제 15 회 고등과학원 기하학 겨울학교

2019년 12월 26일(목) ~ 12월 30일(월)

imagine the impossible

KIAS KOREA INSTITUTE FOR ADVANCED STUDY

제15회 고등과학원 기하학 겨울학교

2019년 12월 26일(목) - 12월 30일(월) 하이원리조트 힐콘도

	26일 (목)	27일 (금)	28일 (토)	29일 (일)		30일 (월)
09:00-09:50		표준철	홍한솔	김현규	09:00-09:40	최학호
09:50-10:05		Break			09:40-09:55	Break
10:05-10:55		서검교	서검교	김현규	09:55-10:35	한지영
10:55-11:10		Break			10:35-10:50	Break
11:10-12:00		표준철	표준철	안병희	10:50-11:30	서동휘
12:00-13:30		점심				
13:30-15:30		토론				
15:30-16:20	이재혁					
16:20-16:40	Break					
16:40-17:30	김영주					
17:30-19:00	저녁					
19:00-19:50	김영주	박경동		안병희		
19:50-20:00	Break			Break		
20:00-20:50	서검교	홍한솔	기획회의	박경동		

서검교 **숙명여자대학교** (1,3,5 강의)

표준철 **부산대학교** (2,4,6 강의)

극소곡면론 소개

유클리드 공간에 놓여있는 곡면들 중 평균곡률이 영인 곡면을 극소곡면이라고 한다. 공간에 놓인 Jordan 곡선에 대해 극소곡면(disk-type)이 항상 존재하고, catenoid, helicoid 등 경계가 없는 완비된 극소곡면들도 존재한다. 유클리드 공간에 풍부한 예들이 있고, 주어진 경계에 대해 넓이가 최소가 된다는 성질로 부터 흥미로운 성질들이 유도된다. 수학의 여러 분야의 이론을 통하여 극소곡면의 구성방법과 중요한 성질을 소개하고자 한다.

목차

1. Definition of minimal surfaces, 1st variational formula
2. Examples and finding properties of minimal surfaces
3. minimal surfaces theory based on PDEs (maximum principle, convex hull property, harmonicity of the coordinate functions, Bernstein theorem)
4. minimal surface theory based on complex analysis (Enneper-Weierstrass representation theorem, reflection principles)
5. 2nd variational formula and stability
6. minimal surfaces in the 3-dim sphere

이재혁 **이화여자대학교**

삼차원 공간의 곡면에 대한 모스함수의 풍부성

모스 함수의 중요성은 기하학과 위상수학의 다양한 분야에서 나타난다. 우리는 모스함수의 기본적인 성질을 소개하고 다양체의 위상에 대한 활용을 살펴본다. 그리고 유클리드 공간의 부분다양체에 대한 모스함수의 풍부성에 대한 결론을 삼차원 곡면에 관한 미분기하학에서 사용하는 용어로 소개한다. 이 강연은 곡선과 곡면 관한 미분기하학과 Milnor의 Morse theory의 제1장을 배경으로 한다.

김영주 **건국대학교**

쌍곡기하 산책

본 강연에서는 쌍곡기하공간의 정의와 모델, 등장사상 등의 기초적인 정의에서 시작하여 쌍곡 구조를 갖는 다양체가 갖는 특징을 살펴본다.

박경동 **IBS-CGP**

Holonomy groups and hyperkaehler manifolds in homogeneous varieties

The holonomy group of the Levi-Civita connection on a Riemannian manifold is a Lie group given by parallel transport maps along loops based at a fixed point, and is a global invariant which measures constant tensors on the manifold. Interesting in various aspects are Riemannian manifolds with special holonomy groups, which include Calabi-Yau, hyperkaehler, and quaternionic Kaehler manifolds. However, it is extremely hard to construct manifolds with a specific holonomy. For instance, there are no known examples of compact quaternionic Kaehler manifold that are not symmetric spaces. I will explain Berger's classification of holonomy groups and explore attractive problems involved in special holonomy groups. The latter half of my talk will be devoted to hyperkaehler manifolds given by zero loci of global sections of equivariant vector bundles on homogeneous varieties.

홍한솔 연세대학교

Toric mirror symmetry and beyond

Mirror symmetry of a toric manifold is now a well-studied subject, in which one can see explicit constructions of the mirror space and mirror functors. I will give a short survey on the toric mirror symmetry, mainly using simple examples, and discuss more recent progresses if time permits.

김현규 이화여자대학교

Goldman symplectic structure on space of surface group representations, and skein algebra

The moduli space of group homomorphisms from the fundamental group of a surface into a Lie group G appears in many areas of mathematics and physics, like Teichmuller theory and gauge theory. In 1980's Goldman constructed a natural symplectic structure on this space, which I will review in this talk. On the other hand, I will review the skein algebra of a surface, which appeared as an approach to link invariants in 3d spaces. I will explain how Turaev showed that the skein algebra can be viewed as a quantization of the above symplectic moduli space, in case when G is SL(2,C). If time allows, I will discuss recent research problems.

안병희 IBS-CGP

Legendrians and constructible sheaves

This series of two lectures will introduce the microlocal study of constructible sheaves and their interactions with Legendrian graphs. More precisely, we will briefly review microlocal sheaf theory firstly introduced by Kashiwara and Schapira in 1980's, and define an invariant for Legendrian graphs by using the microlocal sheaf theory which is related with several questions in symplectic geometry.

최학호 고등과학원

On symplectic fillings of small Seifert 3-manifolds

One of the fundamental problems in symplectic 4-manifold topology is in classifying symplectic fillings of certain 3-manifolds equipped with a natural contact structure. If we get the classification result, then it is natural to ask that Is there any surgery description of those fillings.

In this talk, we discuss classification of minimal symplectic fillings of small Seifert 3-manifolds satisfying certain conditions. Furthermore, we demonstrate that every minimal symplectic filling of small Seifert 3-manifolds satisfying the conditions can be obtained by a sequence of rational blowdowns from the minimal resolution. This is joint work with Jongil Park.

한지영 서울대학교

Homogeneous dynamics and application to the lattice counting problem

Homogeneous dynamics is to study dynamical properties-e.g. ergodicity, equidistribution property-of orbits of a flow defined on the quotient space of G/Γ, where G is a Lie gorup and Γ is a discrete subgroup, for which $Vol(G/\Gamma)$ is finite. The most famous example is $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$, which can be identified with the space of unit-covolume lattices of $\mathbb{R}^n$. Using this relation, we can apply homogeneous dynamics to a problems in the geometry of numbers.

서동휘 KAIST

Shape optimization problems for Steklov eigenvalue

1954년에 Weinstock은 주어진 둘레를 가진 planar simply connected set중 first Steklov eigenvalue가 최대가 되는 경우는 disk임을 보였다. 이 후에 관련 결과들이 다양한 기하학적 방법론으로 증명되었다. 이 강연에서는 최근 연구되고 있는 반지름이 주어진 두 개의 ball로 bound되는 annular domain에서의 first eigenvalue에 대한 shape optimization problem들을 소개하고 이 결과들이 two-point homogeneous space로까지 확장될 수 있음을 보이려고 한다.

Introduction to minimal submanifold theory *

Keomkyo Seo

1 History

- Euler(1744), catenoid (also Lagrange 1760)
- Meusnier(1776), helicoid
- 1830' - 1890': Enneper, Plateau, Riemann, Scherk, Schwarz, Weierstrass...
- 1900' - 1950': Bernstein, Courant, Douglas, Morrey, Morse, Radó, Schiffman...
- 1960' - 1990': Allard, Almgren, Calabi, do Carmo, Chern, Costa, Federer, Finn, Fleming, Gackstatter, Gulliver, Hildebrandt, Hoffman, Houdt, Jenkins, Lawson, Meeks, Nitsche, Osserman, Perez, Ros, Rosenberg, Schoen, Serrin, Simon, Simons, White, Yau...
- 2000' - : Brendle, Colding-Minicozzi, Marques-Neves...

2 Introduction and definition of minimal submanifolds

Setting: $\Sigma^k \subset (M^n, g)$

$$A(X,Y) = (\nabla_X Y)^\perp, \quad \text{Second fundamental form}, \quad X, Y \in T\Sigma$$

Definition 2.1. *$\{e_i\}$: orthonormal basis of $T_p\Sigma$.*

$$\overrightarrow{H} = \mathrm{Tr}_g A = \sum g^{ij} A(e_i, e_j) = (\overline{\nabla}_{e_i} e_i)^\perp, \ \underline{\text{mean}} \ \underline{\text{curvature}}$$

Definition 2.2. *Σ: minimal $\iff$ $H \equiv 0$.*

$\star$ Graph $\Sigma^{n-1} \subset \mathbb{R}^n$

$$\Sigma = \Sigma_u = \{\underbrace{(x, u(x))}_{=F(x)=F(x_1,\cdots,x_{n-1})} : x \in \Omega \subset \mathbb{R}^n\}$$

$$\Rightarrow g_{ij} = \langle F_i, F_j \rangle = \delta_{ij} + u_i u_j$$

and [the unit normal of Σ_u] =

$$\nu = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}$$

*This note is based on lectures in 15th KIAS winter school during December 26-30, 2019. I would like to thank Jihye Lee for typing the manuscript.

($g^{ij} = \delta_{ij} - \nu_i \nu_j$ where $\nu_i = \frac{u_i}{\sqrt{1+|\nabla u|^2}}$, $i = 1, 2, \cdots, n-1$)

$$\therefore \mathrm{Vol}(\Sigma_u) = \int_\Omega \sqrt{1+|\nabla u|^2} dx \left(= \int_\Omega \sqrt{\det(g_{ij})} dx \right)$$

∘ Euler-Lagrange equation for $\mathrm{Vol}(\Sigma_u)$ ($\forall \eta \in C_c^\infty(\Omega)$)
Suppose u is a critical point of $\mathrm{Vol}(\Sigma_u)$.

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(\Sigma_{u+t\eta}) = \int_\Omega \left.\frac{d}{dt}\right|_{t=0} \sqrt{1+|\nabla u + t\nabla\eta|^2} dx \\ &= \int_\Omega \frac{\nabla u \cdot \nabla \eta}{\sqrt{1+|\nabla u|^2}} dx \\ &= -\int_\Omega \frac{d}{dx_i}\left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right)\eta dx \end{aligned}$$

$$\Rightarrow \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i}\left(\frac{u_i}{\sqrt{1+|\nabla u|^2}}\right) = 0 \quad \text{(MSE)}$$

$$\begin{aligned} \text{Expansion: } & \frac{u_{ii}}{\sqrt{1+|\nabla u|^2}} - \frac{u_i u_j u_{ij}}{(1+|\nabla u|^2)^{3/2}} = 0 \\ \iff & \frac{1}{\sqrt{1+|\nabla u|^2}}\left(\delta_{ij} - \frac{u_i}{\sqrt{1+|\nabla u|^2}}\frac{u_j}{\sqrt{1+|\nabla u|^2}}\right) u_{ij} = 0 \\ \iff & \frac{1}{\sqrt{1+|\nabla u|^2}} \sum_{i,j=1}^{n-1} g^{ij} u_{ij} = 0 \end{aligned}$$

On the other hand,

$$A\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}\right)^\perp = (F_{ij})^\perp = \langle F_{ij}, \nu \rangle \nu = h_{ij}\nu,$$

where $h_{ij} = \langle F_{ij}, \nu \rangle$.

$$F_{ij} = (0, \cdots, 0, u_{ij}) \Rightarrow h_{ij} = \frac{u_{ij}}{\sqrt{1+|\nabla u|^2}}$$

$$H = \mathrm{Tr}_g A = \sum g^{ij} A\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{1}{\sqrt{1+(\nabla u)^2}} \sum g^{ij} u_{ij} = 0$$

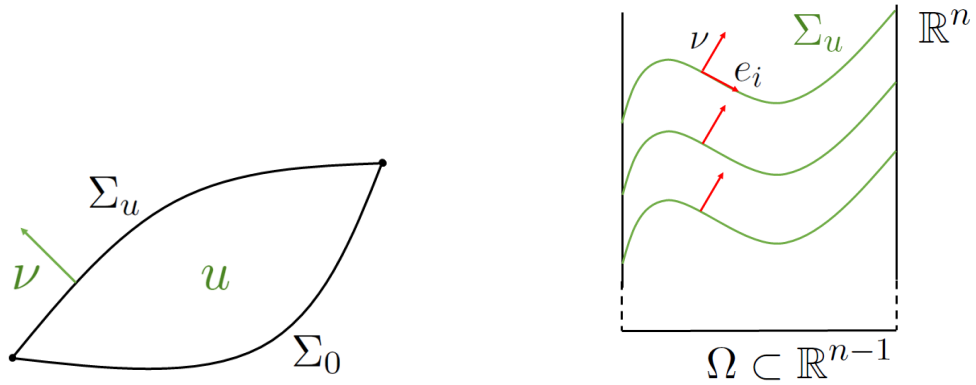
When $n = 2$,

$$(1+u_2^2)u_{11} - 2u_1 u_2 u_{12} + (1+u_1^2)u_{22} = 0 : \quad \text{quasi-linear and elliptic.}$$

Proposition 2.3. *$\Sigma_u^{n-1} \subset \mathbb{R}^n$, minimal graph over $\Omega \Rightarrow \Sigma_u$: volume-minimizing*

Proof. $\Sigma_u \cup (-\Sigma_0) = \partial u$. $\Sigma_u \sim \Sigma_0$, homologous.

Foliation of minimal graphs ⇒ Extend ν in $\mathbb{R}^n$ in a natural way.
dx = the volume form in $\mathbb{R}^n$.
$\omega := \nu \lrcorner dx$ = volume form on Σ_u



(a) (b)

$\Rightarrow d\omega = \mathrm{div}\nu dx = 0$
Note: $\mathrm{div}\nu = \sum \underbrace{\langle \nabla_{e_i}\nu, e_i\rangle}_{=0\ (\because H\equiv 0)} + \langle \nabla_\nu \nu, \nu\rangle = 0$

$$\therefore 0 = \int_u d\omega = \int_{\partial u} \omega = \int_{\Sigma_u} \omega - \int_{\Sigma_0} \omega = \int_{p\in\Sigma_u} \omega(T_p\Sigma_u) - \int_{q\in\Sigma_0} \omega(T_q\Sigma_0)$$

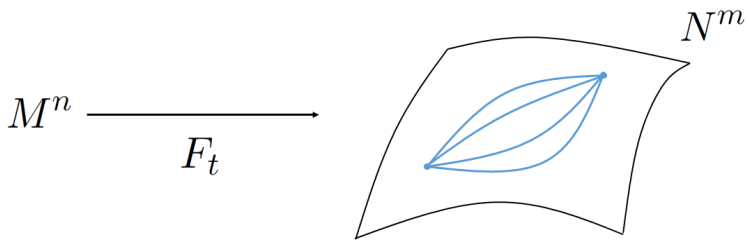
Recall $\omega(V) = \omega(e_1, \cdots, e_{n-1})$ for V^{n-1}

$$\begin{aligned} &\Rightarrow \omega(T_p\Sigma_u) = \omega(e_1,\cdots,e_{n-1}) = \nu \lrcorner dx(e_1,\cdots,e_{n-1}) = dx(e_1,\cdots,e_{n-1},\nu) = 1 \\ &\&\quad \omega(T_q\Sigma_0) = \omega(e_1,\cdots,e_{n-1}) = \nu\lrcorner dx(e_1,\cdots,e_{n-1}) = dx(e_1,\cdots,e_{n-1},\nu) \\ &\qquad = \det(e_1,\cdots,e_{n-1},\nu) = \det(e_1,\cdots,e_{n-1},\nu^\perp) \le 1 \end{aligned}$$

$$\mathrm{Vol}(\Sigma_u) = \int_{\Sigma_u} \omega = \int_{\Sigma_0} \omega \le \int_{\Sigma_0} 1 = \mathrm{Vol}(\Sigma_0)$$

□

3 1st variation formula



$F_t : M^n \times (-\varepsilon, \varepsilon) \to N^m$, $F_0 = F$ (initial immersion)

$$X(p) = \left.\frac{d}{dt}\right|_{t=0} F_t(p), \quad \text{variational vector field.}$$

Theorem 3.1.

$$\left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(F_t M) \overset{(1)}{=} \int_M \mathrm{div}_M(X) dV \overset{(2)}{=} \int_{\partial M} \langle X, \nu \rangle - \int_M \langle H, X \rangle$$

Recall $\mathrm{div}_M(X) = \sum \langle e_i, \nabla_{e_i} X \rangle$, $H = (\overline{\nabla}_{e_i} e_i)^\perp$

Proof. For (1), note

$$\mathrm{Vol}(F_t M) = \int_M J_n(DF_t) = \int_M \sqrt{\det(DF_t^\top \circ DF_t)}$$

$$\therefore \left.\frac{d}{dt}\right|_{t=0} \mathrm{Vol}(F_t M) = \int_M \left.\frac{d}{dt}\right|_{t=0} J_n(DF_t)$$

For simplicity, assume $N^m = \mathbb{R}^m$

$$F_t(p) = p + tX(p) + o(t)$$

$$\therefore DF_t(e_i) = \nabla_{e_i} F_t + \approx \nabla_{e_i}(p + tX(p)) = e_i + t\nabla_{e_i} X$$

$$\begin{aligned} \therefore J_n(DF_t) &= \sqrt{\det(DF_t(e_i) \cdot DF_t(e_j))} \\ &\cong \sqrt{\det(e_i + t\nabla_{e_i} X) \cdot (e_j + t\nabla_{e_j} X)} \\ &\cong \sqrt{\det(\delta_{ij} + t(e_i \cdot \nabla_{e_j} X + e_j \cdot \nabla_{e_i} X)} \end{aligned}$$

Recall: $\det(I + tA) = 1 + t \cdot \mathrm{tr} A + o(t)$

$$\therefore J_n(DF_t) \cong \sqrt{1 + 2t \sum e_i \cdot \nabla_{e_i} X} = \sqrt{1 + 2t\, \mathrm{div}_M X} \cong 1 + t\, \mathrm{div}_M X^\perp$$

