

Introduction to Springer Theory

1. Perverse Sheaves

main reference: Bernstein, Beilinson, Deligne, Gabber "Faisceaux Pervers" Astérisque 100

* Constructible sheaves

Let X be a variety over $k = \bar{k}$ and ℓ be a prime different from char k. Then an ℓ-adic sheaf on X is an inverse system of étale sheaves $\{\mathcal{F}_n\}$ such that

- each $\mathcal{F}_n$ is a $(\mathbb{Z}/\ell^n\mathbb{Z})_X$-module
- $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induces an isomorphism $\mathcal{F}_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}\mathbb{Z}} \mathbb{Z}/\ell^n\mathbb{Z} \simeq \mathcal{F}_n$.

An ℓ-adic sheaf $\mathcal{F} = \{\mathcal{F}_n\}$ is called

- constructible if each $\mathcal{F}_n$ is constructible, i.e. $\exists$ stratification of X (depending on each $\mathcal{F}_n$) s.t. $\mathcal{F}_n$ is locally constant on each stratum.
- lisse if each $\mathcal{F}_n$ is locally constant on X.

Rmk. A lisse sheaf needs not be locally constant for the étale topology since "finite $\neq$ pro-finite". However, $\mathcal{F}$ is constructible if and only if $\exists$ stratification of X s.t. $\mathcal{F}$ is lisse on each stratum.

* Constructible derived category.

For a variety X, we define its **constructible derived category** $D(X)$ to be

$$D(X) = D_c^b(X, \overline{\mathbb{Q}_\ell}) = \left(\varprojlim D_c^b(X, \mathbb{Z}/\ell^n\mathbb{Z})\right) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}.$$

Or, equivalently, $D(X)$ is the full subcategory of $D^b(X, \overline{\mathbb{Q}_\ell})$ consisting of objects $\mathcal{F}^\bullet$ whose cohomology sheaves $\mathcal{H}^i(\mathcal{F}^\bullet)$ are constructible.

For varieties X and Y and a morphism $f: X \to Y$, one can define

- $f_* : D(X) \longrightarrow D(Y)$: "push-forward"
- $f_! : D(X) \longrightarrow D(Y)$: "proper push-forward"
- $f^* : D(Y) \longrightarrow D(X)$: "pull-back"
- $f^! : D(Y) \longrightarrow D(X)$: "proper pull-back"
- $\otimes : D(X) \times D(X) \longrightarrow D(X)$, "internal tensor product"
- $\mathcal{H}om : D(X)^{op} \times D(X) \longrightarrow D(X)$ "internal Hom"
- $\mathbb{D} : D(X)^{op} \longrightarrow D(X)$: "Verdier duality"

such that:

- $f^! = \mathbb{D} f^* \mathbb{D}$
- $f_! = \mathbb{D} f_* \mathbb{D}$
- $\mathbb{D} \cdot \mathbb{D} = Id$
- base change btw f_* and $f^!$
- $f_* = f_!$ if f is proper
- $f^* = f^![-2d]$ if f is smooth and d is the dimension of the fiber of f
- $f_!$ is left adjoint to $f^!$; f^* is left adjoint to f_*
- Tensor – Hom adjunction.

$\rightsquigarrow$ **Grothendieck's six operations**

* Perverse sheaves

A complex $\mathcal{F} \in D(X)$ is called a perverse sheaf if

(1) $\dim \{x \in X \mid \mathcal{H}^i(\mathcal{F})|_x \neq 0\} \leq -i$ for $i \in \mathbb{Z}$,

(2) $\mathbb{D}\mathcal{F}$ satisfies (1).

Let $P(X) \subset D(X)$ be the full subcategory of perverse sheaves on X, called the category of perverse sheaves on X. In fact, this is the heart of the "perverse t-structure" on $D(X)$, thus $P(X)$ is an abelian category.

Rmk. Note that a perverse sheaf is

- not a sheaf, in a sense that it is a complex of sheaves
- a sheaf, in a sense that it can be recovered by local information + glueing maps.

