LLT polynomials in a nutshell: on Schur expansion of LLT polynomials

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The goal

We first define the **LLT polynomials**,
explain **why** they are important by showing **some examples**
and, keep track of what have been known and **what's still open**.

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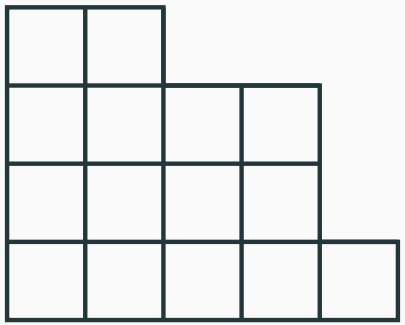
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Definition of LLT polynomials

Basic definitions

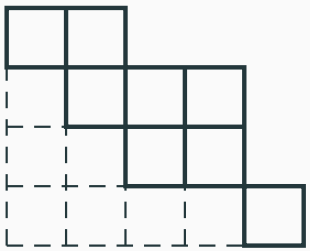
- A **partition** is a sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that
$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0.$$
- We identify a partition λ with its **Young diagram**.

Example. $\lambda = (5, 4, 4, 2)$



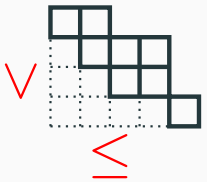
More definitions

- Given two partitions λ and μ such that $\mu \subseteq \lambda$, the **skew shape** λ/μ is defined to be the set theoretic difference $\lambda \setminus \mu$.
  Example.



$(5,4,4,2)/(4,2,1)$

- A **semistandard Young tableau** of shape $\nu = \lambda/\mu$ is a function



  $T : \nu \to \mathbb{Z}_{>0}$ such that
- $\mathrm{SSYT}(\nu) =$ the set of semistandard Young tableaux of shape ν

Symmetric functions

- A **symmetric function** is a polynomial $f(x_1, \ldots, x_n)$ which is invariant under the action of the symmetric group

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n),$$

  or, $\sigma f = f$ for all $\sigma \in S_n$.

- The **elementary** symmetric functions e_λ are defined by

$$e_n = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}, \quad n \geq 1,$$

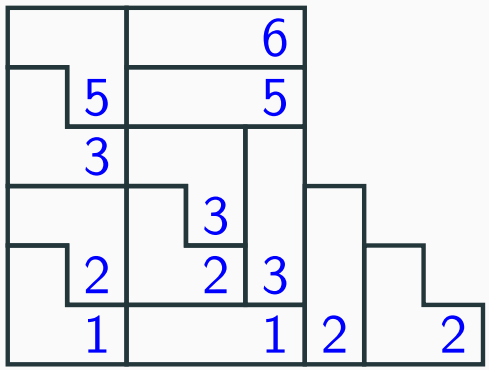
$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots, \quad \text{for } \lambda = (\lambda_1, \lambda_2, \ldots).$$

- The **Schur function** of shape λ/μ is defined by

$$s_{\lambda/\mu}(x) = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} x^T$$

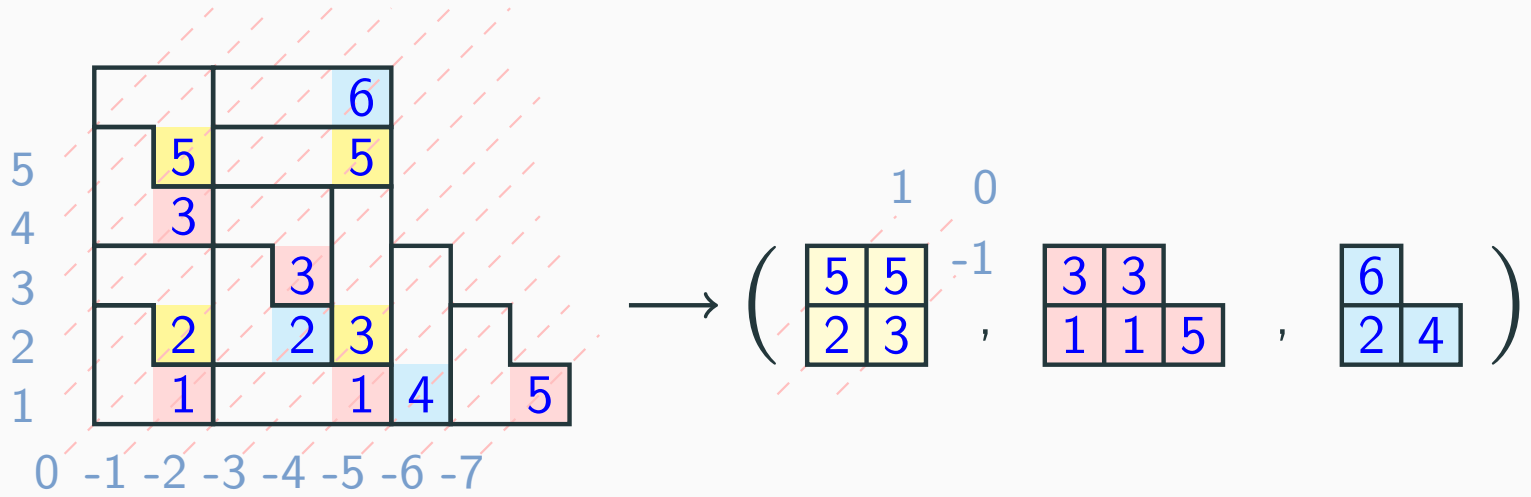
LLT polynomials

- LLT polynomials are a family of symmetric functions introduced by Lascoux, Leclerc and Thibon in '1997.
- The original definition uses **cospin** statistic of ribbon tableaux.



LLT polynomials

- A ribbon tableau maps to a tuple of skew tableaux via **quotient map**.



- Here, we use the variant definition of **Bylund** and **Haiman** using **inversion** statistic.

