[Combinatorics on flag varieties and related topics] 2020. 1. 16

박 경 동

Introduction to Borel-Weil-Bott theorem

§ 0. Motivations

① Equivariant vector bundles E $\overset{\text{lifting}}{\curvearrowright}$ G-action

$\downarrow$

generalized flag variety $G/P \curvearrowright G$

include important examples of vector bundles on G/P (locally free sheaves) : tangent bundle $T_{G/P}$, cotangent bundle $T^*_{G/P}$, sheaf of holomorphic k-forms Ω^k, etc

② Global sections of equivariant vector bundles define interesting subvarieties of G/P :

E

$\pi \downarrow$ $\quad\uparrow s$; s : general global section

zero locus of a global section $Z(s) \subset G/P$

smooth subvariety

For example, Mukai's description about some Fano threefolds

$\mathcal{O}(1)^{\oplus 3}$

$\downarrow$ $\quad\uparrow s$

$V_5 \cong Z(s) \subset Gr(2,5)$

(Fano 3-fold of Picard number 1, index 2, degree 5)

$(\wedge^2 \mathcal{U}^*)^{\oplus 3}$

$\downarrow$ $\quad\uparrow s$

$V_{22} \cong Z(s) \subset Gr(3,7)$

(Fano 3-fold of Picard number 1, index 1, degree 22)

Geometry of $Z(s)$ $\leftsquigarrow$ { • Koszul complex associated to a section s + • Borel-Weil-Bott theorem

§ 1. Generalized flag varieties

$\mathfrak{g}$: semisimple Lie algebra over $\mathbb{C}$

G : the associated simply-connected Lie group

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which means a maximally abelian diagonalizable subalgebra.

Fact The Cartan subalgebras of $\mathfrak{g}$ are conjugate under the adjoint action of G.

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Root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$,

where $\mathfrak{g}_\alpha := \{ X \in \mathfrak{g} \mid [H, X] = \alpha(H) X \text{ for } \forall H \in \mathfrak{h} \}$; root space

and $\Phi := \{ \alpha \in \mathfrak{h}^* \mid \mathfrak{g}_\alpha \neq \{0\}, \ \alpha \neq 0 \}$; root system

We can choose a subset $\Delta = \{\alpha_1, \cdots, \alpha_r\} \subset \Phi$ (simple roots) such that every root $\beta \in \Phi$ is expressed as a linear combination $\beta = \sum_{i=1}^{r} c_i \alpha_i$, where all coefficients c_i are either $c_i \geq 0$ or $c_i \leq 0$.

"positive roots" "negative roots"

Cartan-Killing form

Any $X \in \mathfrak{g}$ gives the corresponding endomorphism of $\mathfrak{g}$ by

$(\mathrm{ad}\, X)(Y) = [X, Y]$.

Thus a Lie algebra $\mathfrak{g}$ admits a natural bilinear form :

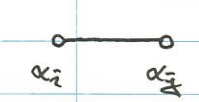
$(X, Y) := \mathrm{tr}((\mathrm{ad}\, X) \circ (\mathrm{ad}\, Y))$ for $X, Y \in \mathfrak{g}$.

Note When $\mathfrak{g}$ is semisimple, this bilinear form is nondegenerate on a Cartan subalgebra $\mathfrak{h}$, and $\mathfrak{h}^*$ is identified with $\mathfrak{h}$ by means of the Cartan-Killing form $(\ ,\ )$.

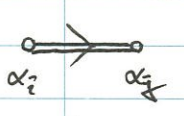
Dynkin diagrams

Fixing an ordering of the simple roots $\alpha_1, \cdots, \alpha_r$, we get the Cartan matrix $C = (c_{ij}) = ((\alpha_i, \alpha_j^\vee))_{i,j=1,\cdots,r}$, where $\alpha_j^\vee = 2\alpha_j / (\alpha_j, \alpha_j)$ is the coroot of α_j

The Dynkin diagram of Φ (or $\mathfrak{g}$) is a directed graph whose nodes corresponds to the simple roots $\alpha_i \in \Delta$ and whose edges have multiplicity $c_{ij} \cdot c_{ji}$ such that



$\iff (\alpha_i, \alpha_j^\vee) = (\alpha_j, \alpha_i^\vee) = -1$



$\iff (\alpha_i, \alpha_j^\vee) = -2, \ (\alpha_j, \alpha_i^\vee) = -1$



$\iff (\alpha_i, \alpha_j^\vee) = -3, \ (\alpha_j, \alpha_i^\vee) = -1$.

Fundamental weights $\{\varpi_i\}_{1 \leq i \leq r}$

The set of simple roots defines a basis for $\mathfrak{h}^*$ satisfying $(\varpi_i, \alpha_j^\vee) = \delta_{ij}$.

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Definition

A weight $\lambda \in \mathfrak{h}^*$ is integral if $\lambda = \sum_{i=1}^{r} a_i \varpi_i$ for $a_i \in \mathbb{Z}$

" dominant if " $a_i \geq 0$

" strictly dominant if " $a_i > 0$

Weyl group W

For $\alpha \in \Phi$, $H_\alpha := \{ x \in \mathfrak{h}^*_{\mathbb{R}} = \Phi \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle x, \alpha \rangle = 0 \}$

is the hyperplane perpendicular to α. "Weyl wall"

Define the reflection σ_α with respect to H_α :

$$\sigma_\alpha : \mathfrak{h}^*_{\mathbb{R}} \longrightarrow \mathfrak{h}^*_{\mathbb{R}}$$

