Kähler-Einstein metrics, Kähler-Ricci flows and their deformations

§1. Basics

Recall

TM, J $\downarrow$ M	$T_X = T\acute{M}$ $\downarrow$ $X \quad (X_{\mathbb{R}} = M)$
g: Riemannian (hermitian if J-inv. Kähler if $\nabla J \equiv 0$	$\omega := g(J\cdot, \cdot)$ $\quad$ (Kähler $\Leftrightarrow d\omega = 0$)
$Ric(g)$: Ricci curv. tensor	$Ric(\omega) := Ric(g)(J\cdot, \cdot)$
$Scal(g) := tr_g Ric(g)$: Scalar curv.	$Scal(\omega) := tr_\omega Ric(\omega) = n \dfrac{\omega^{n-1} \wedge Ric(\omega)}{\omega^n}$

Note: $TM \simeq T_X$, $u \longmapsto v$ $\Rightarrow$ $g(u,v) = \omega(v, \bar{v})$, $Ric(g)(u,u) = Ric(\omega)(v, \bar{v})$, $Scal(g) = Scal(\omega)$

Def.

- g : Einstein if $ric_g(u,u) := \dfrac{Ric(g)(u,u)}{g(u,u)} \equiv \lambda \in \mathbb{R}$
- $g(t)$: (normalized) Ricci flow if $\frac{d}{dt} g(t) = -Ric(g(t)) \pm \lambda \cdot g(t)$
- ω : Kähler-Einstein if $d\omega = 0$ & $Ric(\omega) = \lambda \cdot \omega$ (KE)
- $\omega(t)$: Kähler-Ricci flow if $d(\omega(t)) = 0$ & $\frac{d}{dt}\omega(t) = -Ric(\omega(t)) \pm \lambda \cdot \omega(t)$ (KRF)
- ω is constant scalar curv. Kähler if $d\omega = 0$ & $Scal(\omega) \equiv \lambda$ (cscK)
- $c_1(X) := c_1(T_X) = -c_1(K_X) = \frac{1}{2\pi}[Ric(\omega)] \in H^{1,1}(X, \mathbb{R})$: first Chern class (indep. of ω)

$K_X := \Lambda^n T_X^*$: canonical line bundle

§2. Kähler-Einstein metric & Kähler-Ricci flow

Suppose X : compact Kähler

$\exists\, \omega_{KE}$ on $X \Rightarrow C_1(X) = \frac{1}{2\pi}[Ric(\omega_{KE})] = \frac{\lambda}{2\pi}[\omega_{KE}]$ $\begin{cases} <0 & \text{general type} \\ =0 & \text{Calabi-Yau} \\ >0 & \text{Fano} \end{cases}$

Thm · (Aubin/Yau) $C_1(X) < 0 \Rightarrow \exists!\, \omega_{KE}$ s.t. $Ric(\omega_{KE}) = -\omega_{KE}$

· (Yau) $C_1(X) = 0 \Rightarrow \exists!\, \omega_{KE} \in [\omega]$ s.t. $Ric(\omega_{KE}) = 0$

Fix $\omega_0 \in -2\pi \cdot C_1(X) \underset{\partial\bar\partial}{\Rightarrow} Ric(\omega_0) = -\omega_0 + i\partial\bar\partial f$, $f \in C^\infty(X,\mathbb{R})$

f: Ricci potential

ω: KE $\Leftrightarrow Ric(\omega) = -\omega \Rightarrow [\omega] = [\omega_0] \underset{\partial\bar\partial}{\Rightarrow} \omega = \omega_0 + i\partial\bar\partial\varphi$, $\varphi \in C^\infty(X,\mathbb{R})$

$$Ric(\omega) = -\omega \underset{X:cpt}{\Longleftrightarrow} (\omega_0 + i\partial\bar\partial\varphi)^n = e^{\varphi + f}\omega_0^n$$

$$\omega_0 = i\, g_{j\bar k}\, dz^j \wedge d\bar z^k \Longleftrightarrow \det\left[g_{j\bar k} + \frac{\partial^2}{\partial z^j \partial \bar z^k}\varphi\right] = e^{\varphi + f}\det[g_{j\bar k}]$$

cpx Monge-Ampere eqn (nonlinear elliptic)

Methods ① Continuity method : $Ric(\omega_t) = -t\cdot\omega_t + (1-t)\,Ric(\omega_0)$ $t \in [0,1]$

$$\Longleftrightarrow \det\left[g_{j\bar k} + \frac{\partial^2}{\partial z^j \partial \bar z^k}\varphi\right] = e^{\varphi + t\cdot f}\det[g_{j\bar k}]$$

② Kähler-Ricci flow : $\begin{cases} \frac{d}{dt}\omega_t = -Ric(\omega_t) - \omega_t \\ \omega_t|_{t=0} = \omega_0 \end{cases}$

$$\Longleftrightarrow \det\left[g_{j\bar k} + \frac{\partial^2}{\partial z^j \partial \bar z^k}\varphi\right] = e^{\dot\varphi_t + \varphi + f}\det[g_{j\bar k}]$$

parabolic MA

§3. YTD conjecture

Suppose X: Fano ($\Leftrightarrow C_1(X) > 0$)

- **Thm (Matsushima 1957)** $\exists$ KE ($\underset{(cscK)}{}$) $\Rightarrow$ $\mathrm{Lie}(\mathrm{Aut}(X))$: reductive

  e.g.) $Bl_p(\mathbb{CP}^2)$ no KE

- **Thm (Futaki 1983)** $\exists$ KE $\Rightarrow$ $\mathrm{Fut}_{C_1(X)} : \mathrm{Lie}(\mathrm{Aut}(X)) \longrightarrow \mathbb{C}$ vanish