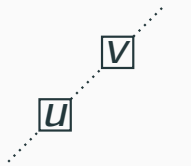
- Consider a k-tuple of skew diagrams $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$, and let

$$\mathrm{SSYT}(\boldsymbol{\nu}) = \mathrm{SSYT}(\nu^{(1)}) \times \cdots \times \mathrm{SSYT}(\nu^{(k)}).$$

- The **content** of a cell $u = (\text{row, column}) = (i, j)$ is the integer

$$c(u) = i - j.$$

- Given $\boldsymbol{T} \in (T^{(1)}, \ldots, T^{(k)}) \in \mathrm{SSYT}(\boldsymbol{\nu})$, a pair of entries $T^{(i)}(u) > T^{(j)}(v)$ form an **inversion** if either

(i) $i < j$ and $c(u) = c(v)$, or 

(ii) $i > j$ and $c(u) = c(v) + 1$. 

- $\mathrm{inv}(\boldsymbol{T}) =$ the number of inversions in $\boldsymbol{T}$.

Definition of LLT polynomials

Definition

The **LLT polynomial** indexed by $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$ is

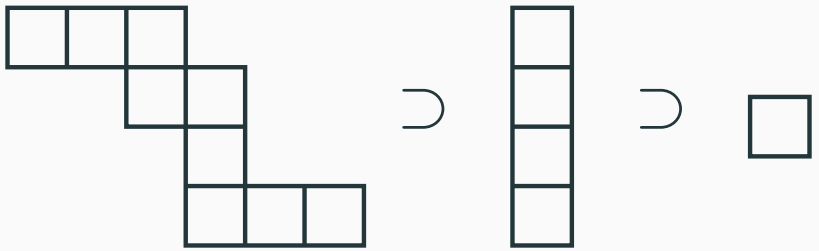
$$\mathrm{LLT}_{\boldsymbol{\nu}}(x; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(\boldsymbol{T})} x^{\boldsymbol{T}}.$$

Note on properties of LLT polynomials

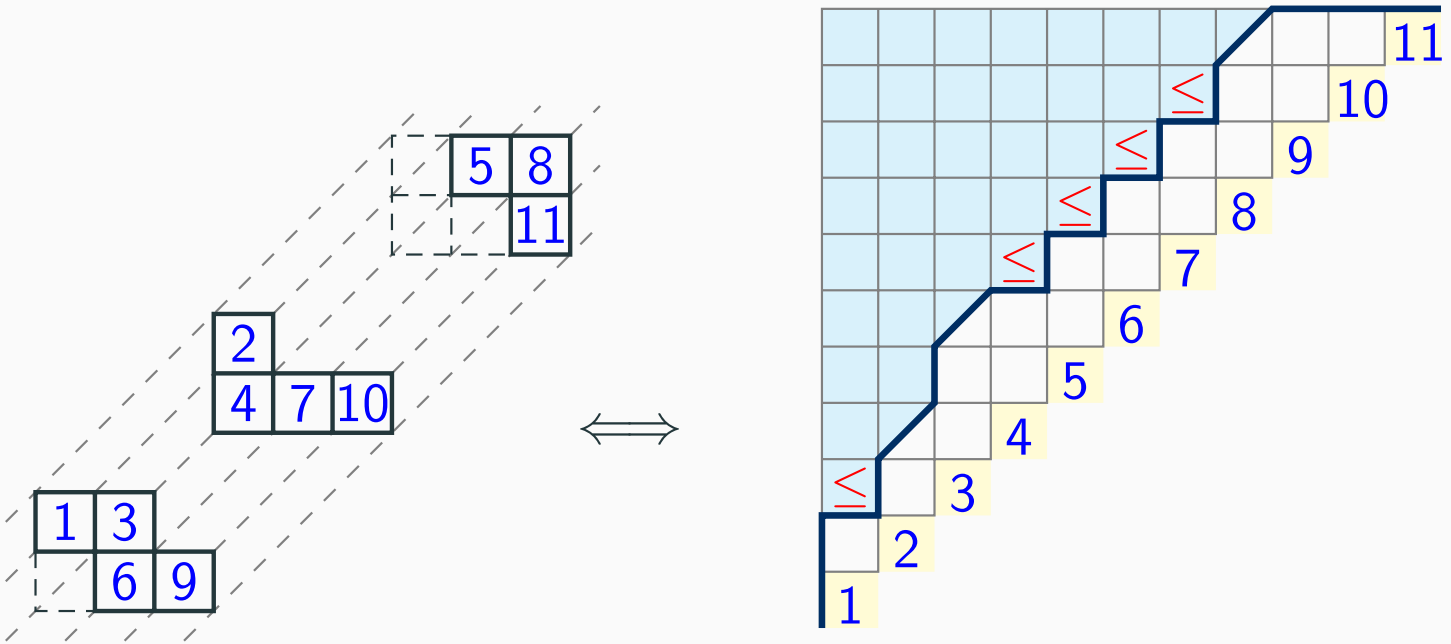
- LLT polynomials are **symmetric**.
- When $q = 1$, $\mathrm{LLT}_{\nu}(x; 1) = \prod_{i=1}^{k} s_{\nu^{(i)}}(x)$.
- Furthermore, they are **Schur positive**. [Grojnowski-Haiman]
- However, combinatorial description for Schur coefficients are **not** known.
- **Macdonald polynomials** $\tilde{H}_{\mu}(x; q, t)$ can be expanded positively in terms of LLT polynomials.
- Hence, the Schur expansion of LLT polynomial gives a combinatorial formula for q, t-**Kostka polynomials**.

Several important subfamilies of LLT polynomials

We consider LLT polynomials indexed by tuple of certain shapes:
ribbons, vertical strips and **unicellular diagrams**