* Intersection cohomology complexes

Let $Z \subset X$ be irr. closed and $U \subset Z^{reg}$ be open. Let $\mathcal{L}$ be an ℓ-adic local system on U. Then up to isomorphism $\exists!\ IC(Z, \mathcal{L}) \in D(X)$ s.t.

(1) $IC(Z, \mathcal{L})|_U \simeq \mathcal{L}[\dim Z]$

(2) $IC(Z, \mathcal{L})|_{X-Z} = 0$

(3) $\mathcal{H}^i(IC(Z, \mathcal{L})) \neq 0 \Rightarrow -\dim Z \leq i < 0$

(4) $\dim \{z \in Z \mid \mathcal{H}^i IC(Z, \mathcal{L})|_z \neq 0\} < -i$ for $-\dim Z < i < 0$

(note that $\operatorname{supp} \mathcal{H}^{-\dim Z} IC(Z, \mathcal{L}) = Z$ by (1))

(5) $\mathbb{D} IC(Z, \mathcal{L})$ satisfies (2) – (4).

($\mathbb{D}\mathcal{L} = \mathcal{L}^\vee$, thus $\mathbb{D} IC(Z, \mathcal{L}) = IC(Z, \mathcal{L}^\vee)$.)

This is called the intersection cohomology complex assoc. w/ $(Z, \mathcal{L})$ or simply an IC sheaf of $(Z, \mathcal{L})$.

Support / Cosupport condition:

deg \ codim	...	-3	-2	-1	0
⋮	⋱	⋮	⋮	⋮	⋮
4	c	c	c	c	c
3		c	c	c	c
2			c	c	c
1				c	c
0					•
-1				s	s
-2			s	s	s
-3		s	s	s	s
-4	s	s	s	s	s
⋮	⋰	⋮	⋮	⋮	⋮

perverse

deg \ codim	...	-3	-2	-1	0
⋮					
4					
3		c	c	c	c
2				c	c
1					c
0					
-1					s
-2				s	s
-3		s	s	s	s
-4					
⋮					

IC

$\rightsquigarrow$ IC sheaf is perverse.

Thm. (BBDG 4.3.1) (1) $P(X)$ is noetherian.

(2) Every simple object in $P(X)$ is of the form $IC(Z, \mathcal{L})$, where $\mathcal{L}$ is an irreducible local system on an open subset of Z^{reg}.

Rmk. 1) One can define $IC(Z, \mathcal{F})$ for any perverse sheaf $\mathcal{F}$. It is given by $Im(^pH^0(j_! \mathcal{F} \to j_* \mathcal{F}))$

2) $IC(Z, \mathcal{L}) \sim IC(Z', \mathcal{L}')$ iff $Z = Z'$ and $\exists$ U (open) s.t. $\mathcal{L}|_U = \mathcal{L}'|_U$ $(\neq 0)$.

3) If X is smooth and $\mathcal{L}$ is a local system on X, then $IC(X, \mathcal{L}) = \mathcal{L}$.

* Decomposition theorem.

Thm (BBDG) Let $f: X \to Y$ be a proper morphism. Then $f_! IC_X = f_* IC_X$ is a direct sum of simple perverse sheaves with possible degree shift.

Here, $IC_X = IC(X, \overline{\mathbb{Q}}_X)$.

In other words, $f_* IC_X$ is "semisimple".

⇝ original proof by BBDG

⇝ mixed Hodge modules by Saito

⇝ classical Hodge theory by de Cataldo–Migliorini

* Semismall maps.

Let X, Y be irreducible algebraic varieties over k and $f: X \to Y$ be a proper surjective morphism. Then we say that f is **semismall** if $\dim X \times_Y X = \dim Y$, or equivalently $\dim \{ y \in Y \mid \dim f^{-1}(y) \geq i \} \leq \dim Y - 2i$ for $i \in \mathbb{N}$. In particular, f is generically finite. We also say that f is small if f is semismall and $\dim \{ y \in Y \mid \dim f^{-1}(y) \geq i \} \leq \dim Y - 2i$ for $i \geq 1$.

e.g. Let $\tilde{X} = \{((a,b,c),(x,y)) \in \mathbb{A}^3 \times \mathbb{P}^1 \mid bx^2 + 2axy - cy^2 = 0\}$, and consider $f: \tilde{X} \to X = \mathbb{A}^3$.