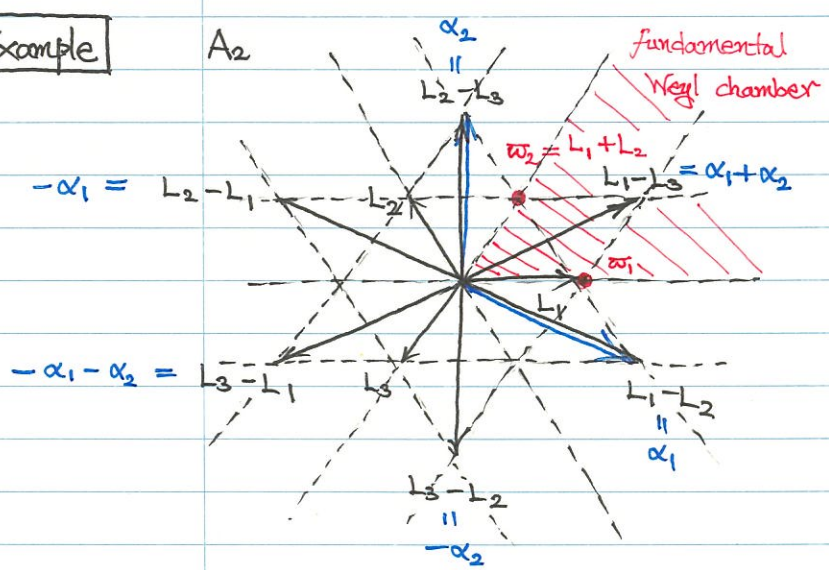
$$\lambda \longmapsto \lambda - 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha = \lambda - (\lambda, \alpha^\vee)\alpha$$

Def The Weyl group W of $\mathfrak{g}$ is the (finite) group generated by the reflections σ_α for $\alpha \in \Phi$.

(In fact, it is sufficient to consider the simple reflections σ_{α_i} for $\alpha_i \in \Delta$.)

Given an element $w \in W$, there exists a minimal integer $\ell(w)$, called the length of w, such that w can be expressed as a product of $\ell(w)$ simple reflections.

Example A_2



$$\Phi = \{ L_i - L_j \mid 1 \leq i \neq j \leq 3 \},$$

where $L_i : \mathfrak{h} \longrightarrow \mathbb{C}$ linear functional

$$\mathrm{diag}(a_1, a_2, a_3) \longmapsto a_i$$

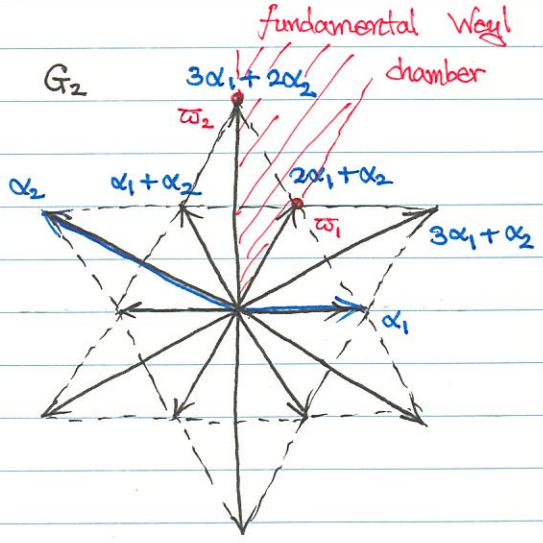
$$\Delta = \{ \alpha_1 = L_1 - L_2,\ \alpha_2 = L_2 - L_3 \}$$

$\varpi_1 = L_1$ and $\varpi_2 = L_1 + L_2$

$$C = (c_{ij}) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad W = \mathfrak{S}_3$$

Dynkin diagram 

G_2



$$\alpha_1 = L_1, \quad \alpha_2 = -\frac{3}{2}L_1 + \frac{\sqrt{3}}{2}L_2$$

$$\begin{cases} \varpi_1 = \frac{1}{2}L_1 + \frac{\sqrt{3}}{2}L_2 = 2\alpha_1 + \alpha_2 \\ \varpi_2 = \sqrt{3}L_2 = 3\alpha_1 + 2\alpha_2 \end{cases}$$

Cartan matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Dynkin diagram 

Weyl group $W = D_6$

: dihedral group of order 12.

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Borel subalgebra = a maximal solvable subalgebra of $\mathfrak{g}$

(eg) standard Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \text{, where } \Phi^+ = \{\alpha \in \Phi \mid \alpha : \text{positive}\}$$

If $\mathfrak{g} = \mathfrak{sl}_n$, then $\mathfrak{b}$ is the subalgebra of upper triangular matrices.

Remark Every Borel subalgebra is G-conjugate to the standard Borel subalgebra.

Def A subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is called **parabolic** if $\mathfrak{p}$ contains a Borel subalgebra

Putting $\Delta_{\mathfrak{p}} = \{\alpha_i \in \Delta \mid \mathfrak{g}_{-\alpha_i} \subset \mathfrak{p}\}$,

$$\mathfrak{p} = \underbrace{\bigoplus_{\alpha \in \Phi \cap \operatorname{span} \Delta_{\mathfrak{p}}} \mathfrak{g}_\alpha}_{\text{reductive}} \oplus \underbrace{\bigoplus_{\beta \in \Phi^+ \setminus \operatorname{span} \Delta_{\mathfrak{p}}} \mathfrak{g}_\beta}_{\text{nilpotent}} \quad ; \text{ Levi decomposition}$$

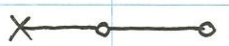
Notation (Marked Dynkin diagrams)

A useful notation for a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is to cross through all nodes in the Dynkin diagram for $\mathfrak{g}$ which correspond to the simple roots in $\Delta \setminus \Delta_{\mathfrak{p}}$.

Def For a parabolic subgroup $P \subset G$ with $\operatorname{Lie}(P) = \mathfrak{p}$, a homogeneous space G/P is called a **generalized flag variety**.