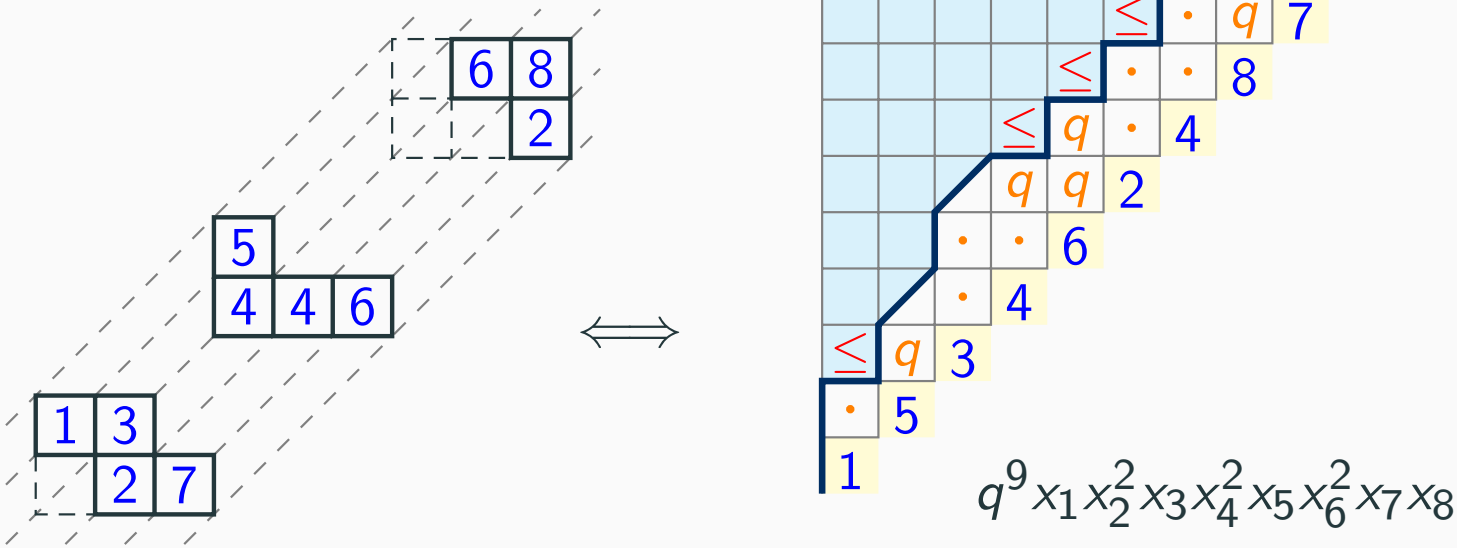


① LLT polynomials indexed by ribbons



① LLT polynomials indexed by ribbons

To read the inversion statistic:



Macdonald polynomials can be expanded positively in terms of ribbon LLT's

$\tilde{H}_\mu(X;q,t)$: the *modified Macdonald polynomial*

Theorem [Haglund-Haiman-Loehr, '2005]

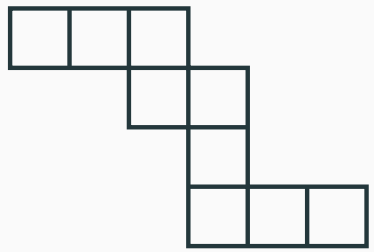
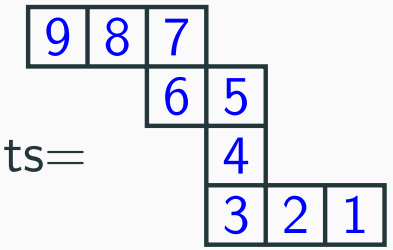
$$\tilde{H}_\mu(X;q,t) = \sum_{\sigma:\,\mu\to\mathbb{Z}_+} q^{\mathsf{inv}(\sigma)}\, t^{\mathsf{maj}(\sigma)}\, x^\sigma$$
$$= \sum_{D\subseteq\{(i,j)\in\mu:\, i>1\}} q^{-a(D)}\, t^{\mathsf{maj}(\sigma)}\, \mathrm{LLT}_{\nu(\mu,D)}(X;q),$$

where $\nu(\mu,D) = (\nu^{(1)},\ldots,\nu^{(k)})$, a tuple of ribbons.

A tuple of ribbons $\nu(\mu, D) = (\nu^{(1)}, \ldots, \nu^{(k)})$

- Define the **descent** set of a ribbon ν to be the set of contents $c(u)$ of those cells $u = (i, j) \in \nu$ such that the cell $v = (i-1, j)$ directly below u also belongs to ν.

Example.

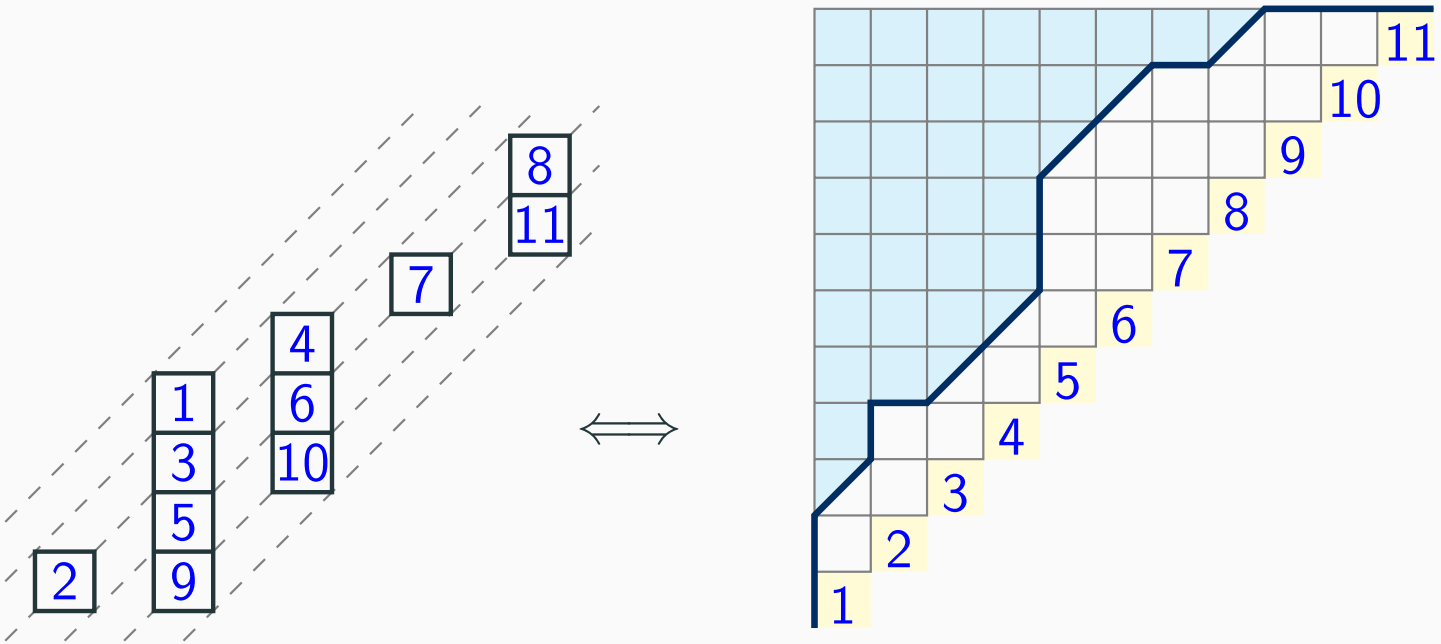
$\nu =$  , contents=  , $\mathrm{Des}(\nu) = \{4, 5, 7\}$

