Symplectic Carleman Approximation on Co-adjoint Orbits

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- Carleman approximation: For any continuous function $\epsilon(x) > 0$ on $\mathbb{R}^{2n}$ there exists ψ such that $\|\psi(x) - \phi(x)\| < \epsilon(x)$ for all $x \in \mathbb{R}^{2n}$. (This is also called *tangential approximation*.)

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- There are also similar results allowing X to have interior due to Arakeljan (1964) and Neresjan (1971/1972), and off course a vast literature on polynomial and rational approximation on compact sets due to Runge, Mergelyan, Vitushkin and others ...

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- Chenoweth, Castro-Infantes (2018/19): Carleman approximation for maps to Oka manifolds and for directed holomorphic curves.

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Theorem (Wold-Deng, 2019): Let $\phi \in \mathrm{Symp}(\mathbb{R}^{2n})$, let $\epsilon(x) > 0$ be a continuous function on $\mathbb{R}^{2n}$, and fix a $k \in \mathbb{N}$. Then there exists $\psi \in \mathrm{Symp}_{hol}(\mathbb{C}^{2n})$ such that

$$\|(\phi - \psi)(x)\|_k < \epsilon(x)$$

for all $x \in \mathbb{R}^{2n}$, and $\psi(\mathbb{R}^{2n}) = \mathbb{R}^{2n}$.

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- The adjoint action of g induces an an action Ad^* on $\mathfrak{g}^*$ defined by

$$\langle \mathrm{Ad}^*(g)\xi, v\rangle := \langle \xi, \mathrm{Ad}(g^{-1})(v)\rangle$$

which is called the *co-adjoint action* of g on $\mathfrak{g}^*$. Orbits of elements ξ in $\mathfrak{g}$ are called *co-adjoint orbits and are denoted by* $\mathcal{O}_\xi$.

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- If we let G be the complex Lie group consisting of matrices as above where we allow complex entries, we get that $G_0 \subset G$ is a closed totally real subgroup, and $\mathfrak{g}^* = \mathfrak{g}_o{}^* + i\mathfrak{g}_o{}^*$.

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- So for a point $\xi \in \mathbb{R}^3$ we see that $\mathcal{O}_{0,\xi}$ is a closed totally real submanifold of the complex co-adjoint orbit $\mathcal{O}_\xi \approx \mathbb{C}^2$.

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- On the other hand, recall that if f is a smooth function on $\mathcal{O}_\xi$ there is a symplectic vector field X on $\mathcal{O}_\xi$ defined by

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  - ω is a symplectic form on Z such that $\omega_0 = \omega|_{Z_0}$ is a symplectic form on Z_0.
  - For each linear map v on $\mathbb{C}^N$ we have that $\tilde{X}_v$ is a complete holomorphic vector field on Z.

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- The potential f may be approximated by real holomorphic polynomials $P(z)$.

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- $P(z) = \sum_{j=1}^{N} a_j \cdot (\Lambda_j(z))^{m_j}$ with a_j and Λ_j real.

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- Then if ϕ_t denotes the flow of X and if ϕ_t^j denotes the flow of X_j it is standard that

$$\lim_{n\to\infty} (\phi_{t/n}^N \circ \circ \circ \phi_{t/n}^1)^n = \phi_t$$

uniformly on compacts as $n \to \infty$.