For (2),

$$\begin{aligned} \int_M \mathrm{div}_M X &= \int_M \mathrm{div}_M(X^\top) + \int_M \mathrm{div}_M(X^\perp) = \int_{\partial M} \left\langle X^\top, \nu \right\rangle + \int_M \mathrm{div}_M X^\perp \\ &= \int_{\partial M} \langle X, \nu \rangle + \int_M \mathrm{div}_M X^\perp, \end{aligned}$$

where ν is the outward unit conormal vector. Here

$$\mathrm{div}_M X^\perp = \left\langle e_i, \nabla_{e_i} X^\perp \right\rangle = \nabla_{e_i} \underbrace{\boxed{\left\langle e_i, X^\perp \right\rangle}}_{=0} - \left\langle \nabla_{e_i} e_i, X^\perp \right\rangle = - \left\langle (\nabla_{e_i} e_i)^\perp, X \right\rangle = - \langle H, X \rangle$$

□

Corollary 3.2. *$M^n \subset \mathbb{R}^m$, minimal $\partial M \neq \emptyset \Rightarrow n \cdot \mathrm{Vol}(M) = \int_{\partial M} \langle X, \nu \rangle$*

Proof. Apply first variation formula to $X(x) = x$. □

4 Convex hull property ($\Leftarrow$ harmonicity of coordinate functions)

Proposition 4.1. *$\Sigma^k \subset \mathbb{R}^n$, x_i: coordinates in $\mathbb{R}^n$.*

$$H \equiv 0 \iff \Delta_\Sigma x_i = 0 \quad \forall i$$

Proof. $\{e_1, \cdots, e_k\}$: orthonormal basis for $T\Sigma$. Let $X : \Sigma \hookrightarrow \mathbb{R}^n$ be an immersion.

$$\Delta_\Sigma x = \sum_{j=1}^{k} \left(e_j e_j x - (\nabla^\Sigma_{e_j}) x \right) = \sum_{j=1}^{k} (\nabla^\perp_{e_j} e_j) = H$$

□

Corollary 4.2. *$\nexists$ closed minimal submanifold in $\mathbb{R}^n$.*

Corollary 4.3. *$\Sigma^k \subset \mathbb{R}^n$, compact minimal $\Rightarrow \Sigma \subset C(\partial\Sigma) = \cap\{H : \text{closed half spaces with } \partial\Sigma \subset H\}$, where $C(\partial\Sigma)$ is a convex hull of $\partial\Sigma$.*

Proof. Let $H = \{x : l(x) \le a\}$, l: linear function ($l = \sum a_i x_i$)

$$\begin{aligned} \partial\Sigma \subset H &\Rightarrow l(x) \le a \quad \forall x \in \partial\Sigma \quad \& \quad \Delta_\Sigma l(x) = 0 \quad \forall x \in \Sigma \\ &\Rightarrow l(x) \le a \quad \forall x \in \Sigma \quad \text{(by maximum principle)} \\ &\Rightarrow \Sigma \subset H \end{aligned}$$

□

5 Monotonicity formula

Lemma 5.1. *$h : \Sigma \to \mathbb{R}_+$, Lipschitz*

$$\int_{\{h \le t\}} f |\nabla^\Sigma h| = \int_{-\infty}^{t} \int_{\{h=t\}} f \, d\sigma$$

Remark 5.2. *When t is a regular value,*

$$dv_\Sigma = \frac{dt \wedge dv_{\{h=t\}}}{|\nabla^\Sigma h|}$$

Theorem 5.3. *$\Sigma^k \subset \mathbb{R}^n$, minimal*

$$\Rightarrow \frac{d}{dt} \frac{|\Sigma \cap B_r(x_0)|}{r^k} \ge 0.$$

In fact,

$$\frac{|B_t(x_0) \cap \Sigma|}{t^k} - \frac{|B_s(x_0) \cap \Sigma|}{s^k} = \int_{\Sigma \cap (B_t(x_0) \setminus B_s(x_0))} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^{k+2}} dv_\Sigma$$

Proof. Take $h(x) := |x - x_0|$. Then $\{h \le t\} = B_t(x_0)$. Let $X = x - x_0$. Then

$$\mathrm{div}_\Sigma(X) = \langle e_i, \nabla_{e_i} X \rangle = k,$$

where $\{e_i\}$ is an orthonormal basis on Σ. By the first variation formula,

$$\left.\frac{d}{dt}\right|_{t=0} |\Sigma \cap B_r(x_0)| = \int_{\Sigma \cap B_r(x_0)} \mathrm{div}_\Sigma(X) = \int_{\Sigma \cap \partial B_r(x_0)} \langle X, \nu \rangle , \tag{5.1}$$

where

$$\nu = \frac{\nabla^\Sigma |x - x_0|}{|\nabla^\Sigma |x - x_0||} = \frac{(x - x_0)^\top}{|(x - x_0)^\top|}.$$

$$\begin{aligned}
\therefore k|\Sigma \cap B_r(x_0)| &\overset{(5.1)}{=} \int_{\Sigma \cap \{|x-x_0|=r\}} |(x - x_0)^\top| = \frac{d}{dr} \int_{\Sigma \cap B_r(x_0)} |(x - x_0)^\top| |\nabla_\Sigma |x - x_0|| \\
&= \frac{d}{dr} \int_{\Sigma \cap B_r(x_0)} \frac{|(x - x_0)^\top|^2}{|x - x_0|} = r \frac{d}{dr} \int_{\Sigma \cap B_r(x_0)} \frac{|(x - x_0)^\top|^2}{|x - x_0|^2} \\
&= r \frac{d}{dr} \int_{\Sigma \cap B_r(x_0)} \left(1 - \frac{|(x - x_0)^\perp|^2}{|x - x_0|^2}\right) \\
&= r \frac{d}{dr} |\Sigma \cap B_r(x_0)| - r \frac{d}{dr} \int_{\Sigma \cap B_r(x_0)} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^2}
\end{aligned}$$

$$\Rightarrow \frac{d}{dr}(r^{-k} |\Sigma \cap B_r(x_0)|) = r^{-k} \frac{d}{dr} \int_{\Sigma \cap B_r(x_0)} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^2} = \frac{d}{dr} \int_{\Sigma \cap B_r(x_0)} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^{k+2}}$$

$$\begin{array}{ccc}
\Rightarrow \displaystyle\int_s^t \frac{d}{dr} \frac{|\Sigma \cap B_r(x_0)|}{r^k} dr = \int_s^t \frac{d}{dr} \int_{\Sigma \cap B_r(x_0)} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^{k+2}} \\
\| \qquad\qquad\qquad\qquad \| \\
\displaystyle\frac{|\Sigma \cap B_t(x_0)|}{t^k} - \frac{|\Sigma \cap B_s(x_0)|}{s^k} \qquad \int_{\Sigma \cap (B_t(x_0) \setminus B_s(x_0))} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^{k+2}}
\end{array}$$

□

Corollary 5.4. *$x_0 \in \Sigma^k \subset B_R(0) \subset \mathbb{R}^n$ and $\sigma < R - |x_0|$*

$$\Rightarrow \omega_k \sigma^k \overset{(1)}{\le} |\Sigma \cap B_\sigma(x_0)| \overset{(2)}{\le} \frac{\sigma^k}{(R - |x_0|)^k} \cdot |\Sigma \cap B_R(0)|$$

where ω_k denotes the volume of the unit ball in $\mathbb{R}^k$.

Proof.

(1) We note that

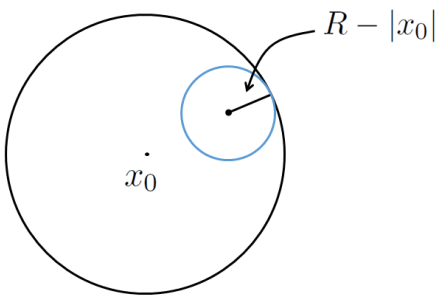
$$\frac{|B_\sigma(x_0) \cap \Sigma|}{\sigma^k} - \frac{|B_s(x_0) \cap \Sigma|}{s^k} \ge 0$$

and

$$\frac{|B_s(x_0) \cap \Sigma|}{s^k} \to \omega_k \quad \text{as } s \to 0.$$

Thus we obtain the result.

(2)



$$\frac{|B_{R-|x_0|}(x_0) \cap \Sigma|}{(R-|x_0|)^k} - \frac{|B_\sigma(x_0) \cap \Sigma|}{\sigma^k} \geq 0.$$

□

6 Bernstein(1912) theorem

Theorem 6.1 (Bernstein, 1912)**.** *Entire minimal graph $\Sigma^2 \subset \mathbb{R}^3$ is a plane.*

⋆ Gauss map n

$$n : \Sigma^2 \to \mathbb{S}^2$$

Lemma 6.2. *$\Sigma^2 \subset \mathbb{R}^3$, minimal. Then*

1. *n : conformal and orientation reversing, i.e., $\forall v, w \in T_p\Sigma$ such that $v \cdot w = 0$ and $|v| = |w|$ $\Rightarrow \nabla_v n \cdot \nabla_w n = 0$, $|\nabla_v n| = |\nabla_w n|$*
2. *$|\nabla_v n| \leq \frac{1}{\sqrt{2}}|A||v|$ and $n^*(\omega_{\mathbb{S}^2}) = K_\Sigma \omega_\Sigma = -\frac{1}{2}|A|^2 \omega_\Sigma$*

Proof. Take an orthonormal basis $\{e_1, e_2\}$ for Σ such that

$$\begin{cases} \nabla_{e_1} n = -\kappa_1 e_1 \\ \nabla_{e_2} n = -\kappa_2 e_2 \end{cases}$$

$$\Rightarrow |\nabla_{e_1} n| = |\kappa_1| = |\kappa_2| = |\nabla_{e_2} n| \quad (\because \kappa_1 \neq \kappa_2 = 0)$$

$$\therefore |\nabla_v n| \leq |\kappa_1||v| = \frac{1}{\sqrt{2}}|A||v|$$

$$|A|^2 = \kappa_1^2 + \kappa_2^2 = 2\kappa_1^2 = -2K$$

$$\& \quad \mathrm{Jac}(n) = \kappa_1\kappa_2 = -\frac{1}{2}|A|^2 \leq 0 \quad \text{(orientation-reversing)}$$

□

Proposition 6.3. *$\Sigma^2 \subset \mathbb{R}^3$, minimal such that $n(\Sigma) \subset \mathbb{S}^2_+ \Rightarrow \exists C > 0$ such that*

$$\int_\Sigma |A|^2\phi^2 \leq C \int_\Sigma |\nabla\phi|^2 \quad \forall \phi \in C_0^\infty(\Sigma) \left(\in W_0^{1,2}(\Sigma)\right)$$

In fact, Σ: stable $\iff$ $c = 1$.

Proof. Note: $\omega_{\mathbb{S}^2} = d\alpha$ (exact) ($\because \mathbb{S}^2_+$: Simply-connected)

$$\Rightarrow -\frac{|A|^2}{2}\omega_\Sigma = n^*(\omega_{\mathbb{S}^2}) = d(n^*\alpha)$$

$$\begin{aligned}
\therefore \int_\Sigma |A|^2\phi^2\omega_\Sigma &= -2\int_\Sigma \phi^2 d(n^*\alpha) = -2\underbrace{\boxed{\textstyle\int_\Sigma d(\phi^2 n^*\alpha)}}_{=0} + 4\int_\tau \phi d\phi \wedge n^*\alpha \\
&\le 4\int_\Sigma |\phi||\nabla\phi||n^*\alpha|\omega_\Sigma \\
&\le 4\int_\Sigma |\phi||\nabla\phi|C|A|\omega_\Sigma \qquad (\because |n^*\alpha| \le |A||\alpha| \le C|A|) \\
&\le 4C\int_\Sigma |\phi||\nabla\phi||A| \\
&\le 4C\left(\frac{\varepsilon}{2}\int_\Sigma |A|^2\phi^2 + \frac{1}{2\varepsilon}\int_\Sigma |\nabla\phi|^2\right)
\end{aligned}$$

□

Proposition 6.4. $\Sigma^2 \subset \mathbb{R}^3$, *entire minimal graph* $\Rightarrow |\Sigma \cap B_R(0)| \le 4\pi R^2$, $\forall R > 0$

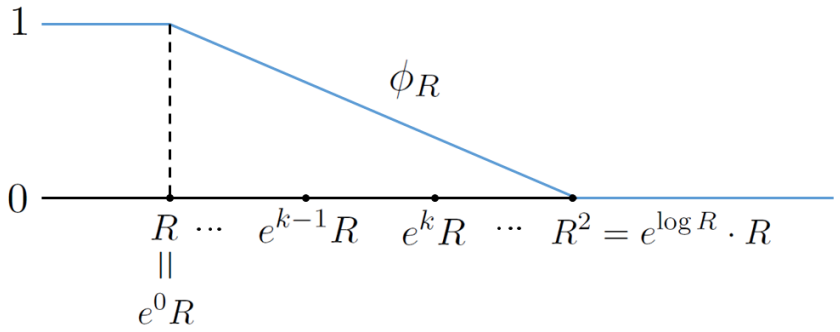
Proof. Σ: area-minimizing □

Proof of Bernstein Theorem. $\Sigma = \Sigma_u$: entire minimal graph $\Rightarrow n(\Sigma) \subset \mathbb{S}^2_+ \Rightarrow \exists C > 0$ such that

$$\int_\Sigma |A|^2\phi^2 \le c\int_\Sigma |\nabla\phi|^2 \quad \forall \phi \in C_c^\infty(\Sigma)$$

◎ logarithmic cutoff trick
Choose

$$\phi_k(r) = \begin{cases} 1, & r \le R \\ 1 - \frac{\log\frac{r}{R}}{\log R}, & R \le r \le R^2 \\ 0, & r \ge R^2. \end{cases}$$



$$B_{R^2} \setminus B_R = \bigcup_{k=1}^{\log R} (B_{e^kR} \setminus B_{e^{k-1}R})$$

$$\begin{aligned}\int_\Sigma |\nabla\phi|^2 &= \frac{1}{(\log R)^2}\int_{\Sigma\cap(B_{R^2}\setminus B_R)}\frac{1}{r^2}\\ &= \frac{1}{(\log R)^2}\sum_{k=1}^{\log R}\int_{\Sigma\cap(B_{e^kR}\setminus B_{e^{k-1}R})}\frac{1}{r^2}\\ &\le \frac{1}{(\log R)^2}\sum\frac{1}{(e^{k-1}R)^2}|\Sigma\cap(B_{e^kR}\setminus B_{e^{k-1}R})|\\ &\le \frac{1}{(\log R)^2}\sum\frac{c(e^kR)^2}{(e^{k-1}R)^2} = \frac{c}{(\log R)^2}\Sigma e^2 = \frac{c}{\log R}\to 0 \quad \text{as } R\to\infty\end{aligned}$$

□

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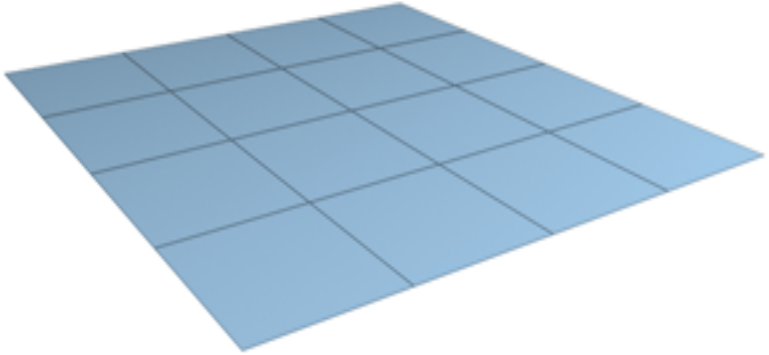
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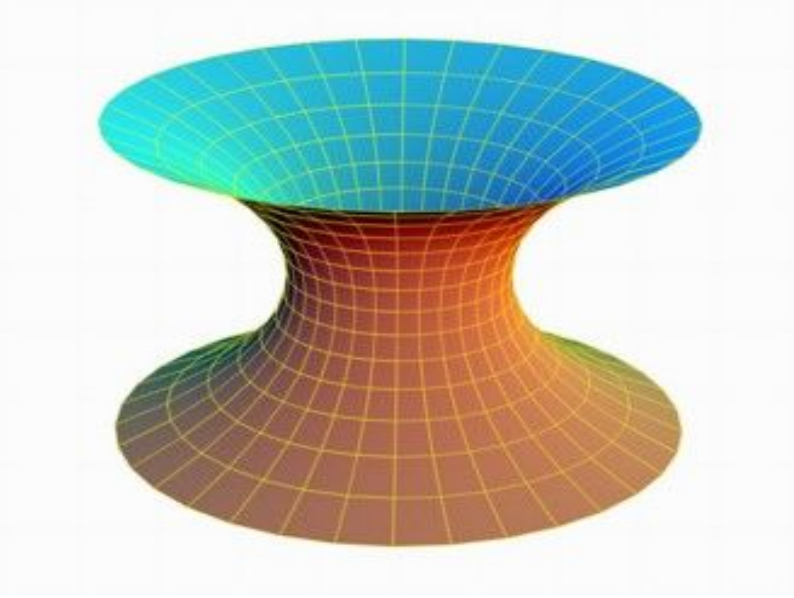
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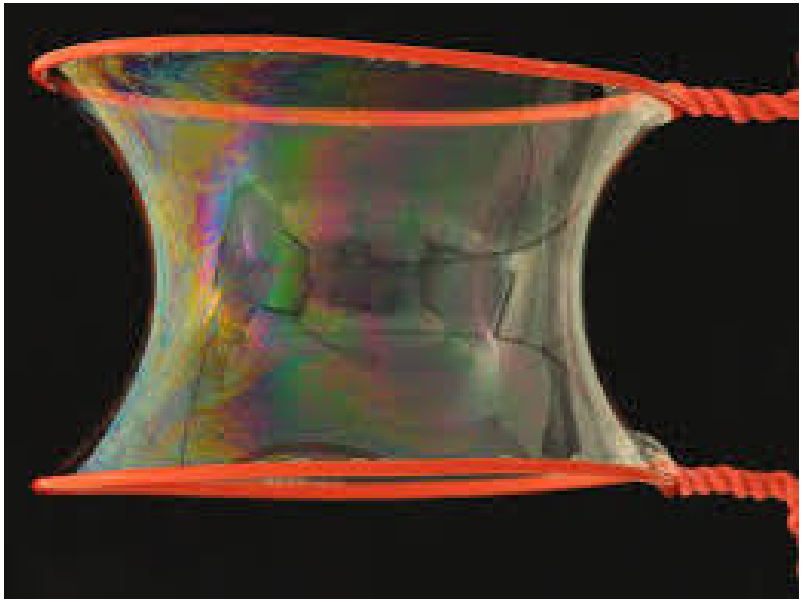
Examples of minimal surfaces in $\mathbb{R}^3$