- Given a partition μ and a subset $D \subseteq \{(i,j) \in \mu : i > 1\}$,

$$\boldsymbol{\nu}(\mu, D) = (\nu^{(1)}, \ldots, \nu^{(k)})$$

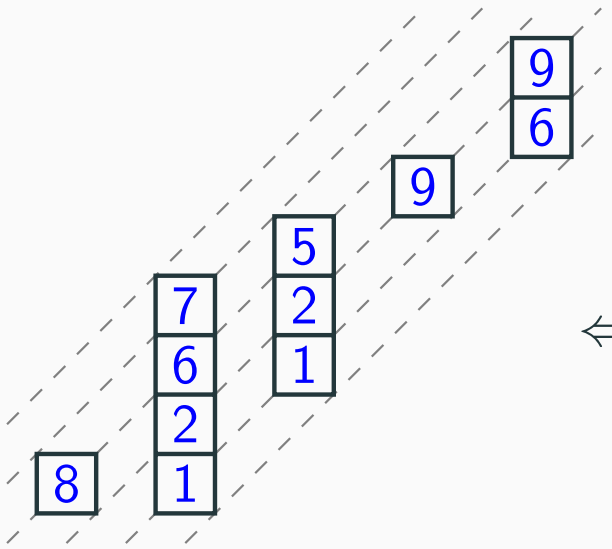
where $k = \mu_1$, $\nu^{(j)}$ has size μ'_j with cell contents $\{1, 2, \ldots, \mu'_j\}$ and descent set $\mathrm{Des}(\nu^{(j)}) = \{i : (i, j) \in D\}$.

② LLT polynomials indexed by vertical strips

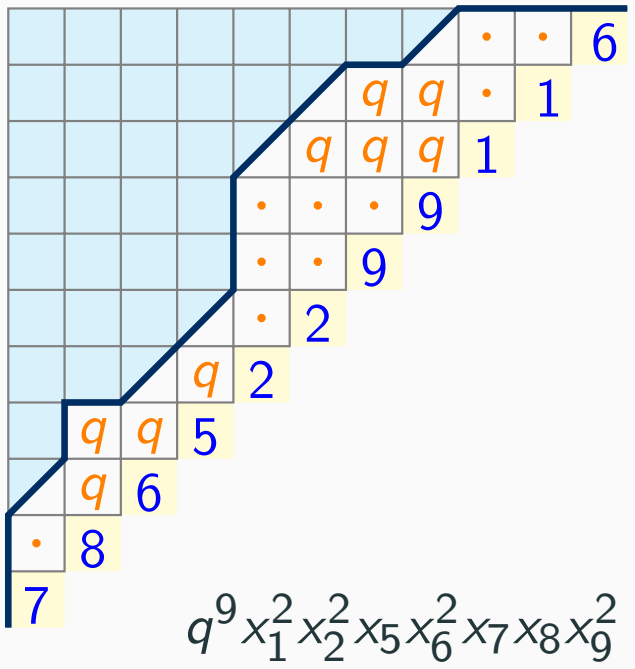


② LLT polynomials indexed by vertical strips

To read the inversion statistic:



$\Longleftrightarrow$



$$q^9 x_1^2 x_2^2 x_5 x_6^2 x_7 x_8 x_9^2$$

Vertical-strip LLT's are related to the Shuffle Conjecture

Shuffle Conjecture [HHLRU, '2005]
$\Rightarrow$ **Shuffle Theorem** [Carlsson-Mellit, '2017]

$$\nabla e_n = \sum_{\sigma \in \mathcal{WP}_n} q^{\mathsf{dinv}(\sigma)} t^{\mathsf{area}(\sigma)} X^{\sigma}$$
$$= \sum_{P} \mathsf{LLT}_P(X; q),$$

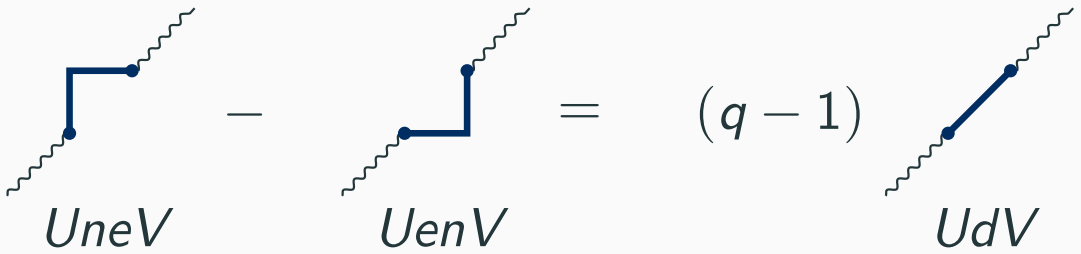
where P's are Schröder paths of size n.