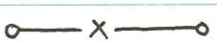
Example $G = SL_4(\mathbb{C})$; maximal parabolic subgroups

$$P_1 = \left\{ \left(\begin{array}{c|ccc} * & * & * & * \\ \hline 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array} \right) \right\}, \quad P_2 = \left\{ \left(\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & * & * \\ 0 & 0 & * & * \end{array} \right) \right\}, \quad P_3 = \left\{ \left(\begin{array}{ccc|c} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ \hline 0 & 0 & 0 & * \end{array} \right) \right\}$$



P_1 = the stabilizer of 1-dimensional subspace $\operatorname{span}\langle e_1 \rangle$, $\{e_1, e_2, e_3, e_4\}$: basis for $\mathbb{C}^4$

$\therefore SL_4/P_1 = \mathbb{P}^3$ (projective space)



P_2 = stabilizer of $\operatorname{span}\langle e_1, e_2 \rangle$

$SL_4/P_2 = Gr(2,4)$ (Grassmannian)



P_3 = stablizer of 3-dimensional subspace $\operatorname{span}\langle e_1, e_2, e_3 \rangle$

$SL_4/P_3 = (\mathbb{P}^3)^\vee$ (dual projective space)

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§ 2. Equivariant vector bundles on G/P

A <u>representation</u> of a Lie algebra $\mathfrak{g}$ on a vector space V is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, i.e., a linear map that preserves Lie brackets

$$[X, Y] . v = X . (Y . v) - Y . (X . v) \quad \text{for } X, Y \in \mathfrak{g},\ v \in V.$$

A nonzero vector $v \in V$ is a <u>weight vector</u> of weight $\lambda \in \mathfrak{h}^*$.

$\overset{\text{def.}}{\iff}$ $\rho(H)(v) = H . v = \lambda(H) v$ for $\forall H \in \mathfrak{h}$

Denote the λ-weight space of V by V_λ.

<u>Note</u> The action of $\mathfrak{g}_\alpha$ carries V_λ to $V_{\lambda+\alpha}$

$$\rho(\mathfrak{g}_\alpha) : V_\lambda \to V_{\lambda+\alpha}$$

($\because$) For $X \in \mathfrak{g}_\alpha$ and $v \in V_\lambda$,

$$\begin{aligned} H . (X . v) &= X . (H . v) + [H, X] . v \\ &= X . (\lambda(H) v) + (\alpha(H) X) . v \\ &= (\lambda(H) + \alpha(H))\, X . v \end{aligned}$$

▨

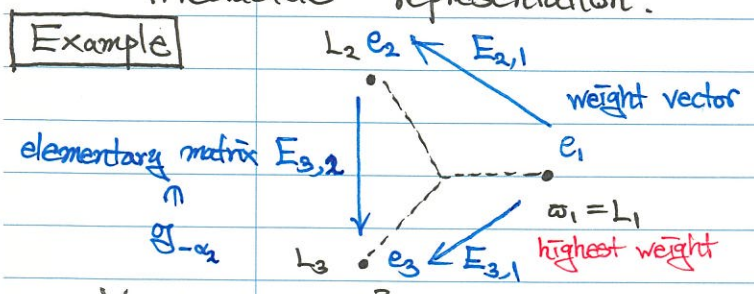
<u>Def</u> A nonzero weight vector in the kernel of the action of $\mathfrak{g}_\alpha$ for all positive roots $\alpha \in \Phi^+$ is called a <u>highest weight vector</u> of V.

<u>Proposition</u> (1) Every finite-dimensional representation V of $\mathfrak{g}$ possesses a highest weight vector.

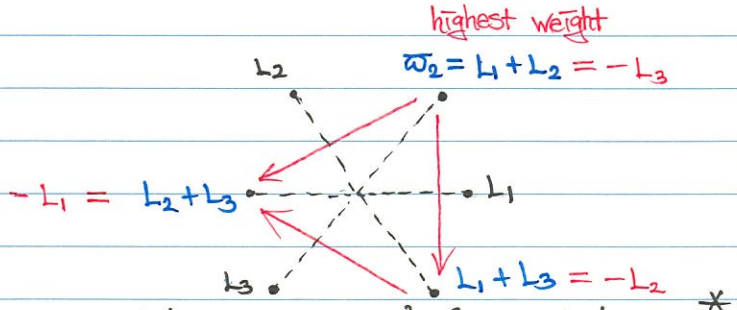
(2) An irreducible representation has a unique highest weight vector up to scalar.

(3) The subspace of V generated by the images of a highest weight vector v under successive application of negative root spaces $\mathfrak{g}_\alpha$ for all negative roots $\alpha \in \Phi^-$ is an irreducible representation.

Example



$V_{SL_3}(\varpi_1) = \mathbb{C}^3$ standard representation



$V_{SL_3}(\varpi_2) = \Lambda^2 \mathbb{C}^3 \cong (V_{SL_3}(\varpi_1))^*$

cf. $V_{SL_n}(\varpi_i) = \Lambda^i \mathbb{C}^n$

Classification theorem of irreducible finite-dimensional representations of a semisimple Lie algebra $\mathfrak{g}$

$$\left\{\begin{array}{c}\text{dominant}\\ \text{integral}\\ \text{weights}\end{array}\right\} \xleftrightarrow[\text{correspondence}]{\text{one-to-one}} \left\{\begin{array}{c}\text{finite-dimensional}\\ \text{irreducible representations}\\ \text{of } \mathfrak{g}\end{array}\right\}$$

$\lambda \in \Lambda^+$ $\qquad$ $V_{\mathfrak{g}}(\lambda)$: irreducible representation with highest weight λ