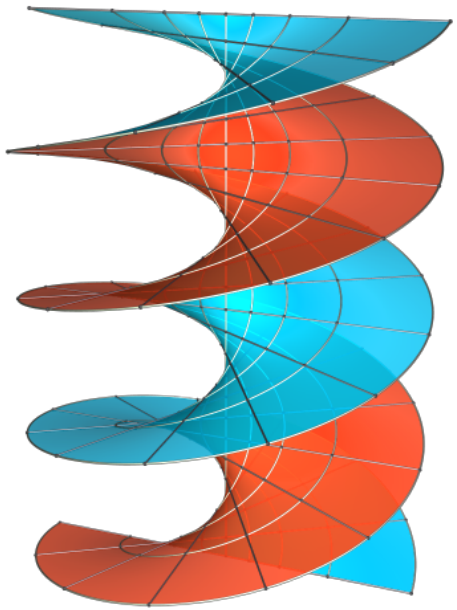


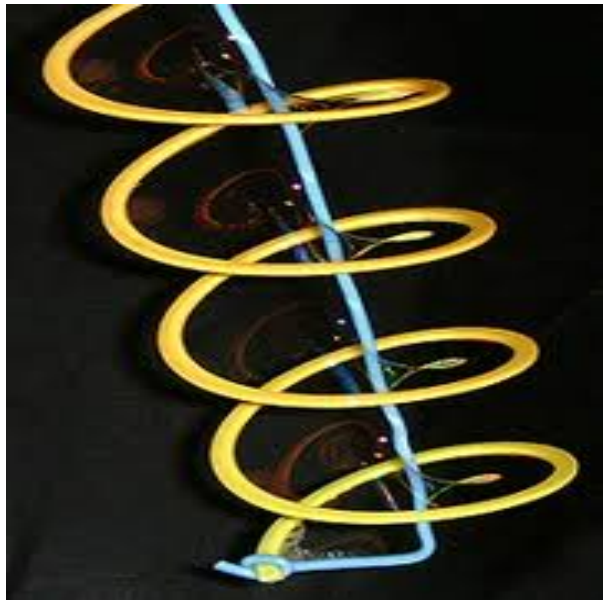
catenoid, Euler(1774) Meusnier(1776)



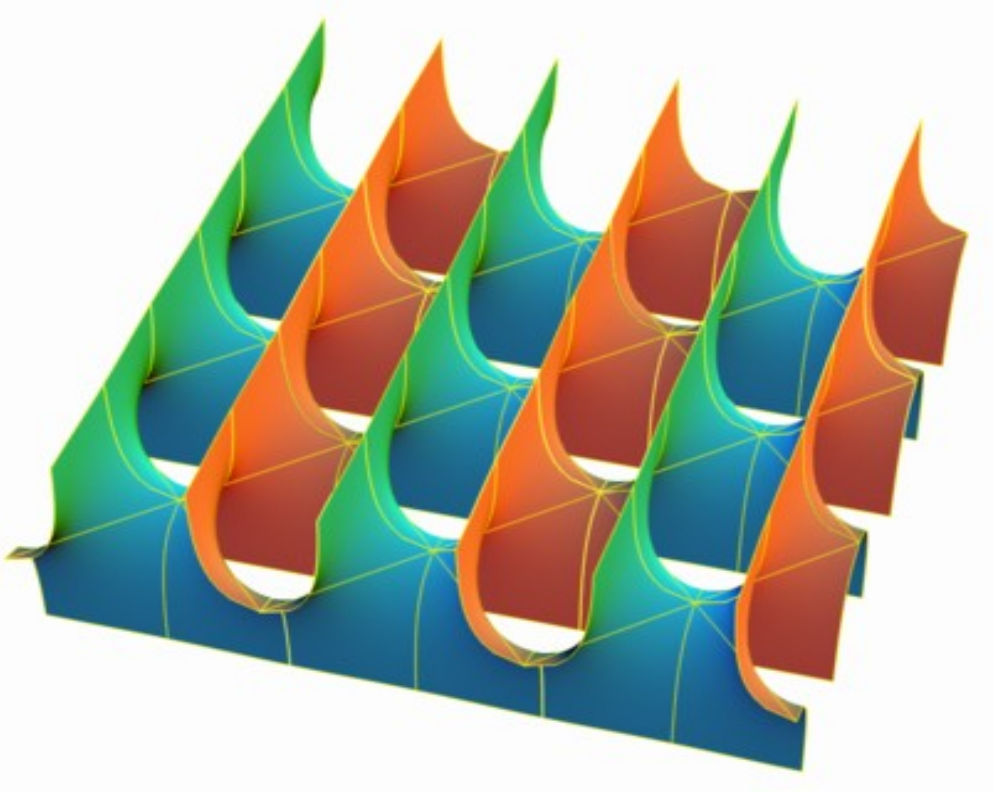


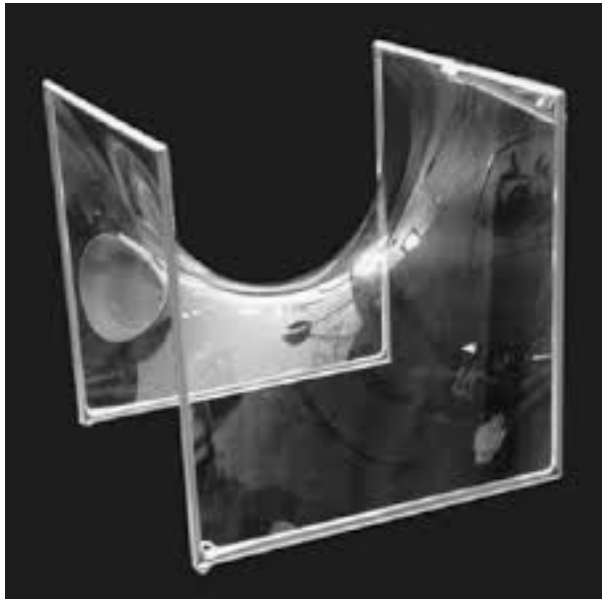
helicoid, Meusnier(1776)



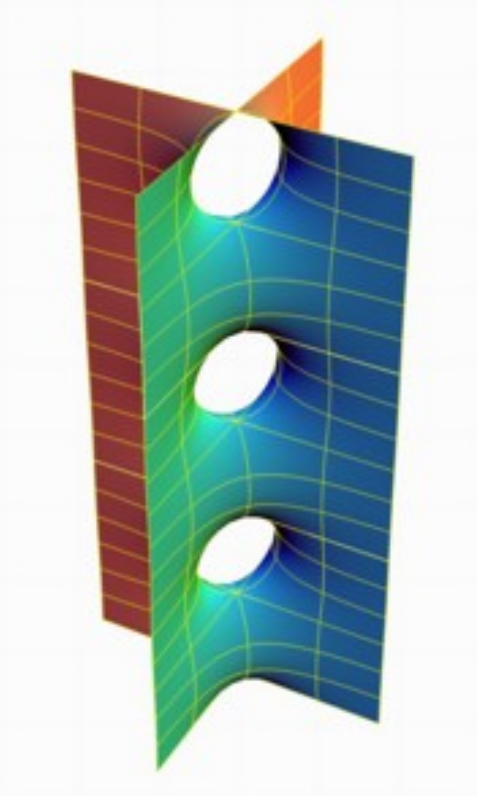


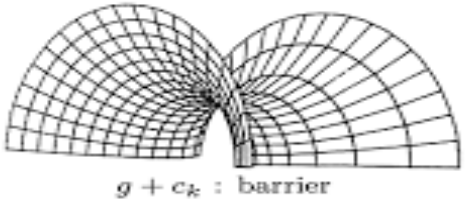
Scherk doubly periodic surface, Scherk(1835)

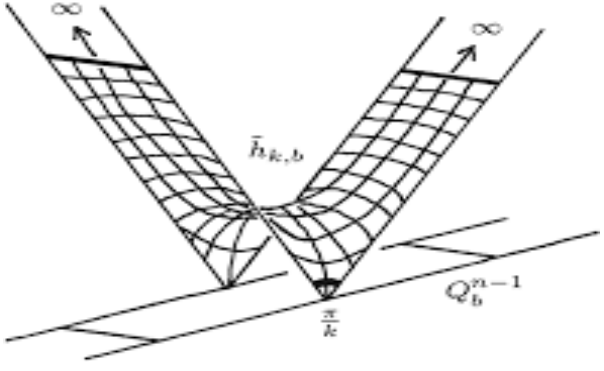




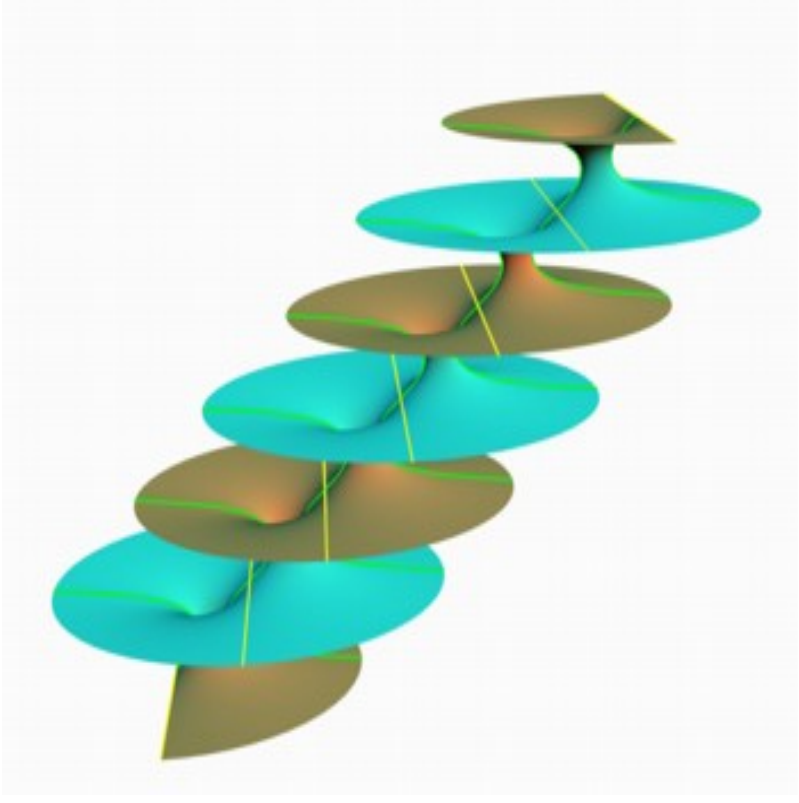
Scherk singly periodic surface, Scherk(1835)

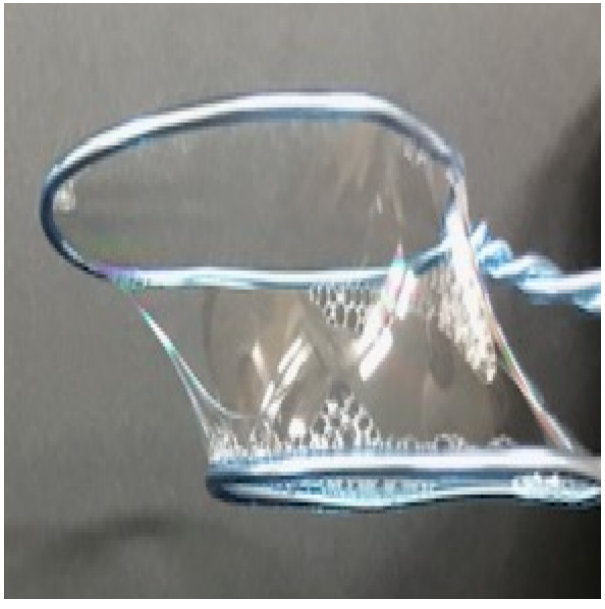






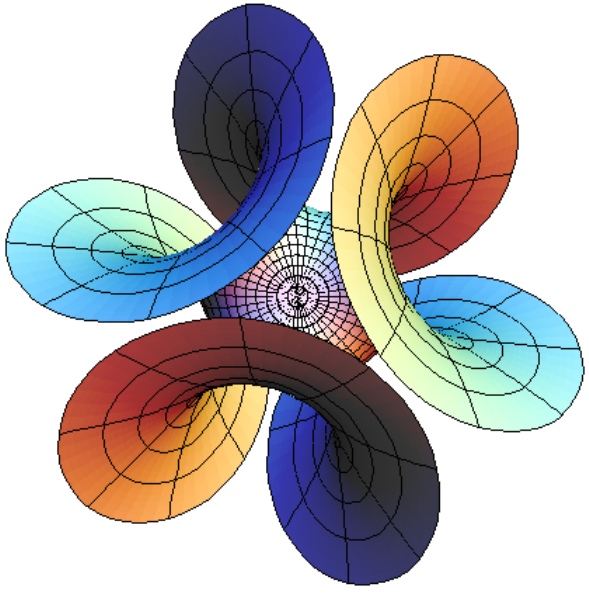
Riemann's example, Riemann(1860s)



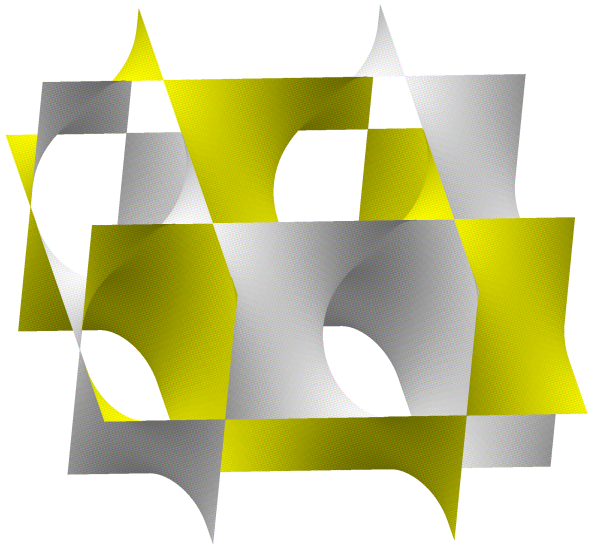


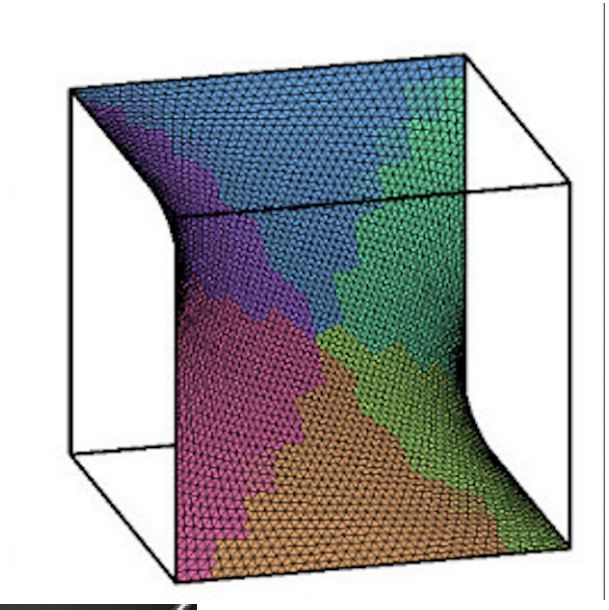
Enneper surfaces, Enneper(1864)

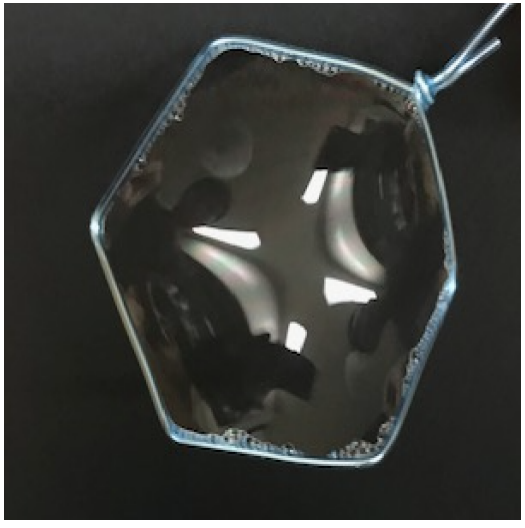




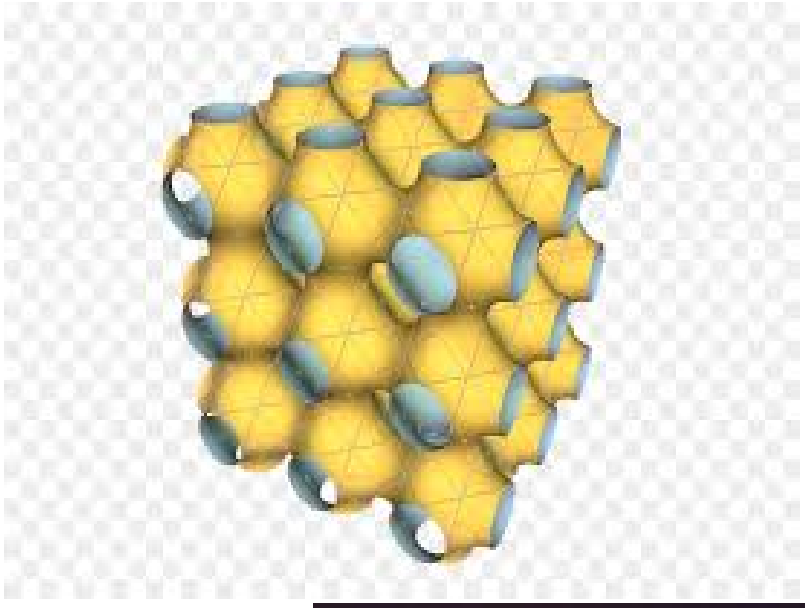
Schwarz D-surface, Schwarz(1865)