1) if $a^2 + bc \neq 0$, then $f^{-1}(a,b,c)$: 2 points

2) if $a^2 + bc = 0$ but $(a,b,c) \neq 0$, then $f^{-1}(a,b,c)$: 1pt

3) $f^{-1}(0,0,0) = \mathbb{P}^1$.

Thus. $\{x \in X \mid \dim f^{-1}(x) \geq 0\}^{-} = X$ codim 0

$\{x \in X \mid \dim f^{-1}(x) \geq 1\}^{-} = \{0\}$ codim 3.

$\{x \in X \mid \dim f^{-1}(x) \geq 2\}^{-} = \emptyset$

$\rightsquigarrow$ $f : \tilde{X} \to X$ is small.

On the other hand if we let $Y \subset X$ be the hyperplane defined by $a^2 + bc = 0$ and set $\tilde{Y} = f^{-1}(Y)$, then Y is 2-dim and

$\{y \in Y \mid \dim f^{-1}(y) \geq 0\}^{-} = Y$ codim 0

$\{y \in Y \mid \dim f^{-1}(y) \geq 1\}^{-} = \{0\}$ codim 2

$\{y \in Y \mid \dim f^{-1}(y) \geq 2\}^{-} = \emptyset$

$\rightsquigarrow$ $f : \tilde{Y} \to Y$ is semismall.

Rmk. In this case,

$\tilde{Y} = \{((a,b,c),(x,y)) \in \mathbb{A}^3 \times \mathbb{P}^1 \mid bx + ay = 0\}$.

which is nothing but $Bl_0 Y$. Note that Y is singular at 0 and $f : \tilde{Y} \to Y$ is a resolution of singularity.

Suppose that $f : X \to Y$ is proper surjective. Then as $f_* \overline{\mathbb{Q}_\ell}_X$ is constructible, $\exists$ stratification $Y = \bigsqcup_\alpha S_\alpha$ s.t. $\mathcal{H}^i(f_* \overline{\mathbb{Q}_\ell}_X)$ is lisse on each S_α. In such a situation $\dim f^{-1}(x)$ is constant for any $x \in S_\alpha$, which we denote by d_α. Now f is semismall $\Leftrightarrow \dim S_\alpha + 2d_\alpha \leq \dim Y \ \forall \alpha$. We say that S_α is a relevant stratum if $\dim S_\alpha + 2d_\alpha = \dim Y$. Then, f is small $\Leftrightarrow \exists!$ relevant stratum, which is dense.

Thm. (Borho - MacPherson) Let $f: X \to Y$ be a semismall morphism between irreducible varieties and suppose that $IC_X \simeq \overline{\mathbb{Q}_\ell}_X[\dim X]$. (Such X is said to be rationally smooth; in particular, it is true if X is smooth) Then, $f_* IC_X$ is a semisimple perverse sheaf (without degree shift). More precisely, we have

$$f_* IC_X = \bigoplus IC(\overline{S_\alpha}, \mathcal{L}_\alpha)$$

where the sum is over all the relevant strata and $\mathcal{L}_\alpha = \mathcal{H}^{2d_\alpha}(f_* \overline{\mathbb{Q}_\ell}_X)|_{S_\alpha}$, i.e.,

$$\mathcal{L}_\alpha|_x = H^{2d_\alpha}(f^{-1}(x)) = H_c^{2d_\alpha}(f^{-1}(x))$$

In particular, if f is small then $f_* IC_X$ is again an IC sheaf.

e.g. Recall the previous example. Let

$$S_1 = \{a^2+bc \neq 0\}. \quad S_2 = \{a^2+bc=0\} - \{0\}. \quad S_3 = \{0\}.$$

Then $f_* IC_{\tilde{X}} = IC(X, \mathcal{H}^\cdot(f_* \overline{\mathbb{Q}_\ell}_{\tilde{X}})|_{S_1})$. As $f^{-1}(x)$ is two pts if $x \in S_1$, $\hookrightarrow$ is a rk 2 local system.