  Def) $\mathrm{Fut}_{C_1(X)}(v) := \int_X v(f)\, \omega^n$ where $\begin{cases} \omega \in C_1(X) \\ \mathrm{Ric}\,\omega - \omega = i\partial\bar{\partial} f \end{cases}$

  Rmk) well-defined on projective normal variety

- **Thm (Tian 1990)** $n=2$ $\exists$ KE $\Leftrightarrow$ reductivity $\Leftrightarrow$ $\mathrm{Fut} \equiv 0$

Consider (X, L) : polarized mfd ($\Leftrightarrow$ cpt Kähler mfd X & $C_1(L) > 0$)

Notes:
- (Kodaira's embedding) X: projective
- $L = K_X^{-1} \Rightarrow X$: Fano

Yau-Tian-Donaldson conjecture. (X, L) : polarized

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$\exists$ cscK $\omega \in C_1(L) \Leftrightarrow$ K-stable.

- Thm (Donaldson '05 / Stoppa '09) "$\Rightarrow$" for $\mathrm{Aut}(X)$: discrete
- Thm (Zhou-Zhu '08 / Donaldson '09) "$\Leftrightarrow$" for toric polarized & $n=2$
- Thm (Chen-Donaldson-Sun / Tian '14) "$\Leftrightarrow$" for Fano ($L = K_X^{-1}$)

Note: cscK $\omega \in C_1(K_X^{-1}) \Leftrightarrow \omega$: KE (∵ $\partial\bar{\partial}$-lemma & max. prin.)

§4. K-Stability

Suppose (X, L): polarized mfd

Def. ① Test configuration of (X,L) is $\mathcal{L} \to \mathcal{X} \xrightarrow{\pi} \mathbb{C}$ with equiv. $\mathbb{C}^*$-action s.t.

- $X_t := \pi^{-1}(t) \cong X$ for $\forall t \neq 0$
- $\mathcal{L}|_{X_t} \cong L$ ample
- $\mathcal{X}$ is normal var. (Li-Xu '14)
- X_0 : normal (special test conf.) Tian

(equivalently, $X \hookrightarrow \mathbb{CP}^N$ & $\lambda: \mathbb{C}^* \hookrightarrow GL(N+1, \mathbb{C})$

$$\mathcal{X} = \overline{\{(\underset{\| }{X_t}, t)\}} \subset \mathbb{CP}^N \times \mathbb{C}$$
$$\lambda(t)\cdot X$$

② Test configuration is trivial if $\mathcal{X} \cong X \times \mathbb{C}$

③ (X, L) is K-semistable if $Fut_{G_1(L)}(\mathcal{X}) := Fut_{G_1(L)}(X_0) \geq 0$ for $\forall \mathcal{X}$

④ (X, L) is K-stable if $Fut_{G_1(L)}(\mathcal{X}) > 0$ for $\forall \mathcal{X} \not\cong X \times \mathbb{C}$

(Motivation) Fix $\omega_0 \in C_1(L)$

$$\mathcal{K} := \{ \varphi \in C^\infty(X, \mathbb{R}) \; ; \; \omega = \omega_0 + i\partial\bar{\partial}\varphi > 0 \}$$

$$M(\varphi) := \int_0^1 \int_X \dot{\varphi}_t (\bar{S} - S_t) \omega_t^n \, dt \quad , \quad \omega_t = \omega_0 + \varphi_t, \; \varphi_t|_0 = 0, \; \varphi_t|_t = \varphi$$

(Mabuchi functional / K-energy)

Critical pt of $M \Leftrightarrow$ cscK in $C_1(L)$

Informaly "pt at ∞" in $\mathcal{K}$ $\longleftrightarrow$ X_0 in $\mathcal{X}$

§5. Manifolds with Large Symmetry

Def X is toric if $(\mathbb{C}^*)^n \curvearrowright X$ & $X_0 := (\mathbb{C}^*)^n \subset X$ (dense)

Thm (Wang–Zhu '04) X: toric Fano $\Rightarrow$ $\exists$ Kähler–Ricci soliton

$\exists$ KE $\Leftrightarrow$ Fut$(X) = 0$ $\Leftrightarrow$ bar$(P) = 0$

(Fut $\Leftrightarrow$ bar: Nakabuchi; P: moment polytope)

$(\mathbb{C}^*)^n \xrightarrow{\text{Log}} \mathbb{R}^n \longrightarrow P \subset \mathbb{R}^n$

{psh fn} $\longleftrightarrow$ {cvx fn}

cpx MA $\longleftrightarrow$ real MA

μ: moment map

$\int_X \omega^n \longmapsto \int_P$

Generalization: G: reductive Lie gp $\supset T = (\mathbb{C}^*)^r$: maximal torus

$B \subset G \curvearrowright X$ & $\exists$ open & dense B-orbit in X

{toric} $\subset$ {group compactification} $\subset$ {symm. var.} $\subset$ {spherical}

Thm (Delcroix '17) X: smooth Fano gp comp.

$\exists$ KE $\Longleftrightarrow$ $\text{bar}_{DH}(P^t) \in 2\rho + \Xi$

Thm (Li–Tian–Zhu '20) generalize to $\mathbb{Q}$-Fano var. gp. comp.

Thm (Delcroix '20) generalize to smooth symm. var.

Thm (Lee–Park–Y.) X: All smooth Fano symm w/ $\text{Pic}\# = 1$ $\Rightarrow$ $\exists$ KE

References: Szekelyhidi – An introduction to extremal Kähler metrics

Donaldson – Stability of alg. var & Kähler geometry

Phong & Sturm – Lectures on Stability & constant scalar curv.