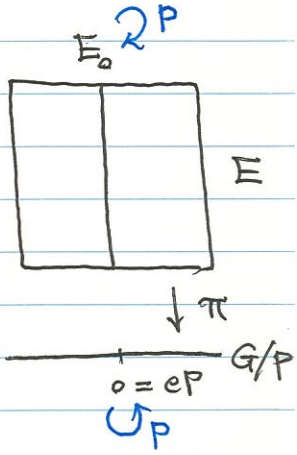
Equivariant vector bundles

Def A holomorphic vector bundle $\pi : E \to G/P$ is G-equivariant if the G-action on the base variety lifts to a compatible action of G on E via holomorphic bundle automorphisms.

$$\left\{\begin{array}{c}\text{Equivariant vector bundles}\\ \text{on } G/P\end{array}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{array}{c}\text{representations}\\ \text{of } P\end{array}\right\}$$

$E = G \times_P E_o$ $\qquad$ $P \to GL(E_o)$

($\to$) Fix a base point $o = eP \in G/P$.
Let $E_o := \pi^{-1}(o)$. Since E is equivariant, the isotropy subgroup P acts linearly on the fiber E_o.



($\leftarrow$) If we have a P-representation $\rho : P \to GL(E_o)$, then P naturally acts on $G \times E_o$ by

$$P \curvearrowright (G \times E_o), \quad p \cdot (g, v) = (gp^{-1}, p \cdot v).$$

$$\Longrightarrow \quad G \times_P E_o := (G \times E_o)/P$$

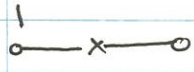
$\downarrow$ G/P — is an equivariant vector bundle on G/P

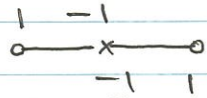
Notation For a P-dominant integral weight $\lambda \in \Lambda_P^+$, denote $\mathcal{E}_\lambda := G \times_P V_P(\lambda)$, where $V_P(\lambda)$ is the irreducible P-module with highest weight λ

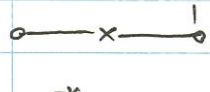
Example $G/P = SL_4(\mathbb{C})/P_2 = Gr(2,4)$

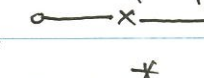
$\mathcal{E}_{\varpi_2}$ =  $= \mathcal{O}(1)$ dual tautological line bundle

$\mathcal{E}_{-\varpi_2}$ =  $= \mathcal{O}(-1)$

$\mathcal{E}_{\varpi_1}$ =  $= \mathcal{U}^*$

$\mathcal{E}_{\varpi_1 - \varpi_2}$ =  $= \mathcal{U}$ universal subbundle

$\mathcal{E}_{\varpi_3}$ =  $= \mathcal{Q}$ universal quotient bundle

$\mathcal{E}_{-\varpi_2+\varpi_3}$ =  $= \mathcal{Q}^*$

$\Rightarrow T_{Gr(2,4)} = \mathcal{U}^* \otimes \mathcal{Q}$  , $\Omega^1 = T^* = \mathcal{U} \otimes \mathcal{Q}^*$ 

tangent bundle; cotangent bundle; P-dominant but NOT G-dominant

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§ 3. Borel-Weil-Bott theorem

Theorem (1957) G : a simply-connected complex semisimple Lie group
$P \subset G$: a parabolic subgroup
$\lambda \in \Lambda_P^+$: an integral weight for G dominant with respect to P

Let $\rho = \sum_{i=1}^{r} \varpi_i = \varpi_1 + \varpi_2 + \cdots + \varpi_r$ denote the sum of fundamental weights of G.

(i) If a weight $\lambda + \rho$ is singular, that is, it lies on a wall of a Weyl chamber,
($\Leftrightarrow$ it is orthogonal to some (positive) root of G)
then every cohomology group $H^i(G/P, \mathcal{E}_\lambda)$ vanishes.

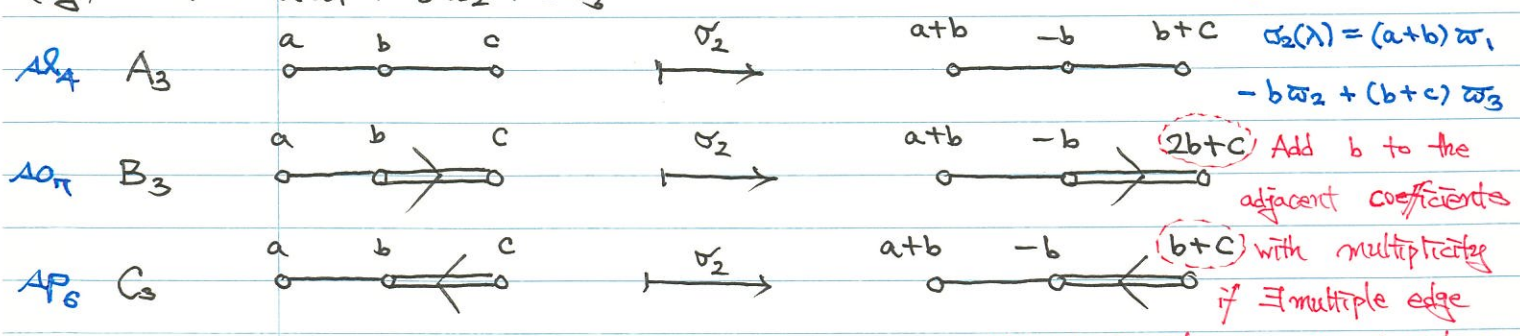
(ii) Otherwise, $\lambda + \rho$ is regular, that is, it lies in the interior of some Weyl chamber,
then $\exists!\ w \in W$ such that $w(\lambda+\rho)$ is strictly dominant,
(an element of the Weyl group)
and
$$H^i(G/P, \mathcal{E}_\lambda) = \begin{cases} V_G(w(\lambda+\rho)-\rho)^* & \text{if } i = \ell(w) \\ 0 & \text{otherwise,} \end{cases}$$
where $\ell(w)$ means the length of $w \in W$.