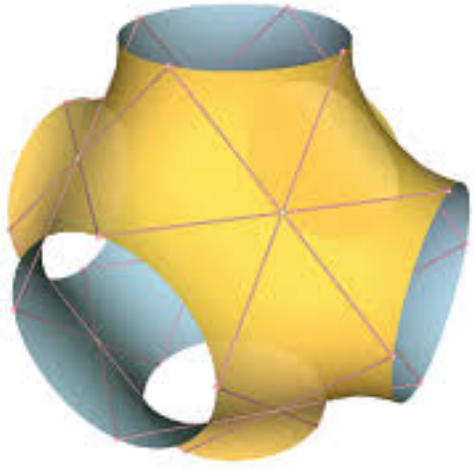


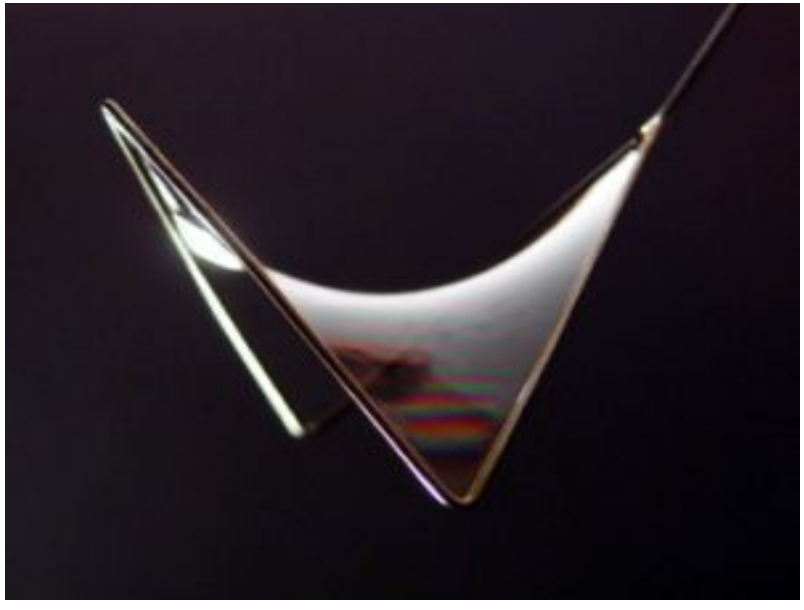




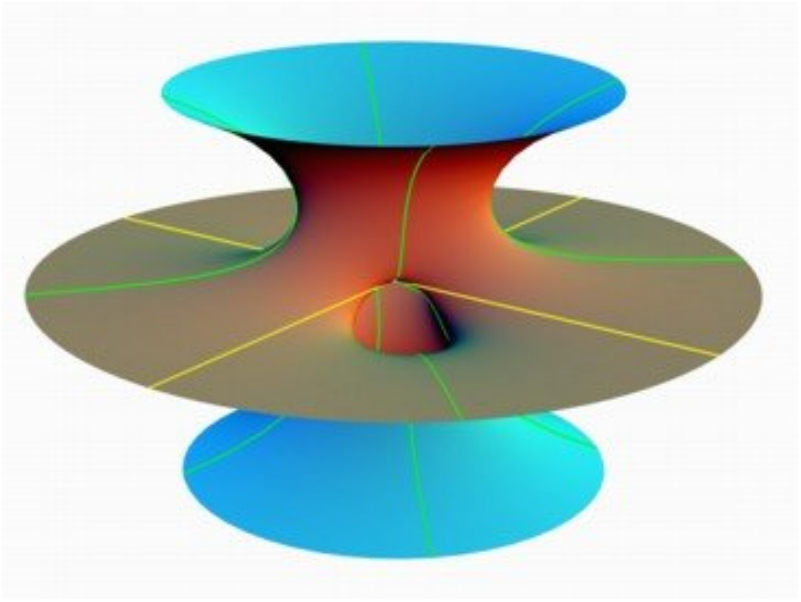
Schwarz P-surface, Schwarz(1880s)

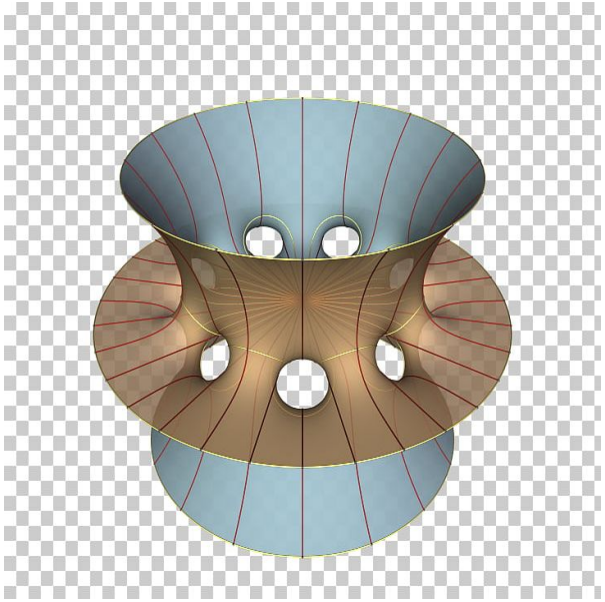


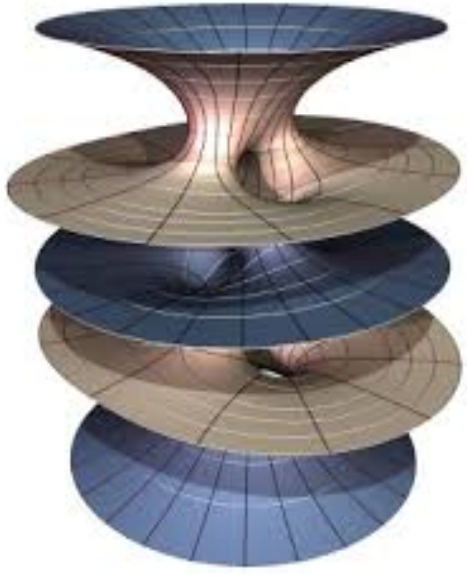




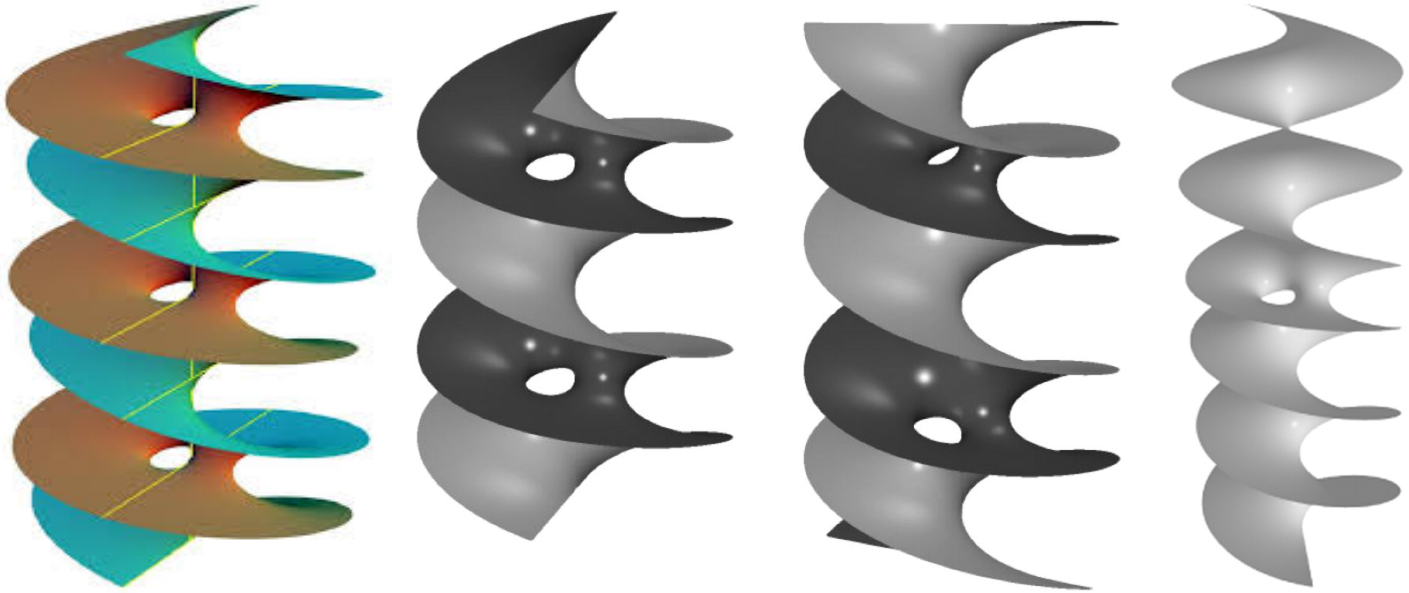
Costa surfaces, Costa(1984), C-H-M(1985)







helicoidal surfaces, 1993, 2016



No.

Lecture 2. Examples and finding properties of minimal surfaces

[1] plane $\Leftrightarrow$ totally geodesic in $\mathbb{R}^3$

(note) homothety property Σ : minimal $\Rightarrow$ $a\Sigma$: minimal

[2] Catenoid, plane $\Leftrightarrow$ surface of revolution

(proof) ($\Rightarrow$) Euler (1744 finding) Meusnier (1776)

$\sigma(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$ [$\cosh z = \sqrt{x^2+y^2}$]

($\Leftarrow$) Bonnet (1860)

w.l.o.g. we can assume that the axis of the surface of revolution is z-axis.

i.e. $\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$, $f(u)>0$, $\dot{f}^2+\dot{g}^2=1$

$I = ds^2 = du^2 + f(u)^2 dv^2$

$II = (\dot{f}\ddot{g} - \ddot{f}\dot{g})du^2 + f\dot{g}dv^2$

$H = \frac{1}{2}\left(\dot{f}\ddot{g} - \ddot{f}\dot{g} + \frac{\dot{g}}{f}\right) = 0$

case 1) $\dot{g}(u) \equiv 0$ $\Rightarrow$ plane

case 2) $\dot{g}(u_0) \neq 0$, $\dot{g}(u) \neq 0$ near u_0

using $\begin{cases} \dot{f}\ddot{f} + \dot{g}\ddot{g} = 0 \\ (\dot{f})^2 + (\dot{g})^2 = 1 \end{cases}$, $f\ddot{f} + (\dot{f})^2 - 1 = 0$

general solutions $f(u) = \frac{1}{a}\sqrt{1 + a^2(u+b)^2}$

using $\dot{f}^2 + \dot{g}^2 = 1$, $g(u) = \pm \frac{1}{a}\sinh^{-1}(au) + c$

Therefor $f(u) = \frac{1}{a}\cosh(a(g(u)-c))$.

$x = \frac{1}{a}\cosh(a(z-c))$ □

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No.

[3] helicoid, plane $\Leftrightarrow$ ruled surface

($\Rightarrow$) (Meusnier 1776)

$\sigma(u,v) = (v\cos u, v\sin u, \alpha u)$ $\alpha \in \mathbb{R}$ $\left[\alpha=1,\ z=\tan^{-1}\left(\frac{y}{x}\right)\right]$

($\Leftarrow$) (Catalan 1842) <Exercise>

[4] level set of a "simple" function

regular algebraic surface $\Leftrightarrow$ plane

algebraic surface $\Rightarrow$ higher order "Enneper surface etc"

[5] graph of a smooth function $z = f(x,y)$

$\sigma(x,y) = (x, y, f(x,y))$

$I = (1+f_x^2)\,dx^2 + 2f_xf_y\,dxdy + (1+f_y^2)\,dy^2$

$II = \frac{1}{\sqrt{1+f_x^2+f_y^2}}\left(f_{xx}\,dx^2 + 2f_{xy}\,dxdy + f_{yy}\,dy^2\right)$

$H=0 \Leftrightarrow (1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy} = 0 \Leftrightarrow \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)=0$

$\begin{vmatrix} 1+f_y^2 & f_xf_y \\ f_xf_y & 1+f_x^2 \end{vmatrix} > 0 \Rightarrow$ 2nd order quasilinear elliptic pde.

From pde theory,

① analytic property

② maximum principle

③ Dirichlet problem (Jenkins-Serrin theorem)

④ locally, $|\nabla f| \sim \varepsilon \Rightarrow \Delta f = 0$, globally $\operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right)=0$ different from $\Delta f = 0$.

(More detail, see in lecture 3)

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No.

- radially symmetric solution $\Rightarrow$ catenoid
- separable solution $z = f(x,y) = g(x) + h(y)$

$$(1+(h')^2)g'' - 2g'h' \times 0 + (1+(g')^2)h'' = 0$$

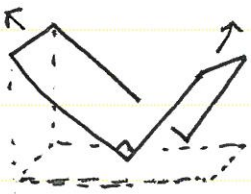
$$\Rightarrow \frac{g''}{1+(g')^2} = -\frac{h''}{1+(h')^2} = \text{constant} \overset{\text{let}}{=} 1 \quad (*)$$

Since $g(x)$, $g(-x)$ satisfy the $(*)$, $g(0)=0$, $g'(0)=0$

$\Rightarrow g(x) = -\ln\cos x$, $h(y) = \ln\cos y$

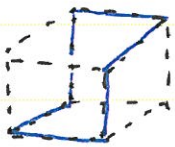
By reflection principle along a line, Scherk's doubly periodic surface.
(Scherk 1835)

[Dirichlet solution + maximum principle]

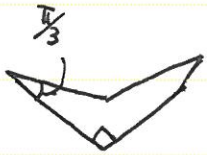


Scherk's singly-periodic surface (1835)

- Schwarz D-surface (1865)



Schwarz P-surface (1880s)



- Riemann's example (1860s)
- Costa surface (1984) Hoffman-Meeks (1985)
- genus-1 helicoid (1993)

- Enneper surface (1864) $(x,y,z) = \left(u - \frac{u^3}{3} + uv^2, -v - u^2v + \frac{v^3}{3}, u^2 - v^2\right)$

- half-space theorem
  - Nadirashvili example
    immersed, complete m.s.
    in a ball



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No.

Lecture 4 minimal surfaces theory based on complex analysis

prop 4.1 Every surface has a conformal patch for each point.



$(u,v) \in \mathbb{R}^2 \leftrightarrow u+iv \in \mathbb{C}$

let $z = u+iv$, $\varphi(z) := \sigma_u - i\sigma_v = (\varphi_1, \varphi_2, \varphi_3)$

prop 4.2 $\sigma : U \to \mathbb{R}^3$ is a conformal parametrization

σ is minimal $\Leftrightarrow \Delta_0 \sigma = \sigma_{uu} + \sigma_{vv} = \vec{0}$

$\Leftrightarrow \varphi$ is holomorphic on U (φ_i : holomorphic $i=1,2,3$)

(proof) σ : minimal $\Leftrightarrow \Delta_0 \sigma = 0$

$$\Delta_0\sigma \cdot \sigma_u = \sigma_{uu}\cdot\sigma_u + \sigma_{vv}\cdot\sigma_u = \frac{1}{2}(\sigma_u\cdot\sigma_u)_u + (\sigma_v\cdot\sigma_u)_v - (\sigma_v\cdot\sigma_{uv})$$
$$= \frac{1}{2}(\sigma_u\cdot\sigma_u - \sigma_v\cdot\sigma_v)_u + (\sigma_u\cdot\sigma_v)_v \overset{\text{conformal}}{=} 0$$

similarly, $\Delta_0\sigma \cdot \sigma_v = 0$

$\Delta_0\sigma = \lambda N \Rightarrow \lambda = \Delta_0\sigma \cdot N = h_{11} + h_{22} = 2EH = 0$ □

σ : minimal $\Leftrightarrow$ φ : holomorphic (Cauchy-Riemann eq)

φ : holomorphic $\Leftrightarrow \begin{cases} (\sigma_u)_u = (-\sigma_v)_v \\ (\sigma_u)_v = -(-\sigma_v)_u \quad \text{trivial} \end{cases}$

$\Leftrightarrow \Delta\sigma = 0$

$\Leftrightarrow \sigma$: minimal □

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No.

prop 4.3 $\sigma : U \to \mathbb{R}^3$ a conformal parametrized minimal surface.

$\varphi = \sigma_u - i\sigma_v = (\varphi_1, \varphi_2, \varphi_3)$ satisfies (1) $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 0$ (2) $|\varphi| > 0$

conversely, U is a simply-connected domain. $\varphi_1, \varphi_2, \varphi_3$ holomuphic on U.

satisfies (1) & (2). Then there is a unique conformal parametrized

minimal surf. $\sigma : U \to \mathbb{R}^3$ s.t. $\varphi = \sigma_u - i\sigma_v$ upto translation.

(proof) $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ $\varphi_k = \sigma_u^k - i\sigma_v^k$, $k = 1, 2, 3$

$$\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = \sum_{k=1}^{3} \left\{ (\sigma_u^k)^2 - (\sigma_v^k)^2 - 2i\,\sigma_u^k \sigma_v^k \right\} = |\sigma_u|^2 - |\sigma_v|^2 - 2i\,\sigma_u \cdot \sigma_v = 0$$ (conformal)

$\varphi(p) = \vec{0} \iff \sigma_u(p) = 0 = \sigma_v(p)$ not regular.

Conversely, for a fixed point $(u_0, v_0) \in U$, $\gamma \subset U$ a path

connecting (u_0, v_0) to $(u, v) \in U$.

Define $\sigma(z) = \mathrm{Re} \int_\gamma \varphi(\xi)\, d\xi$.

It is well-defined by U is simply-connected.

Denote $\Phi(z) = \int_\gamma \varphi(\xi) d\xi$ holomuphic and $\Phi'(z) = \varphi(z)$, $\sigma = \mathrm{Re}\,\Phi$.

$$\sigma_u = \mathrm{Re}(\Phi_u) = \mathrm{Re}(\Phi') = \mathrm{Re}(\varphi)$$

$$\sigma_v = \mathrm{Re}(\Phi_v) = \mathrm{Re}(i\Phi') = -\mathrm{Im}(\varphi)$$

so, $\varphi = \sigma_u - i\sigma_v$.

(1) $\Rightarrow |\sigma_u|^2 = |\sigma_v|^2$, $\sigma_u \cdot \sigma_v \iff$ conformal

(2) $\Rightarrow |\sigma_u|^2 > 0$ immersion

If $\tilde{\sigma}$ is another conformal minimal surface with the same φ.

$\Rightarrow \tilde{\sigma}_u = \sigma_u$, $\tilde{\sigma}_v = \sigma_v$

$\Rightarrow \tilde{\sigma} - \sigma =$ constant

□

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By rearrangement, $\varphi_1 - i\varphi_2 \neq 0$.

$f = \varphi_1 - i\varphi_2$: holomorphic function

$g = \dfrac{\varphi_3}{\varphi_1 - i\varphi_2}$: meromorphic function

$$\left(\frac{1}{2} f(1-g^2),\ \frac{i}{2} f(1+g^2),\ fg \right) = (\varphi_1, \varphi_2, \varphi_3) = \varphi$$

(Note1) $$N = \left(\frac{2\,\mathrm{Re}\, g}{|g|^2+1},\ \frac{2\,\mathrm{Im}\, g}{|g|^2+1},\ \frac{|g|^2-1}{|g|^2+1} \right)$$

g is the stereographic projection of the Gauss map.