	S_1	S_2	S_3
$f_* IC_{\tilde{X}}$:	$\overline{\mathbb{Q}_\ell}^{\oplus 2}[3]$	$\overline{\mathbb{Q}_\ell}[3]$	$\overline{\mathbb{Q}_\ell}[3] \oplus \overline{\mathbb{Q}_\ell}[1]$

If restricted to $Y = S_2 \sqcup S_3$, then S_2 and S_3 are both relevant, thus $f_* IC_{\tilde{Y}} = IC(S_2, \cdots) \oplus IC(S_3, \cdots)$

As $f_* IC_{\tilde{Y}} = f_* IC_{\tilde{X}}|_Y[-1]$ (base change), we have

	S_2	S_3
$IC(S_2, \cdots)$	$\overline{\mathbb{Q}_\ell}[2]$	$\overline{\mathbb{Q}_\ell}[2]$
$IC(S_3, \cdots)$		$\overline{\mathbb{Q}_\ell}$

2. Springer theory.

* Setup

- k: alg. closed field
- G: connected reductive gp / k. e.g. GL_n
- B: a fixed Borel subgp of G.
  e.g. upper-triangular matrices in GL_n
- T: a fixed maximal torus of B
  e.g. diagonal matrices in GL_n
- U: unipotent radical of B
  e.g. unip. upper-triangular matrices in GL_n
- $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}, \mathfrak{n} := \mathrm{Lie}\, G, \mathrm{Lie}\, B, \mathrm{Lie}\, T, \mathrm{Lie}\, U$
  e.g. $\mathrm{Lie}\, GL_n = \mathrm{Mat}_{n\times n}$, ...
- $\mathcal{B}$: the flag variety of G, i.e. the set of Borel subalgebras of $\mathrm{Lie}\, G$. As $\mathcal{B}$ is a single orbit under the adjoint action of G and $\mathrm{Stab}_G(\mathrm{Lie}\, B) = B$, we usually identify $\mathcal{B}$ with G/B.
  e.g. complete flags in k^n
- W: $N(T)/T$, the Weyl gp of G.

The choice of the pair (B, T) endows W with the Coxeter system, i.e. the set of simple reflections $S \subset W$.
e.g. $W = S_n$, S = adjacent transposition

* Springer fiber

For $X \in \mathfrak{g}$, its Springer fiber is defined to be $\{ \mathfrak{b}' \in \mathcal{B} \mid X \in \mathfrak{b}' \} \simeq \{ gB \in G/B \mid \mathrm{Ad}\, g^{-1} X \in \mathfrak{b} \}$.
This is a closed subvariety of $\mathcal{B}$.

e.g. 1) $X = 0 \rightsquigarrow \mathcal{B}_0 = \mathcal{B}$

2) X regular semisimple $\rightsquigarrow \mathcal{B}_X$ consists of $|W|$ pts

e.g. $X = \mathrm{diag}(1, 2, 3, \dots, n)$, then
$\mathcal{B}_X = \{ wB \in G/B \mid w \in W \}$.

3) X regular nilpotent $\rightsquigarrow \mathcal{B}_X = \{ * \}$.

e.g. $X = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$, then $\mathcal{B}_X = \{ \mathfrak{b} \}$

4) X "subregular nilpotent". $\mathcal{B}_X$ = "Dynkin curve"
$\rightsquigarrow$ Union of $\mathbb{P}^1$'s whose "dual graph" is (the unfolding of) the Dynkin diagram of G.

e.g. $X = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$, then

$\mathcal{B}_X \simeq$  $n-1$ $\mathbb{P}^1$'s

In general, $\mathcal{B}_X$ can be (very) singular.

Goal: to define an action of W on $H^i(\mathcal{B}_X)$.

If $X = 0$, then W acts on $H^i(\mathcal{B})$ since $W \curvearrowright G/T$ and $G/T \longrightarrow G/B$ is an affine bdl $\rightsquigarrow$ same cohomology.

If X regular semisimple, then $\mathcal{B}_X$ can be regarded as a W-torsor.