Vertical-strip LLT's are generated by linear relations

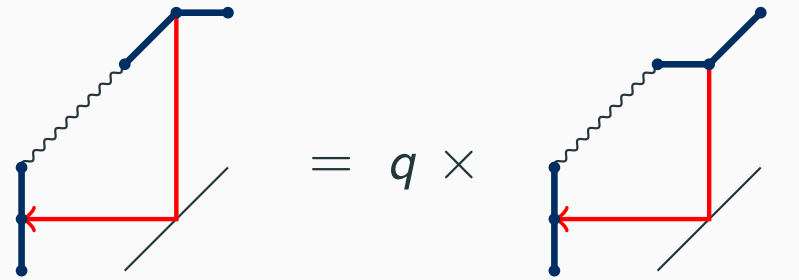
Theorem [Alexandersson-Sulzgruber, '2020]

There exists a unique function $F : \mathcal{S} \to \Lambda_{\mathbb{Q}(q)}$, $P \mapsto F_P$, that satisfies the following conditions:

1. $F_{nd^k e} = e_{k+1}$ for all $k \in \mathbb{N}$
2. $F_{PQ} = F_P F_Q$ for all $P, Q \in \mathcal{S}$
3. (Unicellular relation)

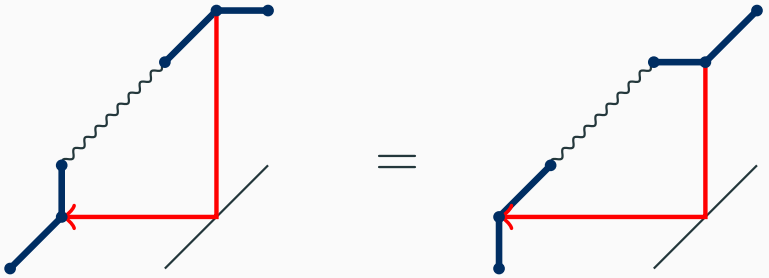


4 (Bounce relations)