(Note2) minimal surface $\leftrightsquigarrow$ (f, g)

$(e^{i\theta} f, g) \rightsquigarrow$ minimal surface (associate surface)

$\theta = \frac{\pi}{2}$ i.e. $(if, g) \rightsquigarrow$ minimal surface (conjugate surface)

(Note3) $$I = \frac{1}{2} \varphi \bar{\varphi} (du^2 + dv^2)$$
$$= \frac{1}{4} |f|^2 (1+|g|^2)^2 (du^2 + dv^2)$$

Every associate surface is isometric each other.

(Note4) $$K = \frac{-16 \left| \frac{dg}{dz} \right|^2}{|f|^2 (1+|g|^2)^4}$$

The zeros of the Gaussian curvature of a minimal surface are isolated.

(Note5) Enneper surface $U = \mathbb{C}$, $f(z) = 1$, $g(z) = z$

$$\mathrm{Re} \left\{ z - \frac{z^3}{6},\ \frac{i}{2}\left(z + \frac{z^3}{3}\right) + \frac{z^2}{2} \right\}$$

(Note6) Schwarz reflection principle.

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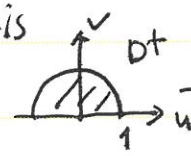
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Schwarz reflection principle.



w.l.o.g $L = x$-axis

$\forall p \in L$, $\sigma(u,v)$: 

$\longrightarrow (x(u,v), y(u,v), z(u,v))$

$y(u,0) = 0 = z(u,0)$, y, z : harmonic

(Schwarz reflection principle)

$y(u,-v) := -y(u,v)$, $z(u,-v) = -z(u,v)$: harmonic D

$\phi_1^2 = -\phi_2^2 - \phi_3^2$

$= -(y_u - i y_v)^2 - (z_u - i z_v)^2$

on $v=0$, $\phi_1^2 = y_v^2 + z_v^2 \geq 0 \Rightarrow x_v(u,0) = 0$

$x(u,-v) := x(u,v)$ harmonic on D

$(x(u,v), y(u,v), z(u,v))$ on $D^+ \rightarrow (x,y,z)$ on D and

$(x,-y,-z)$ on D^- .

It is 180° - rotation along L.

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Lecture 6. minimal surfaces in the 3-dim sphere

Model of $S^3 = \{ x \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \}$

$\Sigma \subset S^3$ a regular surface

$\sigma : (u,v) \in \mathbb{R}^2 \longrightarrow S^3 \subset \mathbb{R}^4$ a conformal parametrization

i.e. $\sigma_u \cdot \sigma_u = \sigma_v \cdot \sigma_v = E$, $\sigma_u \cdot \sigma_v = 0 \iff I = E(du^2 + dv^2)$

$\{ \frac{1}{\sqrt{E}}\sigma_u, \frac{1}{\sqrt{E}}\sigma_v, N, x \}$ O.N.B of $\mathbb{R}^4$

Denote $h_{11} = \sigma_{uu} \cdot N$, $h_{12} = \sigma_{uv} \cdot N$, $h_{22} = \sigma_{vv} \cdot N \iff II = h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2$

$$K = \frac{1}{2E}\Delta_0 \ln E = \frac{h_{11}h_{22} - h_{12}^2}{E^2} + 1$$

($k_{int} \neq k_{Ext}$ in S^3, but $k_{int} = k_{Ext}$ in $\mathbb{R}^3$)

$$H = \frac{h_{11} + h_{22}}{2E}$$

$$\begin{cases} \sigma_{uu} = \frac{E_u}{2E}\sigma_u - \frac{E_v}{2E}\sigma_v + h_{11}N - E\sigma \\ \sigma_{vv} = -\frac{E_u}{2E}\sigma_u + \frac{E_v}{2E}\sigma_v + h_{22}N - E\sigma \end{cases}$$

$$\Delta_0 \sigma = 2EHN - 2E\sigma \iff \Delta_0 \sigma + 2E\sigma = 0 \quad (\Sigma: \text{minimal})$$

$$\iff \Delta_\Sigma \sigma + 2\sigma = 0 \qquad \Delta_\Sigma = \tfrac{1}{E}\Delta_0$$

prop 6.1 σ : minimal in $S^3 \iff \Delta_\Sigma \sigma + 2\sigma = 0$

prop 6.2 $\Sigma \subset S^3$ minimal $\iff 0 * \Sigma$: 3-dim minimal hypersurface in $\mathbb{R}^4$.

(pf) $p \in \Sigma$, $\sigma(u,v) \to \mathbb{R}^4$ a conformal parametrization of p.

$\tilde{\sigma} : (r,u,v) \longrightarrow \mathbb{R}^4$, $\tilde{\sigma}(r,u,v) = r\sigma(u,v)$ a parametrization of $0 * \Sigma$ at p.

$$(\Rightarrow)\ \operatorname{tr}\left(F_I^{-1}F_{II}\right) = \operatorname{tr}\left[\begin{pmatrix} \frac{1}{r^2E} & 0 & 0 \\ 0 & \frac{1}{r^2E} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} rh_{11} & * & * \\ * & rh_{22} & * \\ * & * & 0 \end{pmatrix}\right] = 0$$

$$(\Leftarrow)\ \Delta_{0*\Sigma} = \frac{1}{r^2}\Delta_\Sigma + \frac{1}{r^2E}\frac{\partial}{\partial r}\left(r^2E\frac{\partial}{\partial r}\right)$$

$$0 = \underset{0*\Sigma:\text{minimal}}{\Delta_{0*\Sigma}\tilde{\sigma}} = \frac{1}{r}\Delta_\Sigma\sigma + \frac{2}{r}\sigma, \quad \text{At } p,\ 0 = \Delta_\Sigma\sigma + 2\sigma \quad \square$$

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Examples :

[1] equator $\{ x \in \mathbb{S}^3 \mid x_4 = 0 \}$: totally geodesic

$k_1 = 0 = k_2$, $K = k_1 k_2 + 1 = 1$ [closed minimal surface in $\mathbb{S}^3$]

[2] Clifford torus $\{ x_1^2 + x_2^2 = x_3^2 + x_4^2 = \frac{1}{2} \}$

$\sigma(u,v) = \frac{1}{\sqrt{2}} (\cos u, \sin u, \cos v, \sin v)$

$\sigma_u = \frac{1}{\sqrt{2}} (-\sin u, \cos u, 0, 0)$

$\sigma_v = \frac{1}{\sqrt{2}} (0, 0, -\sin v, \cos v)$

$N = \frac{1}{\sqrt{2}} (\cos u, \sin u, -\cos v, -\sin v)$

$I = \frac{1}{2}(du^2 + dv^2)$ $\quad II = -\frac{1}{2} du^2 + \frac{1}{2} dv^2$ $\quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$k_1 = 1$ ($u = u_0$), $k_2 = -1$ ($v = v_0$), $K = 0$ flat surface

[3] Lawson's examples Σ_g

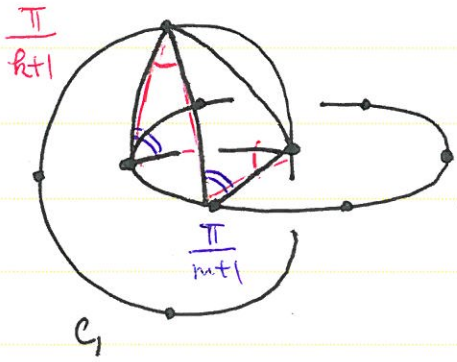
There exists an embedded minimal surface $\Sigma_g \subset \mathbb{S}^3$ with genus g.

(If g is not a prime number then $\exists$ at least 2 different min. surf.)

Another model of $\mathbb{S}^3$: $\mathbb{S}^3 \setminus \{p\} \leftrightarrow \left(\mathbb{R}^3, \frac{4}{(1+x_1^2+x_2^2+x_3^2)^2} (dx_1^2 + dx_2^2 + dx_3^2) \right)$

For positive integers (k, m), $(2k+2)$ pts in $C_1 : x_1^2 + x_2^2 = 1$

$(2m+2)$ pts in $C_2 : x_3^2 + x_4^2 = 1$



$(k=1, m=2)$

- Plateau's solution (More precisely, Morrey solution) Σ_p
- Σ_p : embedded ($\because$ tetrahedron is convex)
- Gauss-Bonnet $\int_{\Sigma_p} K dA = 2 \frac{(-mk+1)}{(k+1)(m+1)} \pi$
- # (total tesselation) $(2k+2)(2m+2)$
- # (Fundamental piece) $\frac{1}{2}(2k+2)(2m+2)$

$\Rightarrow \int_\Sigma K dA = 4(1-mk)\pi = 2\pi \chi(\Sigma)$

$\therefore g = mk$

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④ More examples : Karcher - Pinkall - Sterling $g = 3, 5, 6, 7, 11, 19, 73, 601$

Choe - Sorret $g = 1 + 4m(m-1)l$

Kapouleas - Yang $g \gg 1$

(Note1) (unstability) 2nd variational formula Σ in S^3

$$A''(0) = \int_\Sigma |\nabla f|^2 - (|A|^2 + 2) f^2 \, d\Sigma$$

Σ : closed $\Rightarrow$ take $f = 1$ $\quad A''(0) < 0 \Rightarrow$ unstable

index 1 $\Leftrightarrow$ equator

next index : Clifford torus = index 5

[$\langle N, \vec{a} \rangle$, conformal transformation in R^3]

(Note2) Clifford torus

· An embedded minimal surface in S^3 of genus 1 is congruent to the clifford torus (Brendle 2013, Lawson 1970)

Willmore energy $W(\Sigma) = \int_\Sigma 1 + \frac{H^2}{4}$

{ equator 4π
clifford torus $2\pi^2$

· If Σ is an immersed minimal surface in S^3 of genus at least 1, then $W(\Sigma) \geq 2\pi^2$, "=" holds $\Leftrightarrow$ Clifford torus.
(Marques - Neves 2014, conjectured 1965)

(Note3) $\Delta_\Sigma \sigma + 2\sigma = 0$

(Yau conjecture) The first eigenvalue of the Laplacian on a cpt embedded minimal surface in S^3 is equal to 2.

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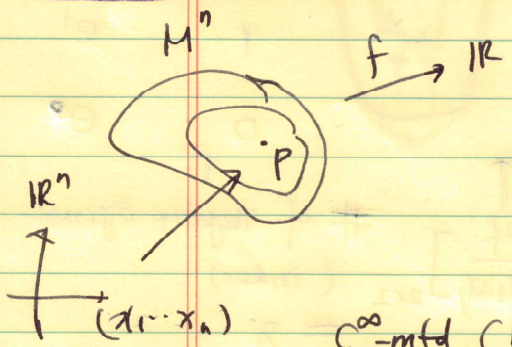
3차원 공간의 곡면에 대한 모스 함수의 동부성

2019. 12. 26. 李載燁

1° Morse function
2° Morse function of $S \subset \mathbb{R}^3$ surface
3° Further .

(iv) $M^a := \{ x \in M \mid f(x) \le a \}$

1° Morse functions



C^∞-mfd. (Riemannian if necessary)

Def For $f : M \to \mathbb{R}$

(i) $p \in M$ is a critical pt if $\nabla f|_p = 0$
and its value $f(p)$ is called a critical value.

(ii) $p \in M$, critical point of f is called "non-degenerate" if $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right]_{n \times n}$ non-singular

Hessian(f)

for local coordinate $(x_1 \cdots x_n)$ near $p \in M$

(iii) A function on M with only non-deg. critical points is called a Morse function.

Note

① Hessian

$f : M \to \mathbb{R}$ with $\nabla f(p) = 0$
Ψ
p

$\mathrm{Hess}(f) : T_pM \times T_pM \to \mathbb{R}$
$(v, w) \mapsto \mathrm{Hess}(f)(v,w)$

where $\mathrm{Hess}(f)(v,w) := \tilde{v}_p(\tilde{w}(f))$

$\tilde{v}, \tilde{w}$ extension of v, w to vector field near p.
$\tilde{v}_p = v$.

check $\mathrm{Hess}(f)$; symmetric bilinear.

observe
$\tilde{v}_p(\tilde{w}(f)) - \tilde{w}_p(\tilde{v}(f))$
$= [\tilde{v}, \tilde{w}]_p(f)$
$= (\nabla f)_p([\tilde{v}, \tilde{w}]_p) = 0$
↑
critial point p.

② $a \notin \{$critical value of $f\}$

$\Rightarrow M^a$ is a smooth mfd with boundary.

↑ implicit function thm

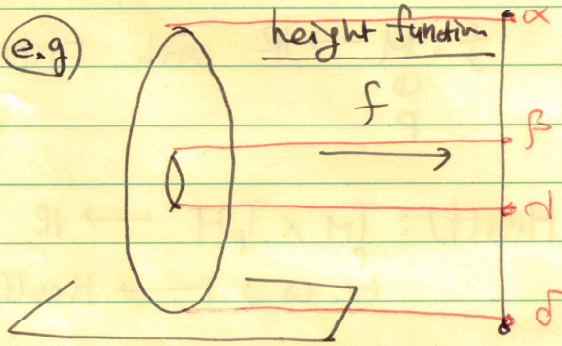
Note

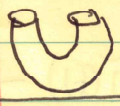
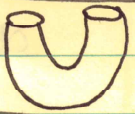
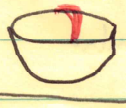
① $e^k := \{x \in \mathbb{R}^k \mid |x| \le 1\}$

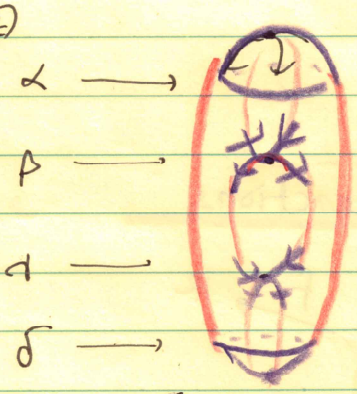
k-cell

Why Morse function ?

e.g) height function



$a \in \mathbb{R}$	M^a	way to understand	
$a \ge \alpha$			attaching 2-cell
$\beta \le a < \alpha$			
$a = \beta$			attaching 1-cell
$\delta < a < \beta$			
$a = \delta$			attaching 1-cell
$a \le \delta$	•	•	0-cell

② 

$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{2\times 2}$

# of negative eigenvalue (index) = λ say

= # of downward principal direction

⇓

Torus $\simeq e^0 \cup e^1 \cup e^1 \cup e^2$

homotopy type ↑ attaching

$H_k(\text{Torus}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & k=2 \\ \mathbb{Z}\oplus\mathbb{Z} & k=1 \\ \mathbb{Z} & k=0 \end{cases}$

Thm $f : M \to \mathbb{R}$ Morse function s.t. each M^a compact

then $M \simeq$ CW complex with one cell of dim $= \lambda$ for each critical point of f with index λ.

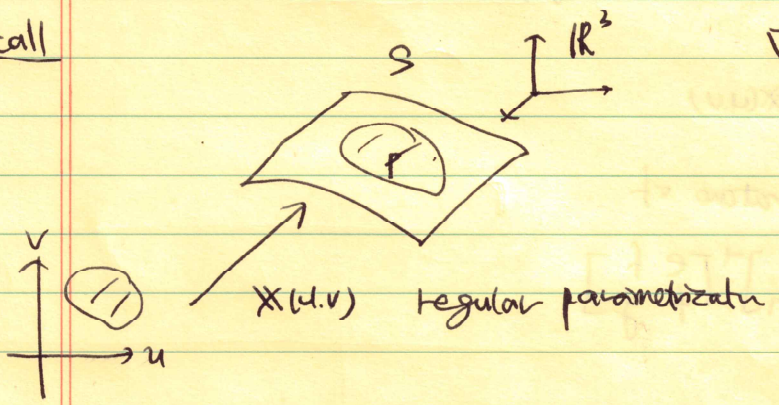
(homotopy type)

e.g. $\mathbb{CP}^n \simeq e^0 \cup e^2 \cup \cdots \cup e^{2n}$ (homotopy type)

$$\Rightarrow H_k(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & k = \text{even} \\ 0 & k = \text{odd} \end{cases}$$

2° Morse functions of $S \subset \mathbb{R}^3$ surface

Recall



1st Fundamental Form

$E = \langle \vec{x}_u, \vec{x}_u \rangle \quad F = \langle \vec{x}_u, \vec{x}_v \rangle \quad G = \langle \vec{x}_v, \vec{x}_v \rangle$

Gauss map (C^0-able or near p only)

$N : S \to \mathbb{R}^3$

$p \mapsto N(p)$: unit normal vector to T_pS

2nd fundamental form

$$e = \langle N_u, \vec{x}_u \rangle = \langle N, \vec{x}_{uu} \rangle$$
$$f = \langle -N_v, \vec{x}_u \rangle = \langle N, \vec{x}_{uv} \rangle = \langle -N_u, \vec{x}_v \rangle \quad \because$$
$$g = \langle -N_v, \vec{x}_v \rangle = \langle N, \vec{x}_{vv} \rangle$$

$$[-dN]_{\{\vec{x}_u, \vec{x}_v\}} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

Def $q \in \mathbb{R}^3 \setminus S$

$$L_q : S \to \mathbb{R}, \quad p \mapsto L_q(p) := |p - q|^2$$

smooth ∨

? critical point ?

$$\nabla\big(\underbrace{L_q(\vec{x}(u,v))}_{f(u,v)}\big) = \big(2(\vec{x}(u,v) - p) \cdot \vec{x}_u,\ 2(\vec{x}(u,v) - p) \cdot \vec{x}_v\big)$$

$$\nabla f = 0 \iff (\vec{x}(u,v) - p) \perp \operatorname{Span}\{\vec{x}_u, \vec{x}_v\}$$
$$\iff (\vec{x}(u,v) - p) \perp T_{\vec{x}(u,v)} S$$

non-degenerate ? $N = \frac{X-g}{|X-g|}$

$f_{uu} = ((X-g)\cdot X_u)_u = X_u \cdot X_u + \underbrace{(X-g)\cdot X_{uu}}_{\uparrow}$

$|X-g| \cdot N \cdot X_{uu}$

$= E + |X-g| e$

measure $\{ \text{☺} \} = 0$

$f_{uv} = ((X-g)\cdot X_u)_v = X_v \cdot X_u + (X-g)\cdot X_{uv}$

$= F + |X-g| f$

$f_{vv} = ((X-g)\cdot X_v)_v = X_v \cdot X_v + (X-g)\cdot X_{vv}$

$= G + |X-g| \cdot g$

Thm

$\mathbb{R}^3 \setminus (\text{☺} \cup S)$

Ψ

g

$\Rightarrow L_g$: Morse function on S

$\mathrm{Hess}(f) = \begin{bmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{bmatrix}$

Rmk

$= \begin{bmatrix} E & F \\ F & G \end{bmatrix} + |X-g| \begin{bmatrix} e & f \\ f & g \end{bmatrix}$

$= \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \left(I_2 + |X-g| \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \right)$

Hess (f) degenerate at $p = X(u,v)$

$\Leftrightarrow -|p-g|^{-1}$ principal curvature of

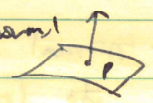
$\{X_u, X_v\}$

$\left[-dN_p \right] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix}$

$\Rightarrow$

$\text{☺} = \left\{ g \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists p \in S \\ \lambda \neq 0 \text{ principal} \\ \text{curvate at } p \end{array} \text{ with } g = p + \frac{1}{\lambda} n_p \right\}$

n_p : unit normal



3° Further

normal bundle $N_{\mathbb{R}^3/S}$.

