i,j=1}^{N} V(u_{ij})$$
$$V(-z) = V(z) \qquad V(z+T) = V(z)$$

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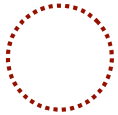
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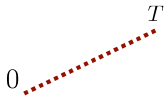
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2. Action of this distribution = average value of V

$$S_{\rm eff}(T) \;=\; \sum_{i,j=1}^{N} V\Big(\frac{i}{N}T\Big) \;=\; N^2 \int_0^1 dx\, V(xT)$$



The potential of our index is quasi-periodic under lattice translations

$$\mathcal{I}(\tau) \;=\; \eta(\tau)^{2N} \int [D\underline{u}] \prod_{i\neq j=1}^{N} \Gamma_{\rm e}(u_{ij}+2\tau;\tau,\tau)\times \\ \times \prod_{i,j=1}^{N} \Gamma_{\rm e}\big(u_{ij}+\frac{1}{3}(2\tau+1);\tau,\tau\big)^3$$

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[Felder-Varchenko '99; Spiridonov '10; Spiridonov-Varatnov '12]

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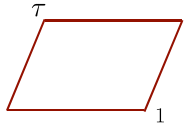
How to make quasi-elliptic functions elliptic

$$\theta_0(z;\tau) \;=\; (1-\zeta)\prod_{n=1}^{\infty}(1-q^n\zeta)\,(1-q^n\zeta^{-1})$$

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[Jacobi]

Lattice $\mathbb{Z}\tau+\mathbb{Z}$



$$q = e^{2\pi{\rm i}\tau}$$
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$$z \;=\; z_1+\tau z_2$$

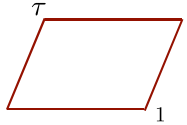
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Construct

$$P(z;\tau) \;:=\; q^{\frac{1}{2}B_2(z_2)}\,\theta_0(z;\tau) \qquad B_2(x)=x^2-x+\frac{1}{6}$$

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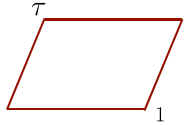
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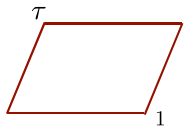
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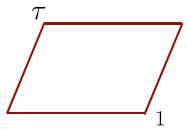
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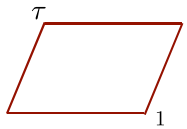
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[Kronecker's second limit formula]

The elliptic gamma function can be deformed to an elliptic function

Construct $Q(z;\tau) = q^{\frac{1}{3}B_3(z_2)-\frac{1}{2}z_2B_2(z_2)}\,\dfrac{P(z;\tau)^{z_2}}{\Gamma_{\mathrm{e}}(z+\tau;\tau,\tau)}$

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We can thus extend the SYM action to an elliptic function (the elliptic SYM action)

$$\begin{aligned}S(\underline{u}) = {} & -2N\log\left(q^{-1/24}\,\eta(\tau)\right) - \frac{1}{6}N\pi\mathrm{i}\tau + \frac{8}{27}\pi\mathrm{i}\tau N^2\\ & + \sum_{i\neq j}\log Q\left(u_{ij}+\tau\right) + 3\sum_{i,j}\log Q\left(u_{ij}-\tfrac{1}{3}\tau+\tfrac{1}{3}\right)\\ & - \sum_{i\neq j}\log P\left(u_{ij}+\tau\right) + \sum_{i,j}\log P\left(u_{ij}-\tfrac{1}{3}\tau+\tfrac{1}{3}\right)\end{aligned}$$

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- Action calculated as the integral along a given direction — use double Fourier expansion!

The action of the (m,n) saddle is a simple rational function of τ

$$\operatorname{Re} S_{\text{eff}}(m,n;\tau) \;=\; \frac{2}{27}N^2\pi\,\tau_2\left(-4+\frac{(m\tau_1+n)}{|m\tau+n|^4}\,\chi_1(m+n) + \frac{1}{|m\tau+n|^2}\,3\chi_0(m+n)\right)$$

The two Dirichlet characters mod 3.