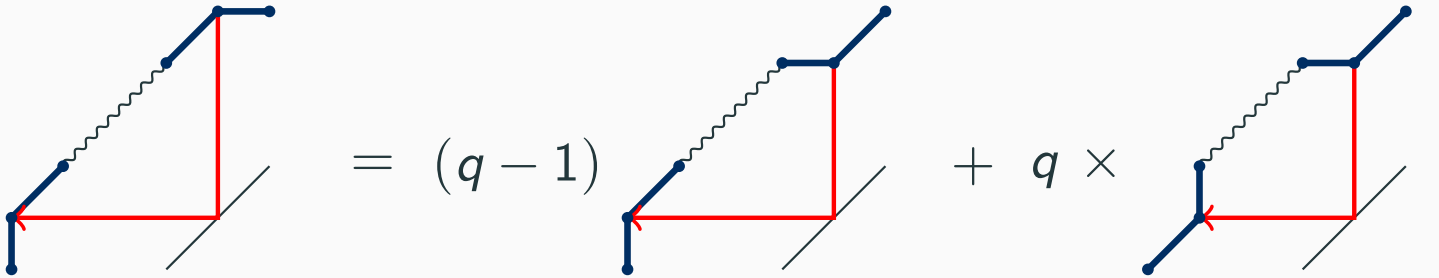
$= q \times$

$Un.nVd.eW \qquad Un.nVe.dW$



$=$

$Ud.nVd.eW \qquad Un.dVe.dW$



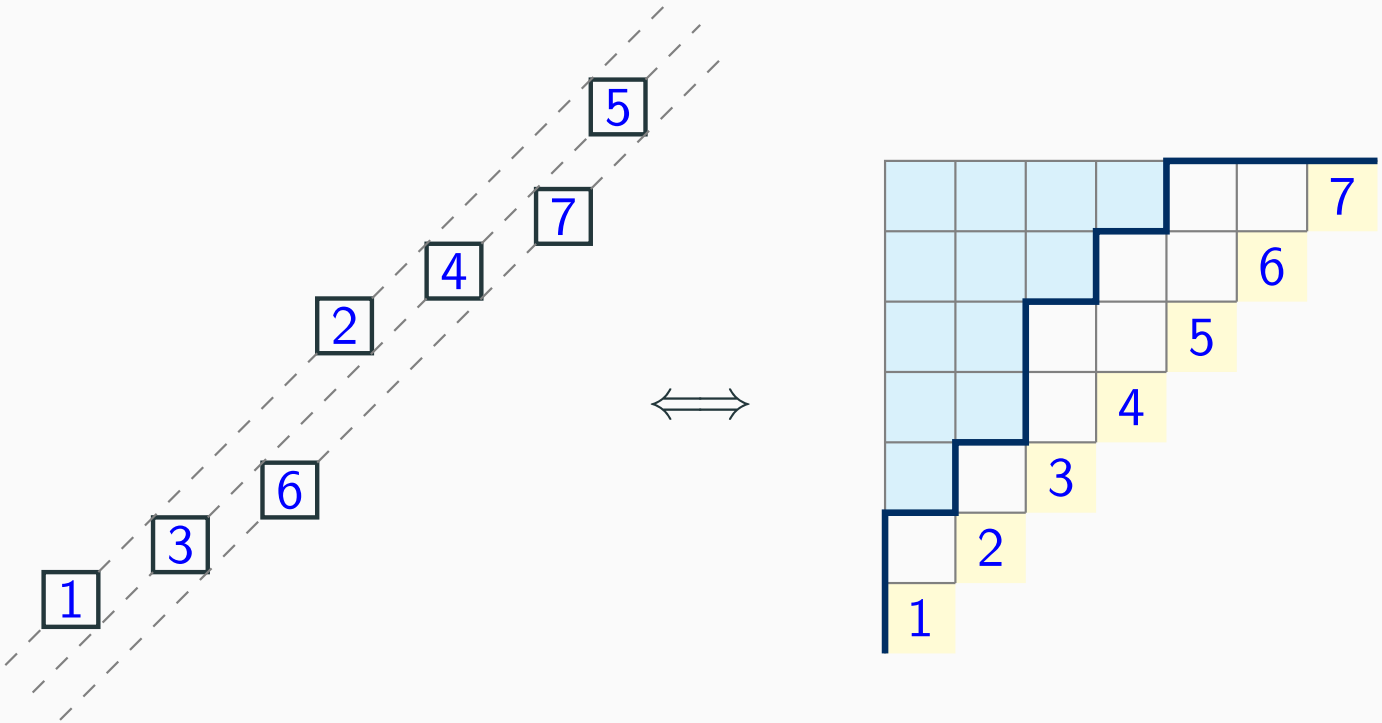
$= (q-1)$ $+ q \times$

$Un.dVd.eW \qquad Un.dVe.dW \qquad Ud.nVe.dW$

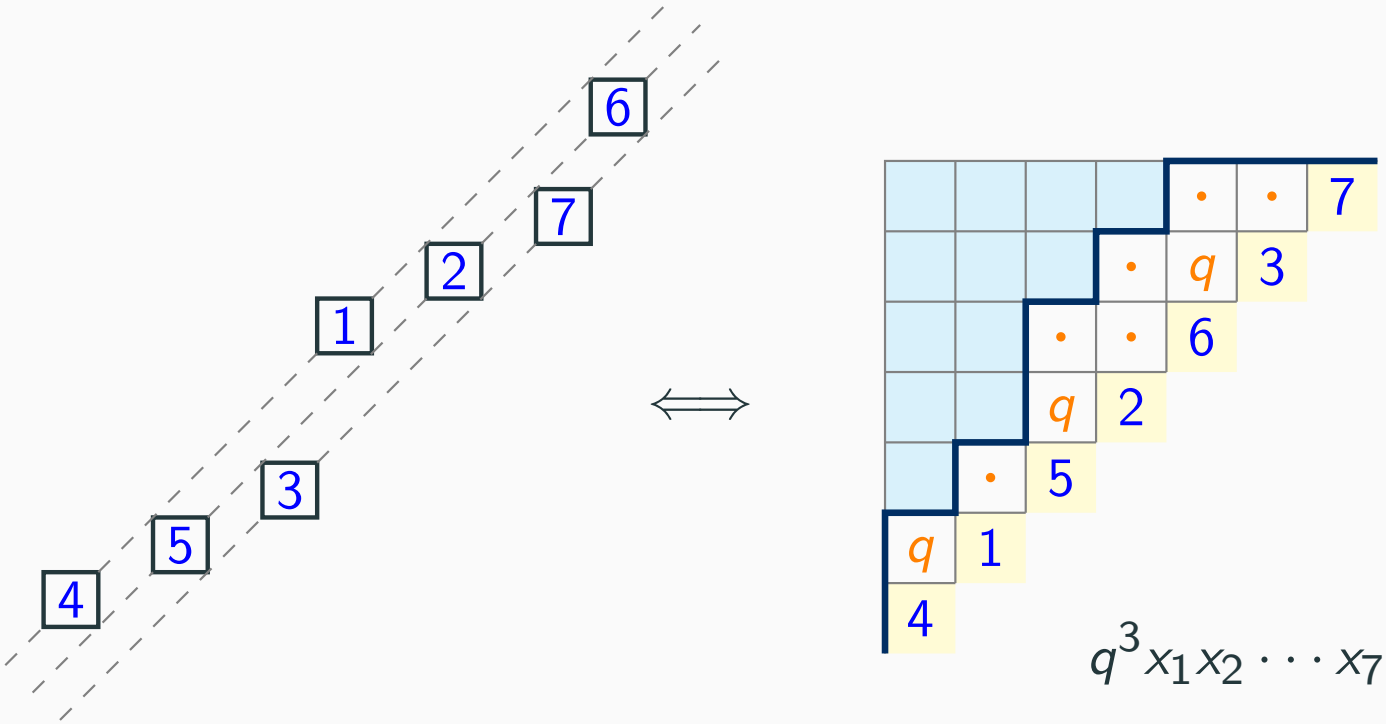
Note on vertical-strip LLT polynomials

- Vertical-strip LLT polynomials can be generated by **linear relations** [Alexandersson-Sulzgruber, '2020].
- As a result, e-positivity after the substitution $q \to q+1$ has been obtained.
- LLT polynomials indexed by certain vertical strips are **Hall-Littlewood** polynomials, after a little bit of modification.

③ Unicellular LLT polynomials



To read the inversion statistic:



Unicellular LLT's are related to chromatic quasisymmetric functions

$f(x_1, x_2, \dots)$ is **quasisymmetric** if for any $a_1, \dots, a_k \in \mathbb{Z}_+$,

$$[x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}] f = [x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}] f$$

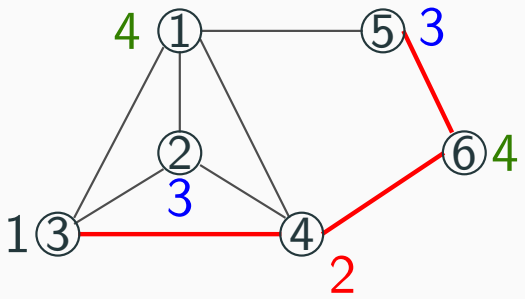
whenever $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$.

Definition. For a simple graph $G = (V, E)$, with $V \subset \mathbb{Z}_+$, the **chromatic quasisymmetric function** is

$$X_G(x; q) = \sum_{\kappa,\ \text{proper}} q^{\text{asc}(\kappa)} x^{\kappa},$$

where $\text{asc}(\kappa) = |\{\{i,j\} \in E \ : \ i < j \text{ and } \kappa(i) < \kappa(j)\}|$.

Example.



Natural unit interval order and e-positivity conjecture

For a list of positive integers $\boldsymbol{m} = (m_1, m_2, \ldots, m_{n-1})$ satisfying $m_1 \leq m_2 \leq \cdots \leq m_{n-1} \leq n$ and $m_i \geq i$, for all i,
the **natural unit interval order** $P(\boldsymbol{m})$ is the poset on $[n]$ with the order relation given by

$$i <_{P(\boldsymbol{m})} j \quad \text{if} \quad i < n \quad \text{and} \quad m_i + 1 \leq j \leq n.$$

Conjecture. [Shareshian-Wachs '2016]
If G is the incomparability graph of a natural unit interval order, then

$$X_G(x; q) \text{ is } e\text{-positive.}$$

Dyck path correspondence

In case of the natural unit interval order, the incomparability graph has a **Dyck path correspondence**.