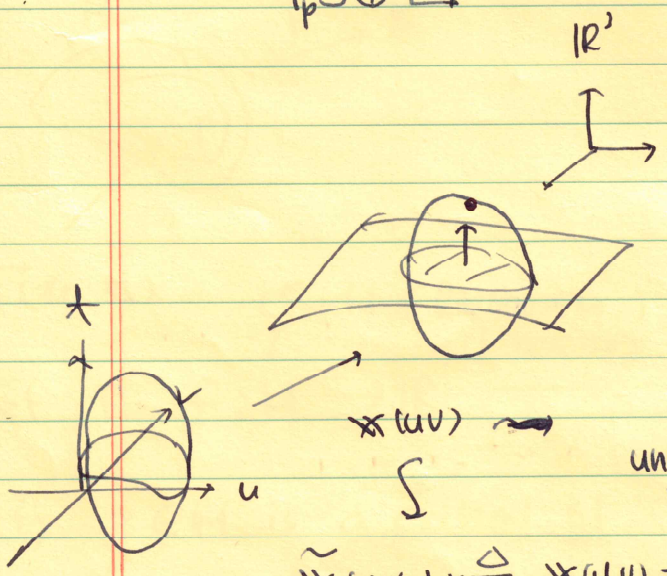
at $p \in S$

$N_{\mathbb{R}^3/S,p}$ (fiber)

$$0 \to T_pS \hookrightarrow T_p\mathbb{R}^3 \to T_p\mathbb{R}^3/T_pS \to 0$$

$$\| \qquad \uparrow \text{with metric}$$

$$T_pS \oplus \square$$



$$\tilde{x}(u,v,t) := x(u,v) + tn$$

$$\Rightarrow J_{\tilde{x}} = \begin{bmatrix} \tilde{x}_u & \tilde{x}_v & \tilde{x}_t \end{bmatrix}$$

↑ Jacobian

for Basis $= \{x_u, x_v, n\}$

$$\Rightarrow J_{\tilde{x}} = \left[\begin{array}{cc|c} \tilde{x}_u \cdot x_u & \tilde{x}_u \cdot x_v & \tilde{x}_u \cdot n \\ \tilde{x}_v \cdot x_v & \tilde{x}_v \cdot x_v & \tilde{x}_v \cdot n \\ \hline \tilde{x}_t \cdot x_u & \tilde{x}_t \cdot x_v & \tilde{x}_t \cdot n \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \begin{bmatrix} E & F \\ F & G \end{bmatrix} + t\begin{bmatrix} e & f \\ f & g \end{bmatrix} & & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 \end{matrix} & & 1 \end{array}\right]$$

$$\tilde{x}_u = x_u$$

$$= \langle x_u + tn_u, x_u \rangle$$

$$= E + te$$

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \left(I_2 - t\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1}\begin{bmatrix} e & f \\ f & g \end{bmatrix} \right)$$

degenerate at $\tilde{x}(u,v,t)$

$\Leftrightarrow \exists \lambda \neq 0$ principal curvature at $x(u,v) \in S$

and $\tilde{x}(u,v,t) = x(u,v, \frac{1}{\lambda})$

called Focal point of $S \subset \mathbb{R}^3$.

쌍곡기하 강의 – Mostow rigidity. 기하겨울학교. 최아론

$\mathbb{H}^n_{\mathbb{R}}$ hyperbolic n-space $(n=2,3)$

$\mathrm{Isom}(\mathbb{H}^n)$ isometry gp of $\mathbb{H}^n$

- $G \leq \mathrm{Isom}(\mathbb{H}^n)$ discrete, torsion free
  (G acts properly discontinuously on $\mathbb{H}^n$)

$\Rightarrow \mathbb{H}^n/G$ hyperbolic mfld. $\pi_1(\mathbb{H}^n/G) = G$.

Remarks. 1) All hyp. mflds come from this construction.

2) Two hyp. mflds $\mathbb{H}^n/G_1$, $\mathbb{H}^n/G_2$ are said to be isometric if $\exists f \in \mathrm{Isom}(\mathbb{H}^n)$ s.t. $fG_1f^{-1} = G_2$.

(cf.
$$\begin{array}{ccc} \mathbb{H}^n & \xrightarrow{f} & \mathbb{H}^n \\ \downarrow & & \downarrow \\ \mathbb{H}^n/G_1 & \dashrightarrow & \mathbb{H}^n/G_2 \end{array}$$
$\exists \bar{f}$: isometry $\Leftrightarrow \exists f$ s.t. $fG_1f^{-1} = G_2$ //

- $\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$, $|ds| = \frac{1}{x_n}|dx|$
  Upper half space model

- all mappings are ori-preserving.

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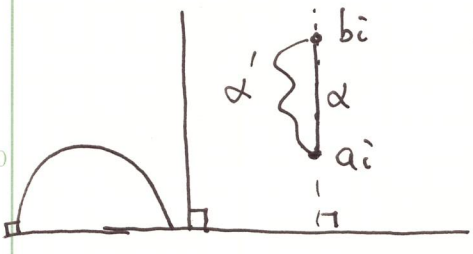
$\mathbb{H}^2$

$$\mathbb{H}^2 = \{ (x,y) \in \mathbb{R}^2 \mid y>0 \} \quad ds^2 = \frac{1}{y^2}(dx^2+dy^2)$$

$$\mathrm{Isom}(\mathbb{H}^2) = PSL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{matrix} a,b,c,d \in \mathbb{R} \\ ad-bc=1 \end{matrix} \right\} / \pm 1$$

$$= \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \right\rangle$$

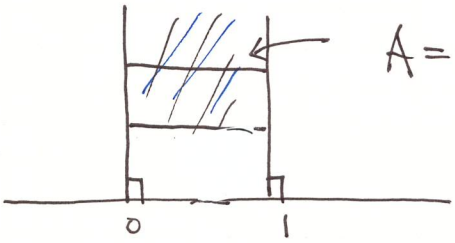
· geodesic



· $d(a_i, b_i) = \ln \frac{b}{a} = \ell(\alpha) \leq \ell(\alpha')$

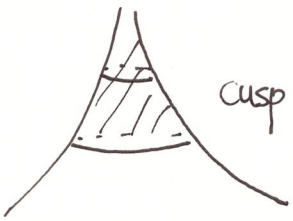
· $\mathrm{Isom}(\mathbb{H}^2)$ acts triply transitively.

· 예제 ① $G = \langle z \xrightarrow{f} z+1 \rangle$ $\quad f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$



$$A = \iint \frac{1}{y^2} dx dy < \infty$$

$\downarrow /G$



· $G' = \langle z \xrightarrow{f'} z+2 \rangle$

$\mathbb{H}^2/G \simeq \mathbb{H}^2/G'$ (isometric)

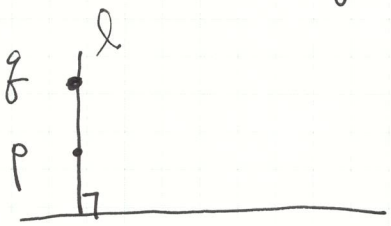
since $\varphi^{-1} f' \varphi = f$, $\varphi(z) = \sqrt{2}z$

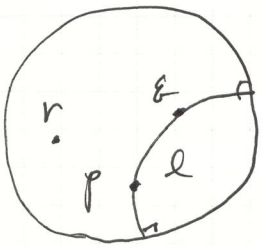
$$\varphi = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \in \mathrm{Isom}(\mathbb{H}^2)$$

$\Rightarrow$ all 2-dim'l cusp are isometric.

Hyperbolic geometry in $\mathbb{H}^2$

properties ① $\forall p \neq q \in \mathbb{H}^2$, $\exists!$ geod. l passing thru p, q

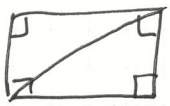




② $l \not\ni r$ $\exists$ infinitely many geod. passing thru r and parallel to l.

③ $\forall l_1, l_2$ ultraparallel, $\exists!$ ortho. geod. γ
$\delta(l_1, l_2) = l(\gamma)$.

정삼각형.

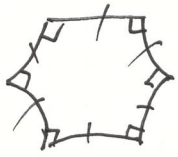
④ Given $\alpha, \beta, \gamma \geq 0$ s.t $\alpha+\beta+\gamma < \pi$,
$\exists \triangle$ with α, β, γ
$\Rightarrow$ Area$\left(\triangle(\alpha,\beta,\gamma)\right) = \pi - (\alpha+\beta+\gamma)$
$\Rightarrow \nexists$ rectangle ($\because$ If so,  Area $= 0$ $*$

Thm. For each $n \geq 3$ and for each $\alpha \in \left(0, \frac{n-2}{n}\pi\right)$,
$\exists$ a (compact) regular hyperbolic n-gon
whose interior angle is α.

For $n \geq 5$, $\frac{n-2}{n}\pi \geq \frac{5-2}{5}\pi > \frac{\pi}{2}$

$\therefore \exists$ a regular right-angled n-gon.

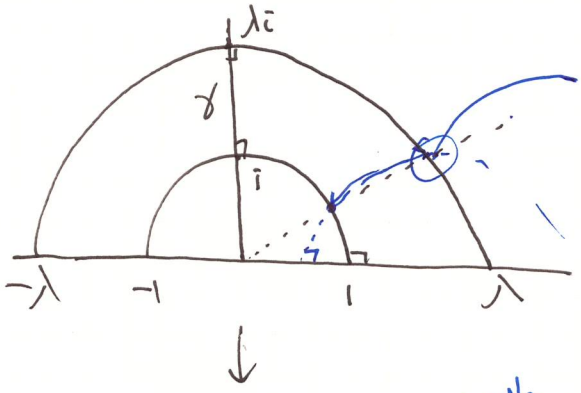
Especially, $\exists$ a regular right-angled Hexagon.

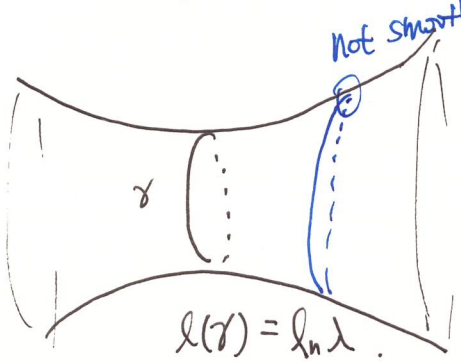


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예제 ② Fix $\lambda > 1$

$G_\lambda = \langle z \mapsto \lambda z \rangle$





① not isometric to a cusp.
(but topologically same)

(∃! geod. loop in every homotopy classes.
(smooth. closed.))

② G_λ is non-isometric for different λ

exer L_1, L_2 two disjoint non-asymptotic geod.



∃! ortho. geod. segment δ

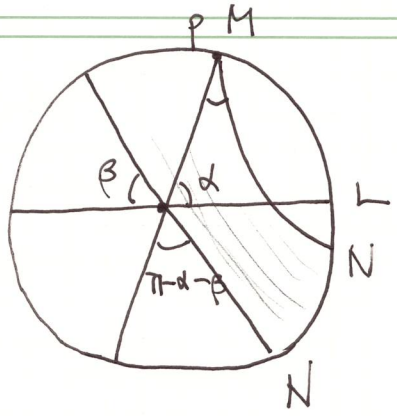
예제 ③ S_2

Fact sum of angels of a hyp. $\triangle < \pi$.

prop. Given $\alpha, \beta, \gamma \geq 0$ s.t $\alpha+\beta+\gamma < \pi$,
∃ a hyp. $\triangle$ with α, β, γ

Sheet 4 Date

proof.



(assume $\alpha, \beta, \gamma < \frac{\pi}{2}$).

consider the family of geod. in $\mathbb{H}^2$ which meet L at angle 0

$\angle(M, N_\cdot) = \pi - \alpha - \beta$ at o

$\angle(M, N) = 0$ at p

By continuity $\exists N$ s.t. $\angle(M, N) = \gamma < \pi - \alpha - \beta$

$o \leq$

□

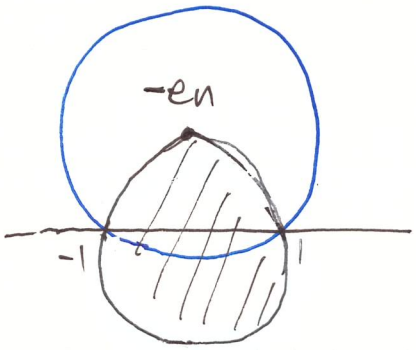
Ball model.

upper half space

- $f : B^n \longrightarrow U^n \in \text{Möb}(\hat{\mathbb{R}}^n)$

$$f(x) = \frac{2(x+e_n)}{|x+e_n|^2} - e_n$$

reflection in the sphere of radius $\sqrt{2}$ centered at $-e_n$.



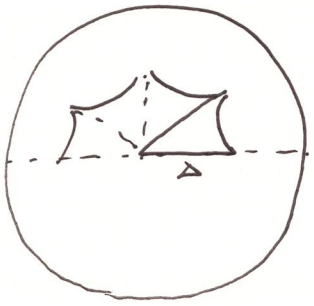
$$Df(0) = 2\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix}$$

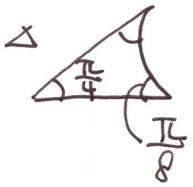
B^n, $ds^2 = \frac{4}{(1-|x|^2)} dx^2$

- Möb. transf. = composition of reflections

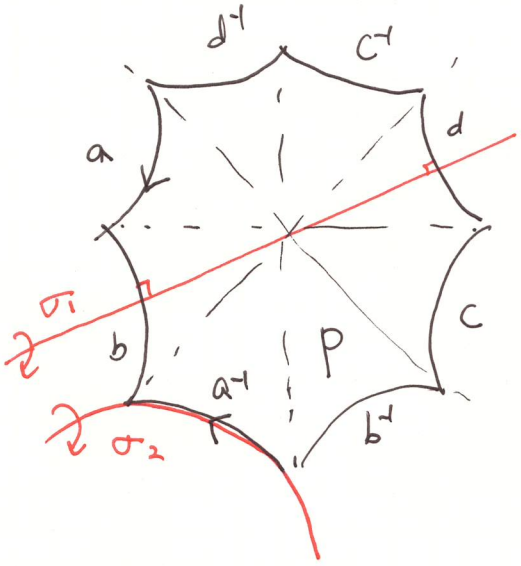
(*)

예제





use 8 copies of △ to construct an regular octagon P in $\mathbb{H}^2$ with interior angle $\frac{\pi}{4}$.



Let $g_a \in Isom(\mathbb{H}^2)$ given by

$g_a = \sigma_2\sigma_1$

Similarly, define $g_b, g_c, g_d \in Isom(\mathbb{H}^2)$

let $G = \langle g_a, g_b, g_c, g_d \rangle$

1) $gP \cap P = \emptyset \quad \forall g \in G - \{id\}$

2) $\bigcup_{g \in G} g\bar{P} = \mathbb{H}^2$

Poincaré's polygon thm

G is discrete (acts properly discontinuously) in $\mathbb{H}^2$

$\Rightarrow \mathbb{H}^2/G \cong$ 

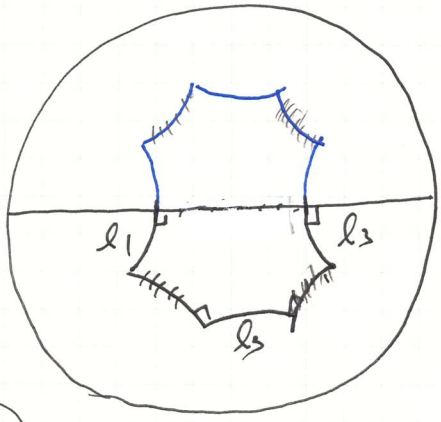
- How many non-isometric S_2 ?

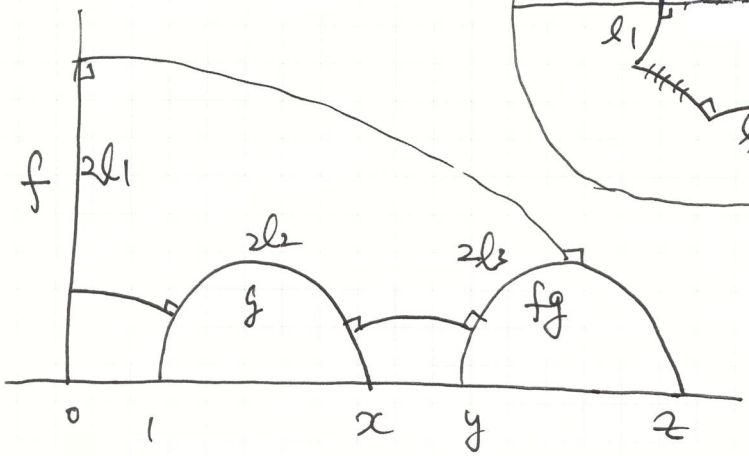
$Teich(S_2) \cong \mathbb{R}^6$

$Teich(S_6) = \mathbb{R}^{6g-6}$

(A) -2

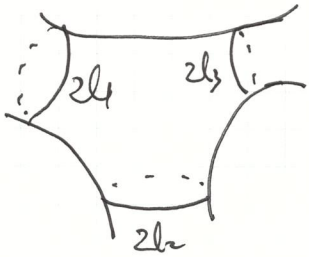
5) $\forall l_1, l_2, l_3 > 0$ $\exists!$ right hexagon with l_1, l_2, l_3





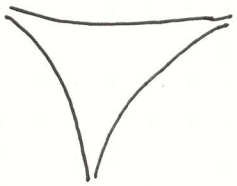
$G = \langle f, g \rangle \leq Isom(\mathbb{H}^2)$ free, discrete.