χ \ ℓ	0	1	2
χ_0	0	1	1
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In particular, $S_{\rm eff}(1,0;\tau) \;=\; \dfrac{N^2\pi\mathrm{i}}{27} \dfrac{(2\tau+1)^3}{\tau^2}$ = the on-shell sugra action of AdS5 BH

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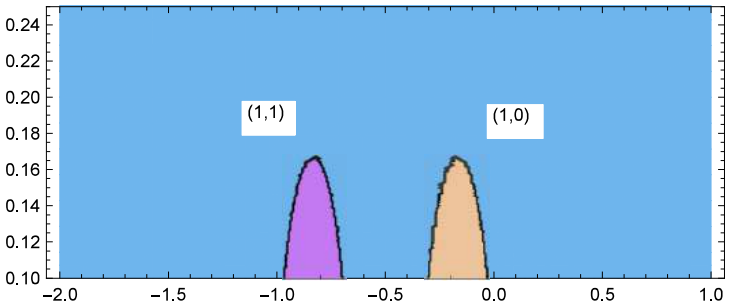
Full action:

$$S_{\rm eff}(m,n;\tau) \;=\; \frac{N^2 \pi \mathrm{i}}{27\, m} \frac{\big(2(m\tau+n)+\chi_1(m+n)\big)^3}{(m\tau+n)^2}$$

In particular, $S_{\rm eff}(1,0;\tau) \;=\; \dfrac{N^2\pi\mathrm{i}}{27} \dfrac{(2\tau+1)^3}{\tau^2}$ = the on-shell sugra action of AdS5 BH

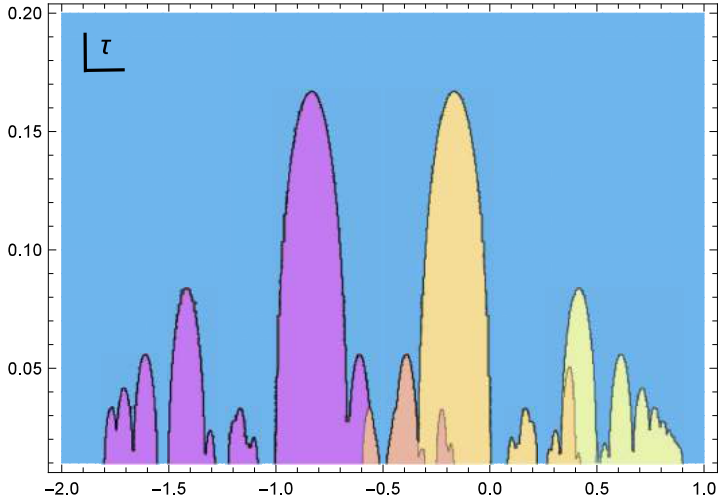
Phase structure: black hole dominates in a region surrounding $\tau = 0$

[Cabo-Bizet, S.M. 1909.]



Phase structure: new phases appear near rational points

[Cabo-Bizet, S.M. 1909.]



The Cardy-like limit is simply the limit of the (1,0) configuration shrinking to origin

As the BH charges $\to \infty$, integral dominated by $\tau \to 0$

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Action in Cardy-like limit

$$S(u) \underset{\tau\to 0}{\longrightarrow} \pi \mathrm{i} \sum_{i,j} \left(\frac{1}{3\tau^2}\left(B_3(\{u_{1ij}\}) + 3B_3(\{u_{1ij}+\tfrac{1}{3}\})\right) + \frac{1}{\tau}\left(B_2(\{u_{1ij}\}) - B_2(\{u_{1ij}+\tfrac{1}{3}\})\right)\right)$$

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- Sequence of Cardy-like limits corresponding to (m,n) saddles!