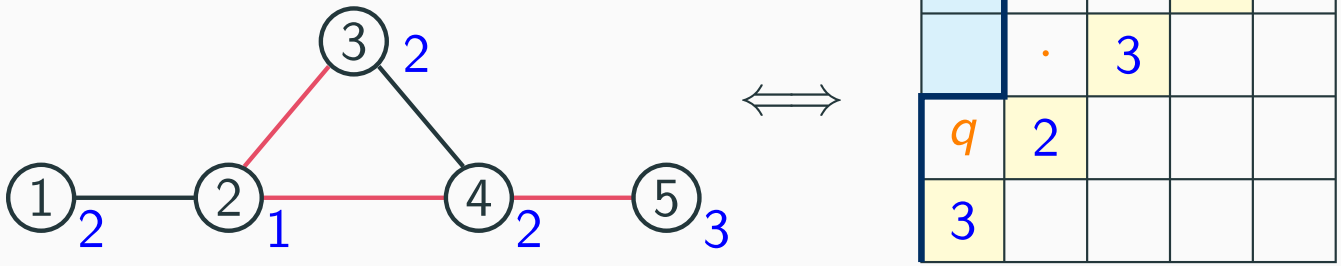
Example.



Dyck path correspondence

In case of the natural unit interval order, the incomparability graph has a **Dyck path correspondence**.

Example.



Relation via plethysm

Proposition. [Carlsson-Mellit '2017]

$$LLT_\nu[(q-1)X;q] = (q-1)^n X_{G_\nu}(x;q),$$

where the square bracket implies the plethystic substitution.

Linear relation

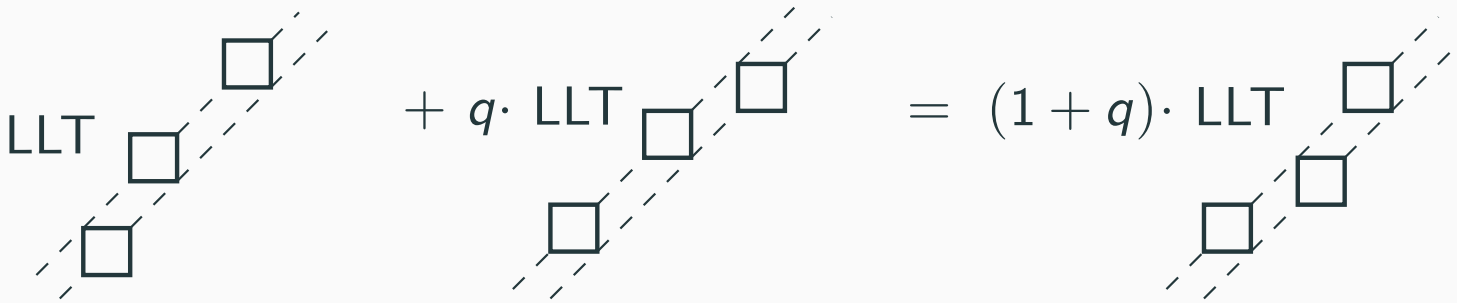
Let $G = (V, E)$ be a graph containing a triangle  with certain conditions. Then

Lemma. (q-version of triple-deletion property)

$$X_G(x;q) + qX_{G-\{e_1,e_2\}}(x;q) = (1+q)X_{G-\{e_1\}}(x;q).$$

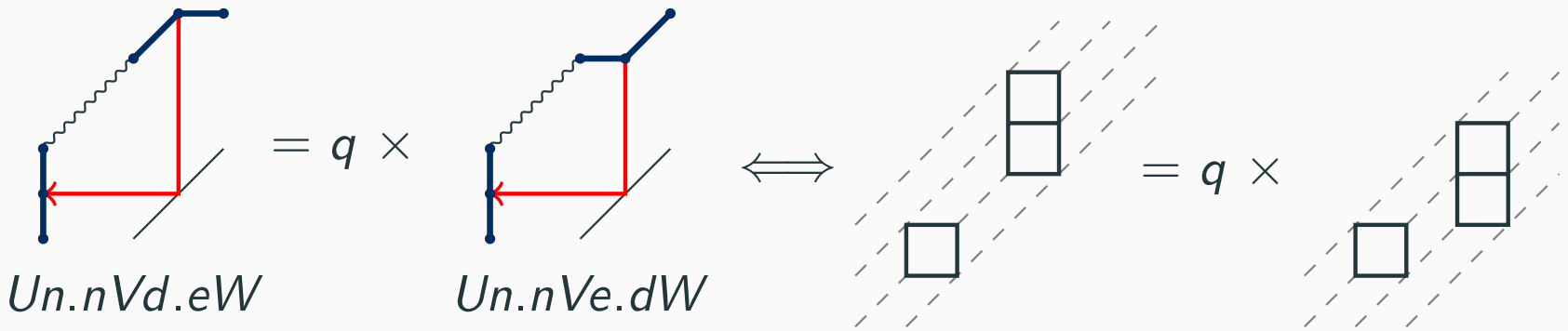
The LLT correspondence :

Lemma. [Lee, '2018$^+$] (Local linear relation)