More precisely, consider $g^{rs} = \{\text{regular s.s.}\}$.
$\tilde{g}^{rs} = \{(X, gB) \in g^{rs} \times G/B \mid Ad g^{-1}(X) \in b\}$.
Then the fiber of $\pi : \tilde{g}^{rs} \to g^{rs}$ at $X \in g^{rs}$ is $\mathcal{B}_X$ = "$|W|$ pts". Now consider
$\mathcal{X} = \{(X, gT) \in g^{rs} \times G/T \mid Ad g^{-1}(X) \in h\}$.
Then $\mathcal{X} \to \tilde{g}^{rs} : (X, gT) \mapsto (X, gB)$ is well-defined and an isomorphism since $N(T) \cap B = T$.
Now the W-action on $\mathcal{X}$ $w \cdot (X, gT) = (X, gw^{-1}T)$ is well-defined, and gives a principal W-bundle structure on $\tilde{g}^{rs} \to g^{rs}$.
$\rightsquigarrow W \curvearrowright \mathcal{B}_X$, $W \curvearrowright H^i(\mathcal{B}_X)$ "continuously on X".

* Springer resolution.
Let us define:
$\tilde{g} := \{(X, b') \in g \times \mathcal{B} \mid X \in b'\}$.
$\pi : \tilde{g} \to g$. Then we have

$$\begin{array}{ccccc} \tilde{g}^{rs} & \hookrightarrow & \tilde{g} & \xrightarrow{\pi'} & \mathcal{B} \\ \downarrow & \lrcorner & \downarrow \pi & & \\ g^{rs} & \hookrightarrow & g & & \end{array}$$

Since π' is an affine bundle (fiber = b) and $\mathcal{B}$ smooth, $\rightsquigarrow \tilde{g}$ smooth.

From now on, we assume that $g^{rs} \neq \emptyset$.
Then it is automatic that g^{rs} is dense in g.
(the only pathological case arises when char $k = 2$ and g has a simple factor of type C, and g is not both semisimple and adjoint).
Thm. $\pi : \tilde{g} \to g$ is small.

e.g. If $G = SL_2 / \mathbb{C}$. then

$\mathfrak{g} = \{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a,b,c \in \mathbb{C} \} \simeq \mathbb{C}^3$

$\mathfrak{g}^{rs} = \{ (a,b,c) \in \mathbb{C}^3 \mid a^2 + bc \neq 0 \}$.

$\mathcal{B} \simeq \mathbb{P}^1 = \{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} B \mid x \in \mathbb{C} \} \cup \{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B \}$

and $\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c - 2ax - bx^2 & * \end{pmatrix}$

Thus $\gamma \in \mathfrak{b}$ iff $bx^2 + 2ax - c = 0$, i.e.,

$\tilde{\mathfrak{g}} \simeq \{ ((a,b,c), (x,y)) \in \mathbb{C}^3 \times \mathbb{P}^1 \mid bx^2 + 2axy - cy^2 = 0 \}$.

Now if $(a,b,c) \in \mathfrak{g}^{rs}$, then $\mathcal{B}_x$ is exactly 2 pts.

If $a^2 + bc = 0$ but $(a,b,c) \neq 0$, then $\mathcal{B}_x = \{*\}$.

If $(a,b,c) = 0$, then $\mathcal{B}_x = \mathcal{B} = \mathbb{P}^1$.

$\rightsquigarrow$ π is small.

Therefore, $\pi_* IC_{\tilde{\mathfrak{g}}} = IC(\mathfrak{g}, \mathcal{L})$ where

$\mathcal{L}$ is the local system $\mathcal{H}^0(\pi_* \overline{\mathbb{Q}}_\ell \mathfrak{g}^{rs}) = \pi_* \overline{\mathbb{Q}}_\ell \mathfrak{g}^{rs}$

which is naturally equipped with the W-action as observed before. Now the functoriality of IC construction guarantees that this W-action extends to $IC(\mathfrak{g}, \mathcal{L}) = \pi_* IC_{\tilde{\mathfrak{g}}} = \pi_* \overline{\mathbb{Q}}_\ell \tilde{\mathfrak{g}}[\dim \tilde{\mathfrak{g}}]$.

i.e., for any $X \in \mathfrak{g}$, W acts on

$\pi_* \overline{\mathbb{Q}}_\ell \tilde{\mathfrak{g}}|_X = \mathcal{H}^\bullet(\pi^{-1}(X)) = H^\bullet(\mathcal{B}_X)$.