The entropy of the (1,0) saddle is found by extremizing in τ and imposing reality

[Cabo-Bizet, Cassani, Martelli, S.M. 1810.] [Cabo-Bizet, S.M. 1909.]

$$e^{\mathcal{S}_{BH}(Q,J)} = \int d\tau \, \exp\big(\mathcal{E}(\tau)\big) \,,$$

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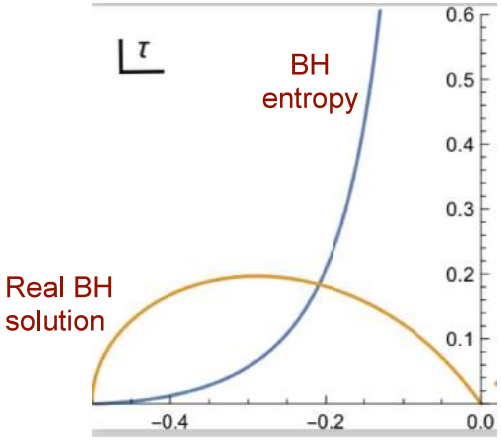
Reality: $\mathrm{Im}(\mathcal{S}_{BH}) = \mathrm{Im}(J) = \mathrm{Im}(Q) = 0$

→ Imposes relation between (Q,J)

$$\mathcal{S}_{BH} = \pi\sqrt{3Q^2 - 4N^2 J}$$

[S. Kim, K.-M. Lee '06]

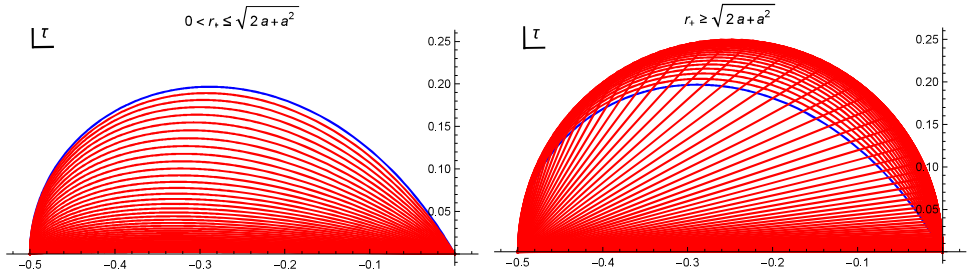
The BH solution exists on a 1d curve in the τ - plane



$$Q^* = -\frac{N^2\,(2\tau_1+1)\,(10\tau_1-1)}{72\,{\tau_1}^2} \qquad J^* = \frac{N^2\,(2\tau_1+1)^2\,(7\tau_1-1)}{432\,{\tau_1}^3}$$

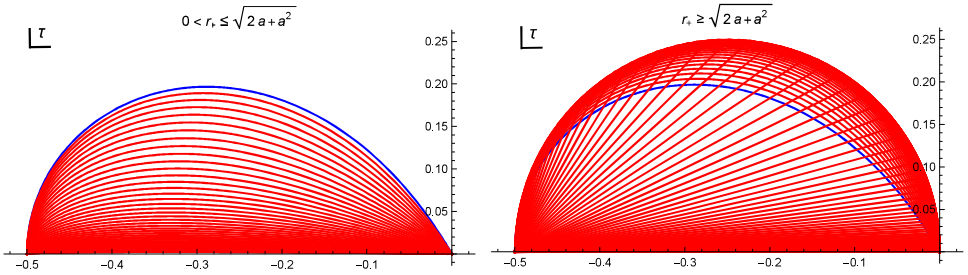
BH solution v/s SYM phase space

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Similar story for (m,n) saddles! What are they in gravity?

Question: do the (m,n) solns tesselate the SYM phase space?

Outlook and Questions

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감사합니다!

“May I be excused for saying with banality that the forest hides the trees, for those who think that they disengage themselves from atomistics by the consideration of differential equations.”



— L. Boltzmann,1905, *Populäre Schriften*

More details

The R-symmetry Wilson line has a subtle effect on the Hamiltonian SCFT index…

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$$\mathcal{I}(\sigma,\tau;n_0) \;=\; \mathrm{Tr}_{\mathcal{H}_{\mathrm{phys}}}\, \mathrm{e}^{\pi \mathrm{i}(n_0+1)F}\, \mathrm{e}^{-\beta\{\mathcal{Q},\overline{\mathcal{Q}}\}+2\pi \mathrm{i}\sigma J_1+2\pi \mathrm{i}\tau J_2+\varphi\, Q}$$

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1. Field theory index with any value of n_0 is protected.