$\mathbb{H}^2/\langle f,g \rangle$



a pair of pants.

↓ right pants

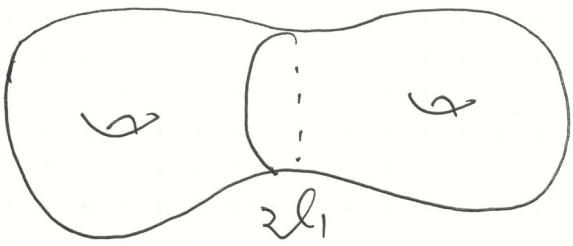


$\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \rangle$

$X2$, $l_2 = l_3$

$\Rightarrow$

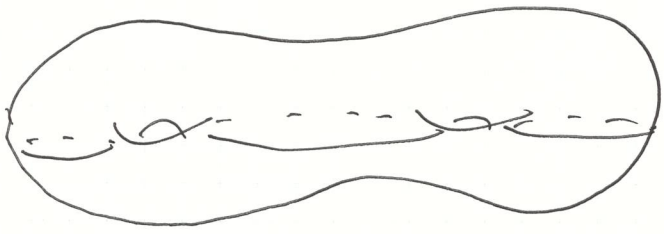
Mashkit Combination thm.



or

$X2$

$\Rightarrow$



pants decomposition

- Every closed orientable surface except the sphere and the torus admits a hyp. metric.

$\chi(S_g) = 2-2g < 0$

$Teich(S_g)$ = all hyp. metric on S up to isometry homotopic to the identity.

$Teich(S_g) \simeq \mathbb{R}^{6g-6}$



However, $Vol(S_g) = |2\pi(2-2g)| < \infty$ by Gauss-Bonnet.

- Mostow rigidity (Ref Foundations of hyp. mflds by Ratcliffe)

$M = \mathbb{H}^n/G$, $N = \mathbb{H}^n/H$, $G, H < Isom(\mathbb{H}^n)$

closed (cpt without ∂), connected, orientable hyp n-mflds with $n \geq 3$

(may assume C^∞ since any conti fctn between diff-mflds is homotopic to a C^∞ map)

$\varphi : M \to N$ a (C^∞) homotopy equivalence is homotopic to an isometry

$M \not\cong N$ homeo $\Longrightarrow$ $M \not\cong N$ isometric

Sheet _____ Date _____

Ref. Hyperbolic geometry, Anderson.

Kleinian gps, Maskit

Geometry and Spectra of compact Riemann surfaces, Buser

Foundations of hyperbolic mflds, Ratcliffe

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< 제 15회 KIAS 기하학 겨울학교 >

Holonomy groups and hyperkähler manifolds in homogeneous varieties

2019년 12월 27일, 29일 박경동 (IBS-CGP)

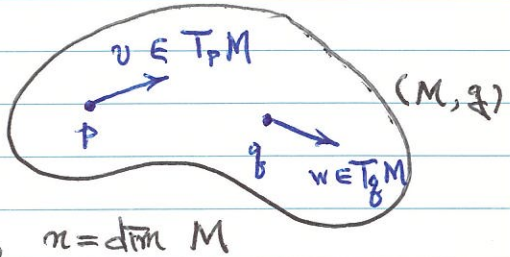
Lecture 1. Holonomy groups and Berger's classification
Lecture 2. Hyperkähler 4-folds in rational homogeneous varieties G/P

§ 1. Holonomy groups

holonomy (홀로노피) 그리스어 어원 : ὅλος (전체) + νόμος (법칙)
Élie Cartan (1869 ~ 1951) 이 1923년 처음 사용한 용어

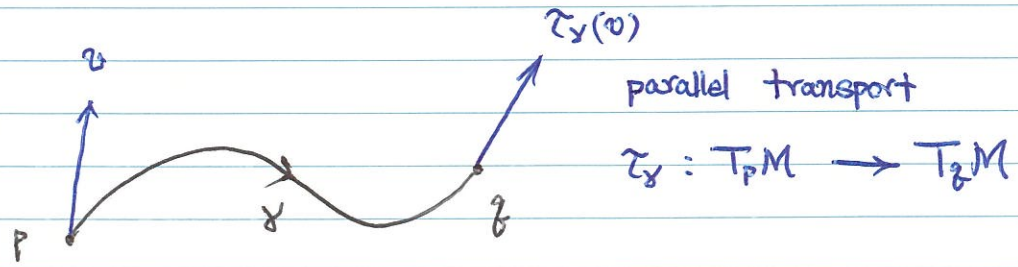
(M, g) : connected Riemannian manifold

Question When are two tangent vectors parallel?



Answer Parallelism on (M, g) is expressed by a subgroup $Hol(M, g) \subset O(n)$, $n = \dim M$
"holonomy group" of (M, g)

Let ∇ be a Levi-Civita connection on (M, g), that is, a torsion-free affine connection compatible with the metric g



Lemma $\gamma : [0, 1] \to M$ smooth curve in M such that $\gamma(0) = p$, $\gamma(1) = q$
Then $\exists!$ parallel vector field V along γ such that $V(0) = v \in T_pM$.

(proof) By definition, the covariant derivative vanishes :

$$\frac{DV}{dt} := \nabla_{\frac{d\gamma}{dt}} V(\gamma(t)) = \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^i \frac{d\gamma^j}{dt} \Gamma^k_{ij} \right) \frac{\partial}{\partial x_k} = 0,$$

where Γ^k_{ij} are the Christoffel symbols.
ODE 기본 정리에 의해 초기 조건이 주어진 연립 선형상미분방정식은 국소적으로 오직 하나의 해를 가진다. QED

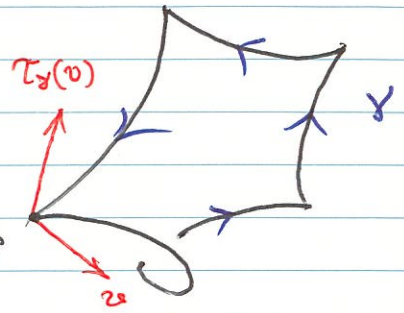
ibis

Definitions. ① If $\gamma : [0,1] \to M$ is a piecewise-smooth curve with $\gamma(0) = \gamma(1) = p$, then we say that γ is a loop based at p.

② parallel transport map along γ

$$\tau_\gamma : T_pM \longrightarrow T_pM, \quad v \longmapsto \tau_\gamma(v)$$

(Check) τ_γ is an invertible linear transformation.



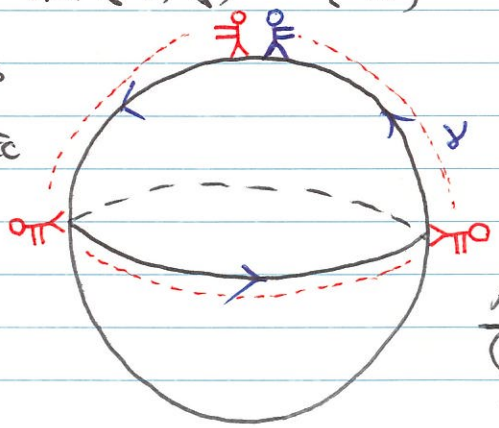
③ the holonomy group of (M, g) based at p

$$Hol_p = Hol_p(M,g) = \{\tau_\gamma \mid \gamma \text{ is a loop based at } p\}$$
$$\subset O(T_pM) \subset GL(T_pM)$$
"orthogonal group" "general linear group"

④ $Hol_p^\circ = \{\tau_\gamma \mid \gamma$ is a contractible loop based at $p\}$
: the restricted holonomy group

Examples (1) $(M,g) = (\text{Euclidean space } \mathbb{R}^n, \text{flat metric})$
$\Longrightarrow Hol(M,g) = \{I_n\}$ trivial group

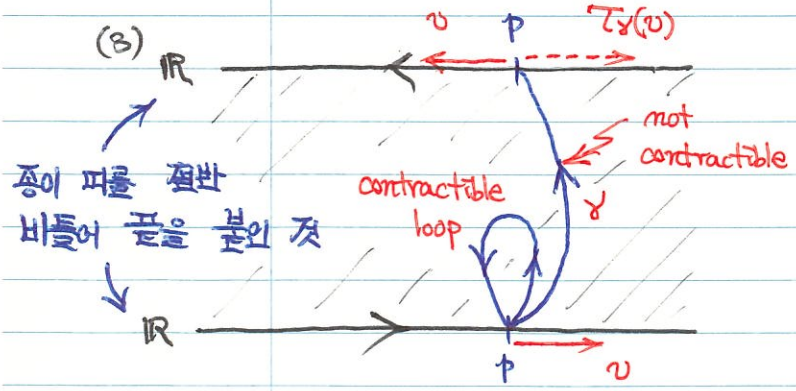
(2) $S^2 \subset \mathbb{R}^3$
g : standard metric



$$Hol(S^2, g_{std}) = SO(2) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\}$$

Note 회전변환 τ_γ 의 회전각은 (구의 반지름이 1일 때) γ로 둘러싸인 구면의 넓이와 같다.

(3)



M = Möbius strip

$$Hol = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset O(2)$$
$$\cong \mathbb{Z}_2$$

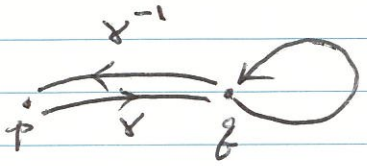
determinant $= -1$

$$Hol^\circ = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset SO(2)$$

Properties of holonomy groups

(1) If M is connected, then $\mathrm{Hol}_p(M,g)$ is independent of a basepoint $p \in M$ (up to isomorphisms).

$\mathrm{Hol}_q = \tau_\gamma \mathrm{Hol}_p \tau_\gamma^{-1}$ (conjugate)



(2) M : orientable $\iff$ $\mathrm{Hol} \subset SO(n)$

(3) reducible case :

$\mathrm{Hol}(M \times N, g \times h) = \mathrm{Hol}(M,g) \times \mathrm{Hol}(N,h)$

$\therefore$ It suffices to consider irreducible Riemannian manifolds.

(4) [Borel - Lichnerowicz, 1952] The restricted holonomy group Hol° is the connected component of the identity in Hol, and is a closed subgroup in $SO(n)$.

$$\pi_1(M) \longrightarrow \mathrm{Hol}/\mathrm{Hol}^\circ$$

surjective homomorphism → the quotient $\mathrm{Hol}/\mathrm{Hol}^\circ$ is countable

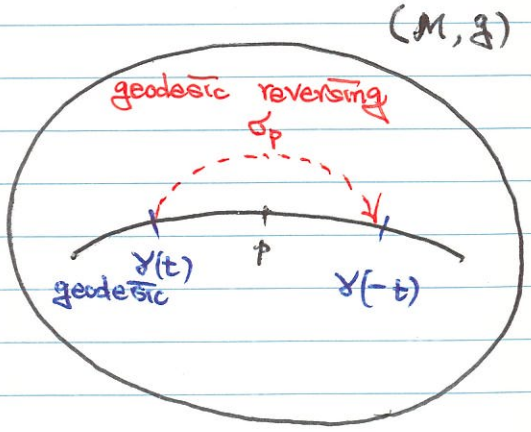
Remark M 이 compact 인 경우 $|\mathrm{Hol}/\mathrm{Hol}^\circ| < \infty$ 일 것으로 추측되었지만, [Wilking, 1999] 이 $|\mathrm{Hol}/\mathrm{Hol}^\circ| = \infty$ 인 simply connected compact Riemannian manifold 의 예를 제시하였다.

§ 2. Symmetric spaces (대칭 공간)

Def A Riemannian manifold (M,g) is said to be a **symmetric space** if for every point $p \in M$ ∃ an isometry $\sigma_p : M \to M$ such that $\sigma_p^2 = \mathrm{id}$ and p is an isolated fixed point of σ_p (점 대칭)

$\Updownarrow$

$\sigma_p(p) = p$ and $d\sigma_p = -\mathrm{Id}$



History 1925년 Élie Cartan 에 의해 도입된 개념이며, Lie theory 를 이용해 Cartan 자신에 의해 완벽히 분류되었다.

ibis

$G = \mathrm{Isom}^\circ(M, g)$: the identity component of Lie group of isometries on (M, g)

$G \curvearrowright M$ acts transitively on M

K = the isotropy subgroup at some point p
$= \{ g \in G \mid g(p) = p \}$

$\Rightarrow$ The symmetric space M is a homogeneous space G/K.

Theorem (M, g) : simply-connected symmetric space
$\longrightarrow M \cong G/K$ and $\mathrm{Hol}(M, g) = K$

Example ① complex Grassmannian of p-dimensional complex subspaces in $\mathbb{C}^{p+q}$

$$Gr(p, p+q) = SU(p+q) / S(U(p) \times U(q))$$

$$\Rightarrow \mathrm{Hol}(Gr(p, p+q)) = S(U(p) \times U(q)) \subset SO(\mathbb{C}^p \otimes \mathbb{C}^q)$$

tensor product representation

② Cayley projective plane

$$\mathbb{OP}^2 = F_4 / \mathrm{Spin}(9)$$

↖ octonionic plane (locally) $\dim \mathbb{OP}^2 = 16$

$$\Rightarrow \mathrm{Hol}(\mathbb{OP}^2) = \mathrm{Spin}(9) \subset SO(16)$$

spin representation

§ 3. Berger's classification

- Marcel Berger (1927 - 2016, 파르셀 베르제르) 프랑스의 미분 기하학자
- 1955년 M. Berger에 의해 분류되었고, 이후 [Jim Simons, 1962]가 간략한 증명 방법을 다시 소개하였다.

↑ 세계에서 가장 부유한 수학자

Theorem [Berger, 1955], [Simons, 1962]
Let (M, g) be an n-dimensional simply-connected Riemannian manifold which is irreducible and not symmetric.
Then $\mathrm{Hol}(M, g)$ is one of the following :

[대칭공간이 아닌 단순연결된 n-차원 기약 리만 다양체의 홀로노미군은 다음 7가지 가운데 하나이다 :]

	$SO(n)$		general Riemannian
$n = 2m$,	$U(m)$	$(m \geq 2)$	general Kähler
$n = 2m$,	$SU(m)$		Calabi-Yau
$n = 4k$,	$Sp(k)$	$(k \geq 2)$	hyperkähler
$n = 4k$,	$Sp(k) \cdot Sp(1)$		quaternionic-Kähler
$n = 7$,	G_2		
$n = 8$,	$Spin(7)$		

(Main idea of the proof)

(i) Ambrose-Singer theorem (1953) : The Lie algebra $\mathfrak{hol}(M, g)$ is exactly the subalgebra of $\mathfrak{so}(T_pM)$ generated by the elements $\tau_\gamma^{-1} \circ R_{\tau_\gamma(v_1), \tau_\gamma(v_2)} \circ \tau_\gamma$

↑ Riemannian curvature tensor

(ii) the 1st and 2nd Bianchi identities on the Riemannian curvature R_{ijkl} of g

(iii) $\nabla R \equiv 0 \iff (M, g)$ is locally symmetric

$\Longrightarrow$ 정리의 조건에 부합하지 않는 Lie algebra들을 배제하고 holonomy group으로 등장할 가능성이 있는 후보들의 목록 끝 ∎

Remarks (1) (i) $U(1) \cong SO(2)$; $e^{i\theta} \longleftrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(ii) $Sp(1) \cong SU(2)$; unit quaternion $z_1 + z_2 j \longleftrightarrow \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}$

where $z_1, z_2 \in \mathbb{C}$, $|z_1|^2 + |z_2|^2 = 1$

In fact, $Sp(1) \cong SU(2) \cong Spin(3) \simeq S^3$,

$\Longrightarrow$ {4-dimensional hyperkähler manifolds} = {Ricci-flat Kähler (complex) surfaces} "K3 surfaces"

(iii) $Sp(1) \cdot Sp(1) = SO(4)$

(2) $Sp(k) \subset SU(2k) \subset U(2k) \subset SO(4k)$

{hyperkähler} ⊂ {Calabi-Yau} ⊂ {Kähler} ⊂ {Riemannian}

but $Sp(k) \cdot Sp(1) \not\subset U(2k)$

$\longrightarrow$ Quaternionic-Kähler manifold is NOT Kähler.

(3) [Alekseevskii, 1968] $Hol(M, g) = Spin(9)$

$\longrightarrow (M, g) \cong (\mathbb{OP}^2, g)$ or its dual,

where $\mathbb{OP}^2 = F_4/Spin(9)$: the Cayley projective plane

∴ $Spin(9)$ 은 분류 Berger 목록에 있었지만 이후에 빠짐.

ibis

Question To construct compact nonsymmetric manifolds with specific holonomy groups $\mathrm{Hol}(M, g)$ in the Berger's list.

① $\mathrm{Hol}(M, g) = SU(m)$

: Yau's theorem on the Calabi conjecture (1954)

Theorem (Yau, 1978) (M, J, g) compact Kähler manifold with Kähler form ω and Ricci form ρ.

Then for every closed real $(1,1)$-form ρ' on M with $[\rho'] \in 2\pi \cdot c_1(M)$, $\exists!$ Kähler metric g' on M with Kähler form ω' such that

- $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$ and
- the Ricci form of g' is ρ'.

(Main idea) · Monge-Ampère equation (2^{nd} order nonlinear elliptic PDE)
· a priori estimate
· Inverse function theorem, ...

⟹ In particular, the first Chern class $c_1(M) = 0$
⟹ M carries a Ricci-flat Kähler metric.

② $Sp(k)$: Beauville, Fujiki, Mukai 등이 1980년대 초에 동일한 hyperkähler 다양체의 예 제시

- $\mathrm{Hilb}^k(K3)$:= the Hilbert scheme of k points on a K3 surface
- generalized Kummer varieties, O'Grady's examples.

③ G_2, $\mathrm{Spin}(7)$: Dominic Joyce (1996)이 예를 건설

④ $Sp(k) \cdot Sp(1)$: still unknown

LeBrun-Salamon Conjecture
All compact quaternionic-Kähler manifolds are symmetric.

Evidences (i) [Poon-Salamon, 1991] proved that this is true when $k=2$.