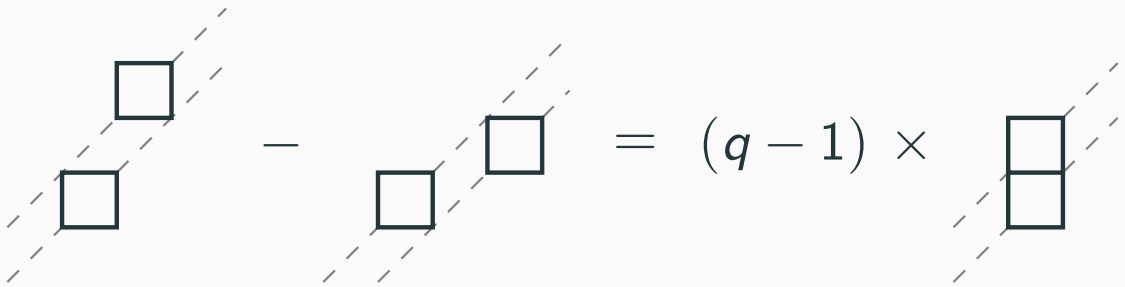
Note on the linear relation

Lee's linear relation is equivalent to the first **bounce relation**



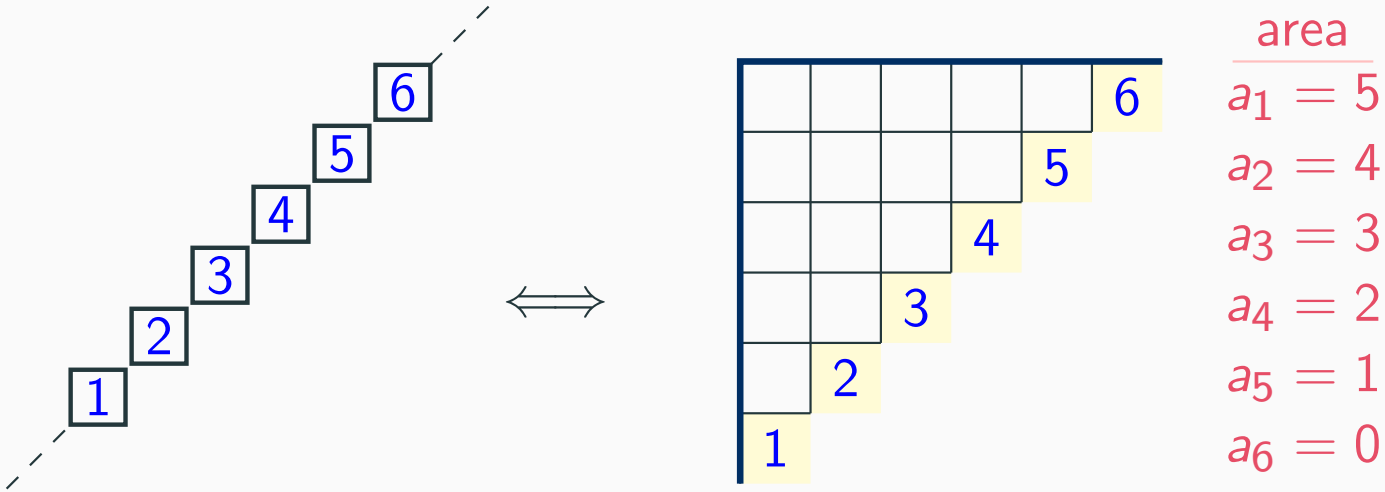
$Un.nVd.eW$ $Un.nVe.dW$

which can be proved by using the relation



Complete graph; K_n

The LLT diagram corresponding to the complete graph K_n is



$$\mathrm{LLT}_{K_n}(x;q) = \tilde{H}_{(n)}(X;q,t) = \tilde{H}_{(1^n)}(X;q)$$
$$= \sum_{\lambda \vdash n} \left(\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{cocharge}(T)} \right) s_\lambda(x)$$

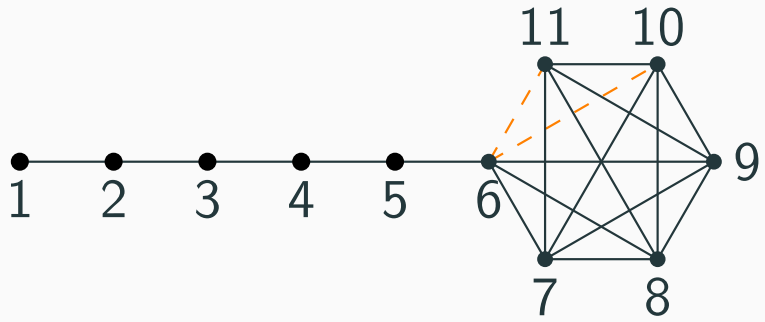
Schur positivity of LLT polynomials

Schur positivity results

- [Grojnowski-Haiman, '2006 (unpublished work)]
  Proved Schur positivity using Hecke algebras and representation theory.
- [J. Blasiak, '2016]; in the case of the tuple consists of $k \leq 3$ ribbons, **explicit** Schur expansion is given (using Kazhdan-Lusztig theory, Lam-algebra and Fomin-Greene machinery).
  Note. Thus this result proves the positivity of the q, t-Kostka polynomials $\tilde{K}_{\lambda\mu}(q, t)$ when μ has three columns.

Unicellular case

- [Huh-Nam-Y., '2020] In the case of LLT diagrams corresponding to the class of **melting lollipop graphs**, explicit combinatorial formula is given.



- [Lee, '2018] LLT polynomials with bandwidth ≤ 2 is proved to be 2-Schur positive.
- [Miller, '2019] LLT polynomials with bandwidth ≤ 3 is proved to be 3-Schur positive.

  Note. k-Schur positivity is stronger than Schur positivity.

Conjecture

Conjecture. [Leclerc-Thibon, '2000]
There is a statistic $c_a(T)$ on standard Young tableaux depending on the area sequence a such that

$$\mathrm{LLT}_a(X;q) = \sum_{\lambda \vdash n} \left(\sum_{T \in \mathrm{SYT}(\lambda)} q^{c_a(T)} \right) s_\lambda(x)$$

which satisfies

$$\sum_{\substack{\kappa \in S_n \\ \text{proper}}} q^{\mathrm{asc}_a(\kappa)} = \sum_{\lambda \vdash n} f^\lambda \sum_{T \in \mathrm{SYT}(\lambda)} q^{c_a(T)}$$

Remark

- [Alexandersson-Sulzgruber, '2018] In the unicellular case, **power-sum** expansion is known.
- [Alexandersson-Sulzgruber, '2020] Unicellular and vertical-strip case, after substituting $q \mapsto q + 1$, e-expansion is proved.
- [Novelli-Thibon, '2019] introduced a **non-commutative** lift of the **unicellular LLT polynomials** and prove that they are **positive** in a non-commutative analogue of the Gessel quasisymmetric basis, and the **relation** to chromatic quasisymmetric function via (non-commutative) plethysm.

Wrap-up

- **Definition:** the **LLT polynomial** indexed by $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(k)})$ is

$$\mathrm{LLT}_{\boldsymbol{\nu}}(x; q) = \sum_{T \in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(\boldsymbol{T})} x^{\boldsymbol{T}}.$$

- We are interested in finding **combinatorial** description for the Schur coefficients.
- When the LLT polynomials are indexed by tuples of **vertical strips** and **unicellular diagrams**, we can use corresponding **paths model** and **linear relations**.
- In the **unicellular LLT** case, the Schur positivity is closely related to the *e*-**positivity** conjecture of Shareshian and Wachs.

Wrap-up

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T	H	A	N	K
		Y	O	U