Corollary (Borel-Weil theorem)
If λ is G-dominant, then $H^0(G/P, \mathcal{E}_\lambda) = V_G(\lambda)^*$
and all other sheaf cohomology vanishes.

The action of a simple reflection on the weight lattice $W \curvearrowright \Lambda_G$

For a simple root $\alpha_i \in \Delta$ and a weight $\lambda \in \Lambda_G$,
$$\sigma_i(\lambda) = \sigma_{\alpha_i}(\lambda) = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$$
Thus the j-th node coefficient for $\sigma_i(\lambda)$ are given by
$$(\sigma_i(\lambda), \alpha_j^\vee) = (\lambda, \alpha_j^\vee) - (\lambda, \alpha_i^\vee)\underbrace{(\alpha_i, \alpha_j^\vee)}_{\text{Cartan integer } c_{ij}}$$

(eg) $\lambda = a\varpi_1 + b\varpi_2 + c\varpi_3$



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Example (1) $SL_2(\mathbb{C})/B = \mathbb{P}^1$; marked Dynkin diagram = $\overset{}{\times}$

Compute the sheaf cohomology of a line bundle on $\mathbb{P}^1$.

i) $\mathcal{O}(-1) = \overset{-1}{\times}$ $\quad \lambda+\rho = -\varpi_1 + \varpi_1 = 0$; singular

$\therefore H^i(\mathbb{P}^1, \mathcal{O}(-1)) = 0, \ \forall i$

ii) $k \geq 0$; $\mathcal{O}(k) = \overset{k}{\times}$ $\quad \lambda+\rho = k\varpi_1 + \varpi_1 = (k+1)\varpi_1$ is strictly dominant.

$H^0(\mathbb{P}^1, \mathcal{O}(k)) = V_{SL_2}(k\varpi_1)^* = (\mathrm{Sym}^k \mathbb{C}^2)^* \cong \mathbb{C}[x_0, x_1]_k$

and $H^1(\mathbb{P}^1, \mathcal{O}(k)) = 0$.

(k-th symmetric power of the standard representation of SL_2; the space of homogeneous polynomials of degree k)

iii) $k \leq -2$;

$\lambda+\rho = (k+1)\varpi_1$ $\overset{k+1}{\times}$ $\overset{\sigma_1}{\longmapsto}$ $\overset{-k-1}{\times}$ strictly dominant if $k \leq -2$

Because $\ell(\sigma_1) = 1$, $H^1(\mathbb{P}^1, \mathcal{O}(k)) = V_{SL_2}((-k-2)\varpi_1)^*$

$= (\mathrm{Sym}^{-k-2}\mathbb{C}^2)^*$

and $H^0(\mathbb{P}^1, \mathcal{O}(k)) = 0$.

(2) $SL_4(\mathbb{C})/P_2 = Gr(2,4)$ $\quad \circ\!\!-\!\!\times\!\!-\!\!\circ$

i) $\mathcal{O}(1) = \overset{1}{\circ\!\!-\!\!\times\!\!-\!\!\circ}\ \mathcal{E}_{\varpi_2}$ $\quad$ Since ϖ_2 is already G-dominant,

$H^0(Gr(2,4), \mathcal{O}(1)) = V_{SL_4}(\varpi_2)$

$= (\Lambda^2 \mathbb{C}^4)^*$

$\Longrightarrow$ it gives the Plücker embedding $Gr(2,4) \hookrightarrow \mathbb{P}(\Lambda^2\mathbb{C}^4)$.

ii) tangent bundle $\mathcal{T}_{Gr(2,4)} = \overset{1}{\circ\!\!-\!\!\times\!\!-\!\!\circ\!\!-\!\!\overset{1}{}}$: G-dominant

By Borel-Weil theorem, $H^0(Gr(2,4), \mathcal{T}_{Gr(2,4)}) = V_{SL_4}(\varpi_1+\varpi_3)^* \cong \mathfrak{sl}_4$

and $H^i(Gr(2,4), \mathcal{T}_{Gr(2,4)}) = 0$ for $\forall i \geq 1$.

From the Kodaira-Spencer deformation theory, we know that the complex structure of the Grassmannian $Gr(2,4)$ is infinitesimally rigid since the first cohomology of the tangent sheaf vanishes.

iii) $\Omega^3 = \Lambda^3 \mathcal{T}^* = \mathcal{T} \otimes K_{Gr(2,4)} = \mathcal{T} \otimes \mathcal{O}(-4) = \overset{1}{\circ\!\!-\!\!\times\!\!-\!\!\circ\!\!-\!\!\overset{-4}{}}\ \mathcal{E}_{\varpi_1 - 4\varpi_2 + \varpi_3}$

$\lambda+\rho$ $\overset{2}{\circ}\overset{-3}{\times}\overset{2}{\circ}$ $\overset{\sigma_2}{\longmapsto}$ $\overset{-1}{\circ}\overset{3}{\times}\overset{-1}{\circ}$ $\overset{\sigma_1}{\longmapsto}$ $\overset{1}{\circ}\overset{2}{\times}\overset{-1}{\circ}$ $\overset{\sigma_3}{\longmapsto}$ $\overset{1}{\circ}\overset{1}{\times}\overset{1}{\circ}$

We have found $w = \sigma_3\sigma_1\sigma_2 \in W$ such that $w(\lambda+\rho)$ is strictly dominant.