2. n_0 only affects the phase of the degeneracies of states

$$\mathcal{I}(\sigma,\tau;n_0) = \sum_{n_1,n_2} d(n_1,n_2;n_0)\, \mathrm{e}^{2\pi \mathrm{i}(\sigma n_1+\tau n_2)}$$

$$|d(n_1,n_2;n_0)| = |d(n_1,n_2;0)|$$

Four-parameter family of (non-susy) BH solutions of minimal sugra

$$\mathcal{L} = (R+12g^2)*1 - \tfrac{2}{3}F\wedge *F + \tfrac{8}{27}F\wedge F\wedge A$$

- BH parameters: (a,b,m,q)

[Gutowski, Reall, '04] [Chong, Cvetic, Lu, Pope, '05; Kunduri, Lucietti, Reall, '06]

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- Killing horizon generated by Killing vector

$$V = \frac{\partial}{\partial t} + \Omega_1 \frac{\partial}{\partial \phi} + \Omega_2 \frac{\partial}{\partial \psi}$$

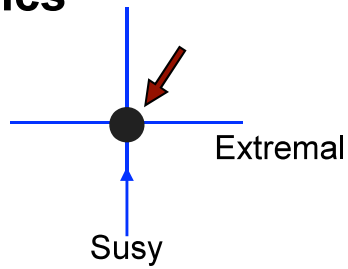
Angular velocities at horizon

- Electrostatic potential at horizon

$$\Phi = V \cdot A|_{r_+} - V \cdot A|_\infty$$

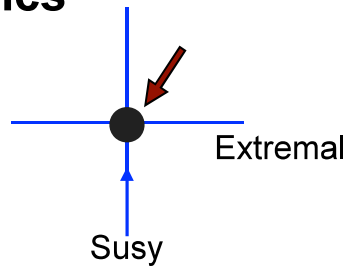
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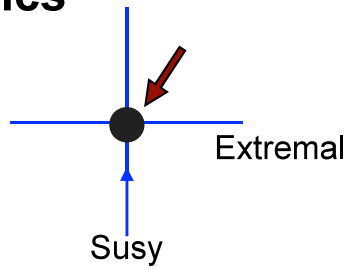


- Quantum statistical relation in new variables:

$$I = \beta E - S_{\rm BH} - \beta\Omega_1 J_1 - \beta\Omega_2 J_2 - \beta\Phi Q$$

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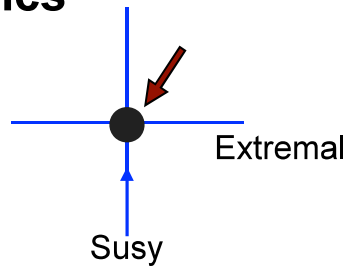


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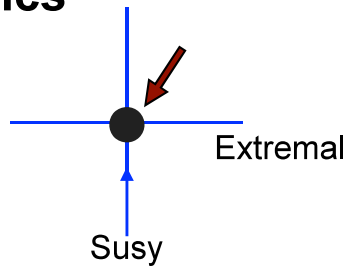


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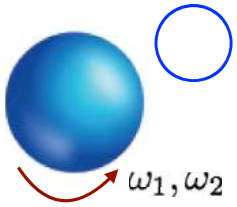
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$$\Rightarrow \quad I = -S_{\rm BH} - \omega_1 J_1 - \omega_2 J_2 - \varphi Q$$

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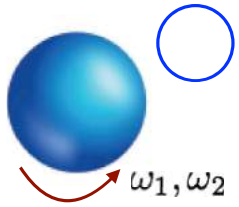
Calculate in presence of Wilson line [Cabo-Bizet, Cassani, Martelli, S.M. 1810.]

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Suggests Cardy-like formula and associated modular symmetry