$\rightsquigarrow$ Springer representation.

We are mainly interested when N is nilpotent

* Springer resolution.

Let $\mathcal{N} \subset \mathfrak{g}$ be the set of nilpotent elements, called the nilpotent cone of $\mathfrak{g}$. It is not smooth. On the other hand, $\tilde{\mathcal{N}} := \pi^{-1}\mathcal{N}$, i.e., $\tilde{\mathcal{N}} := \{(X, \mathfrak{b}') \in \mathcal{N} \times \mathcal{B} \mid X \in \mathfrak{b}'\}$ is smooth as $\tilde{\mathcal{N}} \xrightarrow{\pi'} \mathcal{B}$ is an affine bundle (fiber $= n$).

Let $\mathcal{N}^{reg}$ be the set of regular nilpotent elements in $\mathcal{N}$, which is open dense in $\mathcal{N}$. Then as observed, $\pi : \tilde{\mathcal{N}}^{reg} \to \mathcal{N}^{reg}$ is an iso. In other words, we have

Thm. $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularity. This is called the Springer resolution.

Unlike $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$, $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ is not small. But it has nice properties as described below.

Thm. 1) There are finitely many adjoint G-orbits in $\mathcal{N}$.

2) $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ is G-equivariant, thus π is lisse on each G-orbit on $\mathcal{N}$.

3) $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ is semisimple, and each G-orbit is a relevant stratum.

Thus by decomposition theorem we have

$\pi_* IC_{\tilde{\mathcal{N}}} = \oplus_{\mathcal{O}} IC(\mathcal{O}, \mathcal{L}_{\mathcal{O}})$ where $\mathcal{L}_{\mathcal{O}}$ is the local system corresponding to $\mathcal{H}^{2d_{\mathcal{O}}}(\pi^{-1}\mathcal{O})$.

Note that we have an one-to-one correspondence
"G-equivariant local system on G/H"
$\longleftrightarrow$ "(continuous) $\pi_0(H)$-representation"
($0 \to H \to G \to G/H \to 0$
$\rightsquigarrow \cdots \to \pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) \to \cdots$,
thus $\pi_1(G/H) / \operatorname{im} \pi_1(G) \simeq \pi_0(H)$).
Now when G/H is an orbit of $N \in \mathcal{N}$, then $H \simeq Z(N)$, thus $\mathcal{L}_\sigma$ corresponds to some $\pi_0(Z(N)) =: A_N$-representation.
It can be shown that it is equal to
$Z(N) \curvearrowright \mathcal{B}_N \rightsquigarrow Z(N) \curvearrowright H^\bullet(\mathcal{B}_N) \rightsquigarrow A_N \curvearrowright H^\bullet(\mathcal{B}_N)$.

$\therefore \pi_* IC_{\tilde{\mathcal{N}}} = \bigoplus_\sigma \bigoplus_{\rho \in \hat{A}_N} IC(\mathcal{O}, \mathcal{L}_\rho) \otimes V_\rho$,
where $\bigoplus_\rho \mathcal{L}_\rho \otimes V_\rho \simeq \mathcal{L}_\sigma$.
Now the W-actions factor through each V_ρ part.

Thm. (Springer correspondence) The correspondence V_ρ (as W-mod) $\longmapsto (\mathcal{O}, \rho)$ gives an injection from $\operatorname{Irr}(W)$ to
$\{ (\mathcal{O}, \rho) \mid \rho \in \hat{A}_N \text{ where } N \in \mathcal{O} \}$.
In particular, each V_ρ is injective.

e.g. Type A: each A_N is trivial, in which case the Springer correspondence is given by
(orbit of Jordan type λ) $\longmapsto \chi_\lambda \in \operatorname{Irr}(S_n)$.