Since $\ell(w) = 3$, $H^3(Gr(2,4), \overset{1}{\circ\!\!-\!\!\times\!\!-\!\!\circ\!\!-\!\!\overset{-4}{}}) = V_{SL_4}(0)^* \cong \mathbb{C}$

cf. $H^2(Gr(2,4), \Omega^3) \cong H^{3,3}(Gr(2,4)) = \mathbb{C}\langle \omega \wedge \omega \wedge \omega \rangle$

(Ω^3: sheaf of holomorphic 3-forms; $H^{3,3}$: Dolbeault cohomology defined by $\bar{\partial}$; ω: Kähler $(1,1)$-form)

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§ 4. Applications of Borel-Weil-Bott theorem

Application 1 : Irreducible equivariant Ulrich bundles

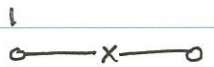
Def Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension d. A vector bundle E on X is called **Ulrich** if the cohomology groups $H^i(X, E \otimes \mathcal{O}(-t)) = 0$ for all $0 \le i \le d$ and $1 \le t \le d$.

Remark This notion is originated from Commutative Algebra :
[Eisenbud-Schreyer, 2003] E is Ulrich
$\iff \exists$ a linear resolution $0 \to L_c \to L_{c-1} \to \cdots \to L_0 \to E \to 0$
with $c = \operatorname{codim}(X, \mathbb{P}^N)$ and $L_i = \mathcal{O}_{\mathbb{P}^N}(-i)^{b_i}$.

Eisenbud-Schreyer Conjecture ; Every smooth projective variety admits an Ulrich bundle.

Their question has been answered positively for several algebraic varieties, e.g. del Pezzo surfaces, K3 surfaces, abelian surfaces, ...

Question Equivariant Ulrich bundles on G/P ?

Example (1) $\mathcal{U}^*$ =  $\mathcal{E}_{\varpi_1}$

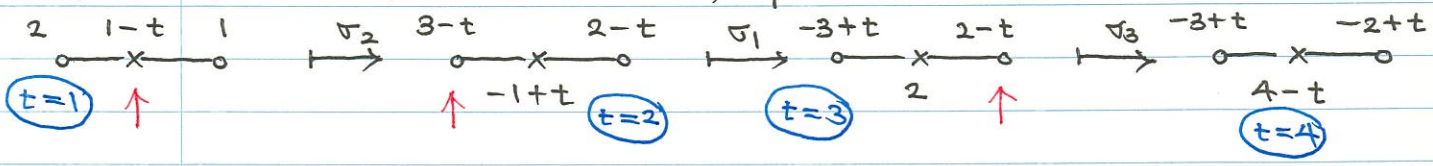
$\downarrow$

$Gr(2,4) = SL_4/P_2 \subset \mathbb{P}(V_{SL_4}\varpi_2) = \mathbb{P}(\Lambda^2\mathbb{C}^4)$

Plücker embedding by $\mathcal{O}(1)$

Claim $\mathcal{U}^*$ is an Ulrich bundle on $Gr(2,4)$.

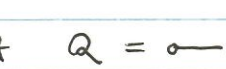
Since $\mathcal{O}(1) = \mathcal{E}_{\varpi_2}$, the defining weight of $\mathcal{U}^*(-t) = \mathcal{E}_{\varpi_1}(-t)$ is $\varpi_1 - t\varpi_2$, and $(\varpi_1 - t\varpi_2) + \rho = 2\varpi_1 + (1-t)\varpi_2 + \varpi_3$



After adding ρ, we apply simple reflections on the weight $\varpi_1 - t\varpi_2 + \rho$ obtaining singular weights for $t = 1, 2, 3, 4$.

By Borel-Weil-Bott theorem,

$H^\bullet(Gr(2,4), \mathcal{U}^*(-t)) = 0$ for all $1 \le t \le 4$.

Similarly, we can prove that Q =  is also Ulrich.

ibis

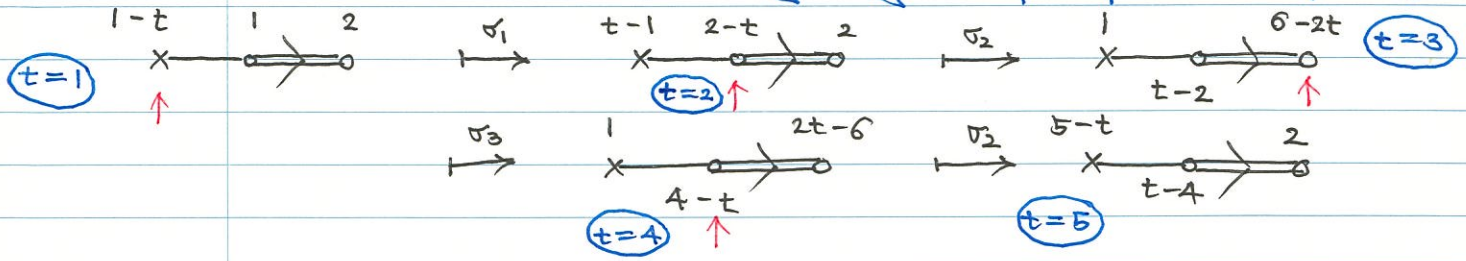
(2) $B_3/P_1 = Spin(7,\mathbb{C})/P_1 = OGr(1,7) = Q^5 \subset \mathbb{P}^6$



Claim The spinor bundle $\mathcal{S} = \mathcal{E}_{\varpi_3}$ is Ulrich.

rank 4 ↓ ↑ irreducible equivariant bundle given by the spin representation of so_5

Q^5



Because the weights $(\varpi_3 - t\varpi_1) + \rho = (1-t)\varpi_1 + \varpi_2 + 2\varpi_3$ are singular for $t = 1,2,3,4,5$, $H^\bullet(Q^5, \mathcal{E}_{\varpi_3}(-t)) = 0 \quad \forall\, 1 \le t \le 5$.

Classification of irreducible equivariant Ulrich bundles on G/P

[Costa and Miró-Roig, 2015] : A Grassmannian admits an Ulrich bundle, and they give a full classification of all irreducible equivariant Ulrich bundles on $Gr(k,n)$. (type A)

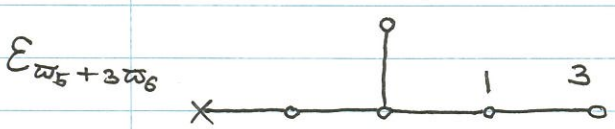
(type B, C, D)

[Fonarev, 2016] (symplectic or orthogonal) isotropic Grassmannians : The only isotropic Grassmannians admitting an irreducible equivariant Ulrich bundles are

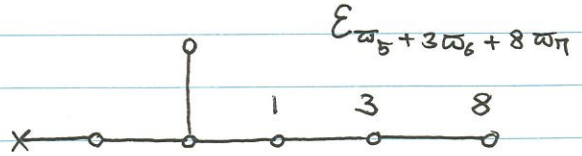
(i) the symplectic Grassmannians of planes $SGr(2,2n)$;
(ii) the hyperquadrics $Q^d = OGr(1, d+2)$;
(iii) the orthogonal Grassmannians of planes $OGr(2,n)$ for $n \ge 4$;
(iv) the orthogonal Grassmannians of 3-spaces of the form $OGr(3, 4g+6)$ for $g \ge 0$.

[Kyoung-Seog Lee and Park, 2017] exceptional flag varieties G/P_{max} (type E, F, G)
: G a complex Lie group of exceptional type
$P \subset G$: a maximal parabolic subgroup
If G/P admits an irreducible equivariant Ulrich bundle, then $G/P = E_6/P_1$ or E_7/P_1.

16-dimensional Cayley plane — 33-dimensional E_7-adjoint variety



rank = 4,608



rank = 3,700,494,720 by the Weyl's character formula

ibis

Notation P_k : a maximal parabolic subgroup associated to a simple root $\alpha_k \in \Delta$

For a P_k-dominant integral weight $\lambda \in \Lambda^+_{P_k}$, we define a set

$$\mathrm{Sing}(\lambda) := \{ t \in \mathbb{Z} \mid \lambda + \rho - t\,\varpi_k \text{ is singular} \}.$$

Proposition (Fonarev's criterion) *Main tool for the classification*

Irreducible equivariant vector bundle $\mathcal{E}_\lambda \to G/P_k$ is Ulrich $\iff$ $\mathrm{Sing}(\lambda) = \{1, 2, \cdots, d-1, d\}$, where $d = \dim(G/P_k)$

(proof) ($\Longleftarrow$) Assume $\mathrm{Sing}(\lambda) = \{1, 2, \cdots, d\}$. Since $\mathcal{E}_\lambda(-t) = \mathcal{E}_{\lambda - t\varpi_k}$, Borel-Weil-Bott theorem implies that

$$H^\bullet(G/P_k, \mathcal{E}_\lambda(-t)) = 0 \quad \text{for all } 1 \le t \le d.$$

($\Longrightarrow$) If $\mathcal{E}_\lambda$ is Ulrich, then $\{1, 2, \cdots, d\} \subset \mathrm{Sing}(\lambda)$.

So it suffices to show that $|\mathrm{Sing}(\lambda)| \le d = \dim(G/P_k)$.

Given an integer m, we define

$$\Phi_m := \left\{ \alpha = \sum_{i=1}^{r} c_i \alpha_i \in \Phi \;\middle|\; c_k = m \right\}.$$

Then we have the decomposition of $\mathfrak{p}_k$:

$$\mathfrak{p}_k = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha \oplus \bigoplus_{\beta \in \Phi^+ \setminus \Phi_0} \mathfrak{g}_\beta$$

So we can identify $T_0(G/P_k) \cong \bigoplus_{\beta \in \Phi^+ \setminus \Phi_0} \mathfrak{g}_{-\beta}$

and $\dim(G/P_k) = |\Phi^+ \setminus \Phi_0|$.

For any $\alpha \in \Phi^+ \cap \Phi_0$, $(\varpi_k, \alpha^\vee) = 0$ from $(\varpi_i, \alpha_j^\vee) = \delta_{ij}$

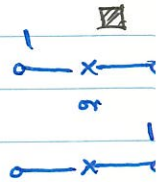
$$(\lambda + \rho - t\varpi_k, \alpha^\vee) = \sum_{i \neq k} (a_i + 1)(\varpi_i, \alpha^\vee) > 0$$

since $\lambda = \sum_{i=1}^{r} a_i \varpi_i$ is a P-dominant weight, that is, $a_i \ge 0$ for all $i \neq k$.

Hence, we only need to consider positive roots in $\Phi^+ \setminus \Phi_0$ for the set $\mathrm{Sing}(\lambda)$.

$$\therefore |\mathrm{Sing}(\lambda)| \le |\Phi^+ \setminus \Phi_0| = \dim(G/P_k) = d.$$

Example $\mathrm{Gr}(2,4) = SL_4/P_2$, $\lambda = a_1\varpi_1 + a_2\varpi_2 + a_3\varpi_3$



$\Phi^+ \setminus \Phi_0 = \{\alpha_2, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$

$\mathrm{Sing}(\lambda) = \{a_2 + 1, a_1 + a_2 + 2, a_2 + a_3 + 2, a_1 + a_2 + a_3 + 3\}$

$= \{1, 2, 3, 4\}$ $\therefore a_2 = 0$, a_1 or $a_3 = 1$

Thus $\mathrm{Gr}(2,4)$ admits exactly two irreducible equivariant Ulrich bundle $\mathcal{E}_{\varpi_1} = \mathcal{U}^*$ and $\mathcal{E}_{\varpi_3} = \mathcal{Q}$.

ibis

Application 2 : Exceptional collections in $D^b(\mathrm{Coh}\, X)$

X : a smooth projective variety
$D^b(X) = D^b(\mathrm{Coh}\, X)$: the bounded derived category of coherent sheaves on X

Def A collection of objects $(E_1, E_2, \cdots, E_\ell)$ in $D^b(X)$ is called <u>exceptional</u> if
(i) for each i, the object is exceptional, that is, $\mathrm{Hom}(E_i, E_i) = \mathbb{C}$ and $\mathrm{Ext}^p(E_i, E_i) = 0$ for $p \neq 0$.
(ii) for all $i > j$, $\mathrm{Ext}^\bullet(E_i, E_j) = 0$
$\qquad\qquad\qquad

Lemma $X = Gr(2, V)$, $\dim V = n$

If $0 \le l_1, l_2 \le \frac{n}{2} - 1$ and $0 \le k \le n-1$, then we have

$$\mathrm{Ext}^p(S^{l_1}\mathcal{U}^*(k), S^{l_2}\mathcal{U}^*) = \begin{cases} S^{l_2 - l_1}V^* & \text{if } l_1 \le l_2,\ k = 0,\ p = 0, \\ \mathbb{C} & \text{if } l_1 = l_2 = \frac{n}{2} - 1,\ k = \frac{n}{2},\ p = n-2, \\ 0 & \text{otherwise.} \end{cases}$$

(proof) $\mathrm{Ext}^p(S^{l_1}\mathcal{U}^*(k), S^{l_2}\mathcal{U}^*) = H^p(X, S^{l_1}\mathcal{U} \otimes S^{l_2}\mathcal{U}^*(-k))$

$\mathcal{U}^*$ $\quad$ 1 $\quad$ $\circ\!-\!\times\!-\!\circ\!-\cdots-\!\circ$ $\quad \mathcal{E}_{\varpi_1}$ $\qquad$ $\mathcal{U}$ $\quad$ 1 $\ -1$ $\quad$ $\circ\!-\!\times\!-\!\circ\!-\cdots-\!\circ$ $\quad \mathcal{E}_{\varpi_1 - \varpi_2}$

$\varpi_1 = L_1$ $\qquad$ $\varpi_1 - \varpi_2 = L_1 - (L_1 + L_2) = -L_2$

$\therefore\ S^{l_2}\mathcal{U}^* = \mathcal{E}_{l_2 L_1}$ and $S^{l_1}\mathcal{U} = \mathcal{E}_{-l_1 L_2}$

By the Littlewood-Richardson rule,

$$S^{l_1}\mathcal{U} \otimes S^{l_2}\mathcal{U}^* = \mathcal{E}_{l_2 L_1 - l_1 L_2} \oplus \left(S^{l_1 - 1}\mathcal{U} \otimes S^{l_2 - 1}\mathcal{U}^*\right).$$

So it suffices to compute $H^p(X, \mathcal{E}_{(l_2 - k)L_1 + (-l_1 - k)L_2})$.

$$\lambda = (l_2 - k)L_1 + (-l_1 - k)L_2$$
$$= (l_2 - k,\ -l_1 - k,\ 0,\ \cdots,\ 0)$$
$$\lambda + \rho = (n + l_2 - k,\ n - l_1 - k - 1,\ n-2,\ n-3,\ \cdots,\ 2,\ 1)$$

$\circlearrowleft$ acts as permutation

$\lambda + \rho$ is regular. $\qquad W_{SL_n} = \mathfrak{S}_n$

$\iff$ all entries of $\lambda + \rho$ are distinct.

$\iff$ $n + l_2 - k \notin \{n-2, n-3, \cdots, 2, 1\}$ and $n - l_1 - k - 1 \notin \{n-2, n-3, \cdots, 2, 1\}$

Case I : $\quad n - l_1 - k - 1 = n - 1 \iff l_1 = k = 0$

$\lambda + \rho = (n + l_2,\ n-1,\ n-2,\ \cdots,\ 2,\ 1)$ strictly decreasing sequence

$\therefore\ H^0(X, \mathcal{E}_\lambda) = S^{l_2}V^*$ $\qquad$ (strictly dominant weight)

Case II. $\quad n - l_1 - k - 1 \le 0 \iff l_1 + k \ge n - 1$

$l_1 \le \frac{n}{2} - 1$ (assumption) $\Longrightarrow k \ge n - 1 - l_1 \ge (n-1) - \left(\frac{n}{2} - 1\right) = \frac{n}{2}$

$l_2 \le \frac{n}{2} - 1$ $\quad$ 〃 $\quad \Longrightarrow n + l_2 - k \le n + \left(\frac{n}{2} - 1\right) - \frac{n}{2} = n - 1$

$k \le n - 1$ $\quad$ 〃 $\quad \Longrightarrow n + l_2 - k \ge l_2 + 1 \ge 1$

Thus $\quad n + l_2 - k = n - 1$ and equalities hold ;

$l_1 = l_2 = \frac{n}{2} - 1$, $\quad k = \frac{n}{2}$

$\therefore\ \lambda + \rho = (n-1,\ \underline{0},\ n-2,\ n-3,\ \cdots,\ 2,\ 1)$

$\Longrightarrow\ \underbrace{(\sigma_{n-1} \cdots \sigma_3 \sigma_2)}_{w}(\lambda + \rho) = (n-1, n-2, n-3, \cdots, 1, \underline{0})$

$l(w) = n - 2$,

$w(\lambda + \rho) - \rho = (-1, -1, \cdots, -1)$

$\therefore\ H^{n-2}(X, \mathcal{E}_\lambda) = V_{SL(V)}(0)^* \cong \mathbb{C}$ $\qquad \blacksquare$

ibis