(ii) Algebraic geometric version of this conjecture:
X : Fano contact manifold
⟹ ?? X is homogeneous

[이 분야의 매우 중요한 난제이지만, 해결은 요원해 보인다.]

ibis

§4. Hyperkähler fourfolds in a rational homogeneous variety G/P

Def Let (M, J) be a complex manifold of complex dimension $2k$ ($=$ real dimension $4k$)

① A holomorphic symplectic structure on M is a closed $(2,0)$-form $\sigma \in H^0(M, \Omega_M^2)$ such that
(Ω_M^2: the sheaf of holomorphic 2-forms)

k^{th} wedge power σ^k is nonzero at every point.
$\Rightarrow$ Since the canonical line bundle $K_M := \Lambda^{2k} T_M^*$ has a nonvanishing global section, $K_M \cong \mathcal{O}_M$ (trivial bundle)
$\Rightarrow$ Ricci-flat (Calabi-Yau)

② An irreducible holomorphic symplectic manifold (IHS) is a simply-connected compact complex manifold such that $H^0(M, \Omega_M^2)$ is generated by σ, i.e., $H^0(M, \Omega_M^2) = \langle \sigma \rangle$.

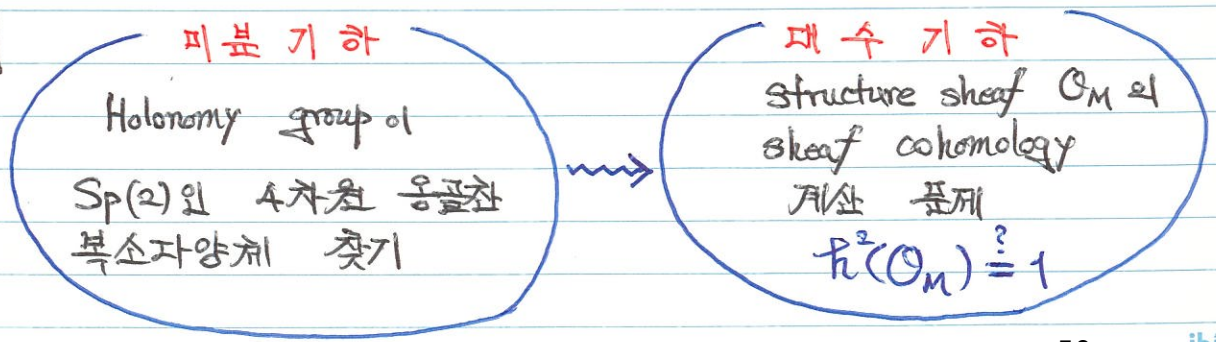
Theorem (Beauville, 1983) M: compact simply-connected
(M, g): hyperkähler $\Longleftrightarrow$ (M, I): IHS
(I: complex structure)

(Sketch of the proof) ($\Rightarrow$) $\sigma(X,Y) = g(JX, Y) + i\, g(IJX, Y)$ is a holomorphic symplectic structure.
($\Leftarrow$) Use Yau's theorem to construct a hyperkähler metric g.

Corollary If (M, J) is a (complex) Kähler manifold of complex dimension 4 such that
$h^0(M, \Omega_M^2) = \dim H^0(M, \Omega_M^2) = 1$,
then M carries a Riemannian metric g with $\mathrm{Hol}(M, g) = Sp(2)$.

Note $h^0(M, \Omega_M^2) = h^{2,0}(M) \overset{\text{by conjugation}}{=} h^{0,2}(M) = h^2(M, \mathcal{O}_M)$
($h^{2,0}$, $h^{0,2}$: Dolbeault cohomology)

Summary



ibis

Examples ① [Beauville-Donagi, 1985]

Z = the Fano scheme of (complex projective) lines in a smooth cubic hypersurface in $\mathbb{P}^5$

$\Longleftrightarrow$ $\quad \mathcal{E} = S^3\mathcal{U}^\vee$, $\quad \mathcal{E} \downarrow \uparrow s$

$Z = Z(s) \subset Gr(2,6) = G(\mathbb{P}^1, \mathbb{P}^5)$

: the zero locus of general global section s

rank $S^3\mathcal{U}^\vee = 2H_3 = 4$, where $\mathcal{U}^\vee$ means the dual of the universal subbundle of rank 2

Note $\quad s \in H^0(Gr(2,6), S^3\mathcal{U}^\vee) \cong Sym^3(\mathbb{C}^6)^\vee = \mathbb{C}[z_1, \cdots, z_6]_3$

Computation of the sheaf cohomology $H^2(\mathcal{O}_Z)$

Koszul complex associated to s of $\mathcal{E} = S^3\mathcal{U}^\vee$:

$$0 \to \Lambda^4\mathcal{E}^\vee \to \Lambda^3\mathcal{E}^\vee \to \Lambda^2\mathcal{E}^\vee \to \mathcal{E}^\vee \to \mathcal{O}_{Gr(2,6)} \to \mathcal{O}_Z \to 0$$

Using the Littlewood-Richardson rule,

$$0 \to \mathcal{O}(-6) \to S^3\mathcal{U}(-3) \to S^4\mathcal{U}(-1) \oplus \mathcal{O}(-3) \to S^3\mathcal{U} \to \mathcal{O}_{Gr(3,6)} \to \mathcal{O}_Z \to 0$$

By the Borel-Weil-Bott theorem, $H^4(Gr(2,6), S^4\mathcal{U}(-1)) = \mathbb{C}$

$\therefore H^2(Z, \mathcal{O}_Z) = \mathbb{C}$

$\Longrightarrow$ Z : hyperkähler 4-fold

② [Debarre-Voisin, 2010]

hyperkähler 4-fold $\quad \Lambda^3\mathcal{U}^\vee \downarrow \uparrow \theta$

$Z(\theta) \subset Gr(6,10)$

rank $\Lambda^3\mathcal{U}^\vee = \binom{6}{3} = 20$

$\dim Gr(6,10) = 6 \times 4 = 24$

Theorem [Park] G/P : a rational homogeneous variety of Picard number 1

$\mathcal{E}$: completely reducible equivariant vector bundle over G/P

$\mathcal{E} \downarrow \uparrow s$

$Z = Z(s) \subset G/P$

If the zero locus Z of general global section of $\mathcal{E}$ is a hyperkähler 4-fold, then Z is of either Beauville-Donagi or Debarre-Voisin.

더 읽을 거리

- Besse, Einstein manifolds, 1987. Chapter 10 & 14.
- Joyce, Compact manifolds with special holonomy, 2000.
- 김홍종, 리만 다양체의 홀로노미, CKMS, 2000.

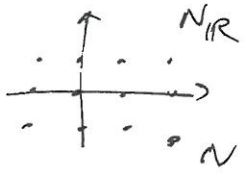
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Toric mirror symm.

1. Toric manifold

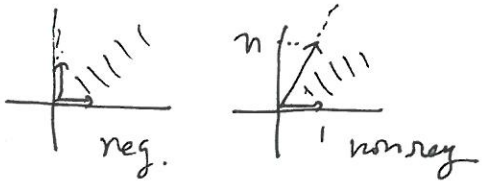
N : lattice ($\cong \mathbb{Z}^n$), $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ ($\cong \mathbb{R}^n$)



Def. $\sigma \subset N_{\mathbb{R}}$: k-dim'l reg cone if

$\sigma = \{ a_1 v_1 + \cdots + a_k v_k \mid a_i \geq 0 \}$ for $v_1, \cdots, v_k \in N$ subset of a $\mathbb{Z}$-basis.

ex.





face of σ (low dim'l cone)

Def. $\Sigma = \{\sigma_1, \cdots, \sigma_s\}$: fan if
cones

(I) closed under taking faces

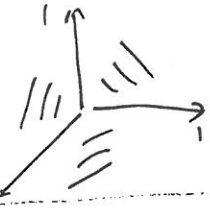
(II) two cones in Σ intersect along a face

not allowed



Σ is complete if $N_{\mathbb{R}} = \sigma_1 \cup \cdots \cup \sigma_s$

ex.



2-dim cones × 3
1-dim cones × 3
0-dim cones × 1
(it's complete)

⊛ For a fan Σ, gen'd by $v_1, \cdots, v_N$. ↪ 1-1

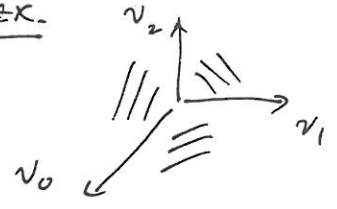
Step 1. (i) take $\mathbb{C}^N$ w/ coord. $z_1, \cdots, z_N$

(ii) If $\{v_{i_1}, \cdots, v_{i_p}\}$ (P: prim coll) does not span a p-dim'l cone but it's proper spans the one w/ the right dim,

Remove $\{z_{i_1} = \cdots = z_{i_p} = 0\}$ ($A(P)$) from $\mathbb{C}^N$.

$\Rightarrow U(\Sigma) = \mathbb{C}^N \setminus Z(\Sigma)$ where $Z(\Sigma) = \bigcup_P A(P)$

ex.



∃ a single prim coll. $\{v_0, v_1, v_2\} = P$

$\Rightarrow A(P) = \{z_0 = z_1 = z_2 = 0\}$

$\Rightarrow U(\Sigma) = \mathbb{C}^3 \setminus \{(0,0,0)\}$.

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Step 2 consider

$$0 \to K \xrightarrow{\cong \mathbb{Z}^{N-n}} \mathbb{Z}^N \to \mathbb{Z}^n \to 0$$

K = kernel $= \{(a_1,\cdots,a_N) \text{ if } a_1v_1+\cdots+a_Nv_N=0\}$.

$e_i \longmapsto v_i$

$\Rightarrow$ obtain $D(\Sigma) \subset (\mathbb{C}^*)^N$ gen'd by $\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^N$, $t \longmapsto (t^{a_1},\cdots,t^{a_N})$

$D(\Sigma) \cong (\mathbb{C}^*)^{N-n}$

Exer. $D(\Sigma) \curvearrowright U(\Sigma)\ (= \mathbb{C}^N \setminus Z(\Sigma))$ is a free action.

Def. $X_\Sigma := U(\Sigma)/D(\Sigma)$: toric mfd assoc. to Σ.

$\dim = n$, $N-n$

(Exer: Σ: complete $\Rightarrow X_\Sigma$: cpt)

ex.



$1\cdot v_0 + 1\cdot v_1 + 1\cdot v_2 = 0 \iff (a_1,a_2,a_3)=(1,1,1)$

$\Rightarrow \mathbb{C}^* \curvearrowright U(\Sigma) = \mathbb{C}^3 \setminus \{0\}$

$(z_1,z_2,z_3) \longmapsto (tz_1, tz_2, tz_3)$

$\therefore X_\Sigma = \mathbb{C}^3\setminus\{0\}/\mathbb{C}^* = \mathbb{P}^2$.

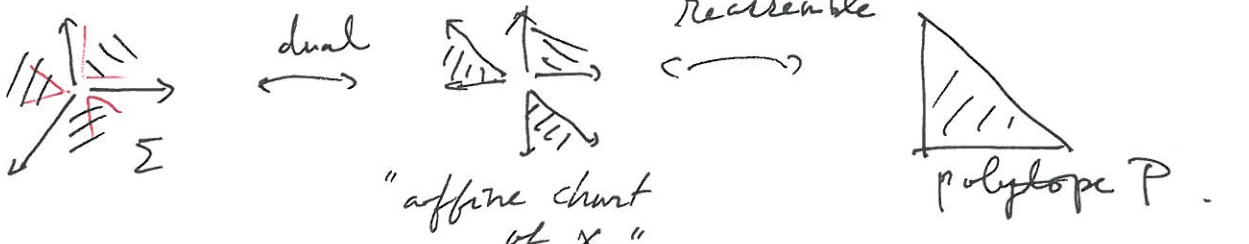
* X_Σ is kähler

kähler quot.

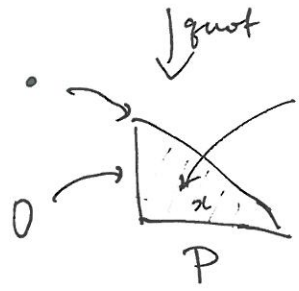
rmk. (i) $\exists (\mathbb{C}^*)^n \curvearrowright X_\Sigma \rightsquigarrow T^n \curvearrowright X_\Sigma^{2n}$

(ii) X_Σ has (toric) homog. coord. $[z_1:\cdots:z_N]$ (up to $D(\Sigma)$-action) coord

* T^n-action on X_Σ



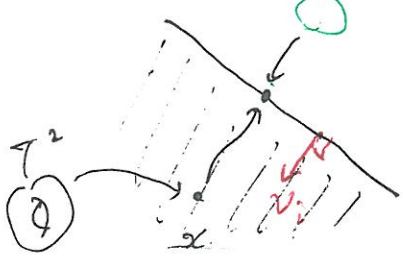
$\Rightarrow$ $X_\Sigma \supset (\mathbb{C}^*)^N$ admits a Lag. T^n-fib'n

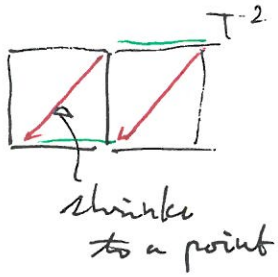


Ralf dim'l.

L_x : Lag Torus

$\begin{cases} |z_1| = c_1 \\ \vdots \\ |z_N| = c_N \end{cases}$ in homog. coord



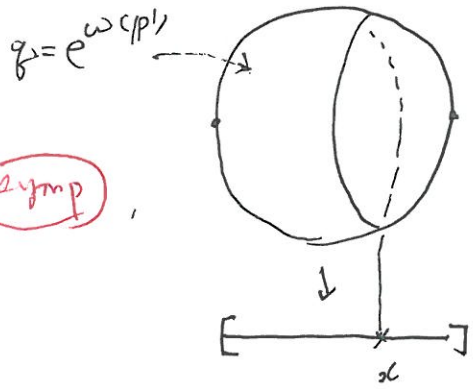


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Mirror symmetry

$\Sigma : \longleftrightarrow \rightsquigarrow X_\Sigma = \mathbb{P}^1$



symp.

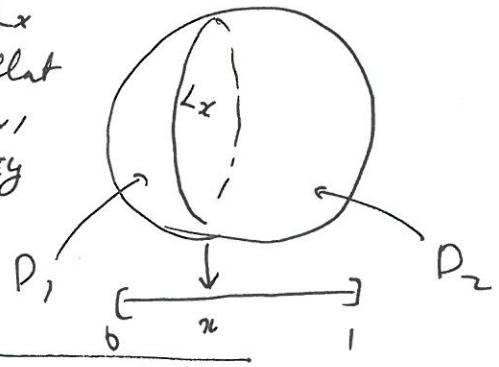
Cohom gen'd by $H = PD[pt]$.

In $QH(\mathbb{P}^1)$, $H \star H = q \cdot 1$

count



equip L_x w/ U(1)-flat bundle w/ hol $= e^{iy}$



vs.

$W : \mathbb{C}^* \to \mathbb{C}$

$z \longmapsto z + \frac{q}{z}$.

$\partial_z W = 1 - \frac{q}{z^2}$

$\Rightarrow Jac(W) = \frac{\mathbb{C}[z,z^{-1}]}{\langle \partial_z W \rangle} = \frac{\mathbb{C}[z,z^{-1}]}{z^2 = q}$

How do we get W?

e.g. Floer theory.

Weighted count of



$D_1 \rightsquigarrow q^\alpha e^{iy} =: z$

$D_2 \rightsquigarrow q^{1-\alpha} e^{-iy} = \frac{q}{z}$

$W = z + \frac{q}{z}$.

- Important to understand hol. disks bounding L_x mirror for toric X_Σ, such disks are all classified. [Cho-Oh]

In homog. coord, $z_i = c_i \prod \frac{z - \alpha_{i,k}}{1 - \bar{\alpha}_{i,k} z}$. ← diff of L_x

i.e. $\widetilde{W} : (D^2, \partial D^2) \to (\mathbb{C}^N \setminus Z(\Sigma), |z_i| = \text{const})$

$z \longmapsto (z_1, \cdots, z_N)$

& z_i must be a product of Blaschke factors c. $\frac{z-\alpha}{1-\bar{\alpha}z}$.

Thm [Cho-Oh]. For X_Σ : toric Fano, W is obtained by counting disks w/ a single Blaschke factor. $W : (\mathbb{C}^*)^n \to \mathbb{C}$

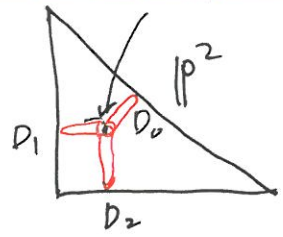
"basic disks" : $\mu = 2$

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L_x $|z_i| = c_i$

ex.



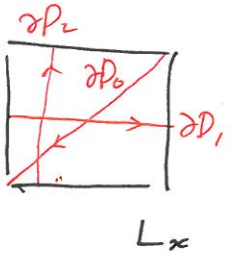
$D_1 \left[c_1 \frac{z-\alpha}{1-\bar{\alpha}z} : c_2 : c_3 \right]$

$D_2 \left[c_1 : c_2 \frac{z-\beta}{1-\bar{\beta}z} : c_3 \right]$

$D_0 \left[c_1 : c_2 : c_3 \frac{z-\gamma}{1-\bar{\gamma}z} \right]$

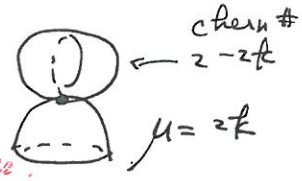
$= \left[c_1 \left(\frac{z-\gamma}{1-\bar{\gamma}z}\right)^{-1} : c_2 \left(\frac{z-\gamma}{1-\bar{\gamma}z}\right)^{-1} : c_3 \right]$

lifting (4) toric divisor exactly once



$W = z_1 + z_2 + \frac{q}{z_1 z_2}$.

rmk. For non-Fano, one needs to count in addition to basic disks!



Lag. Floer theory

(X, ω) : symp. $\supset L, L'$ Lag.

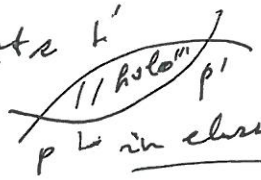


$CF(L, L') = \bigoplus_{p \in L \cap L'} \mathbb{K}\langle p \rangle$

C

d. $\quad dp = \sum_{p', \beta} n_\beta(p, p') q^{\omega(\beta)} \cdot p'$

counts L' // holom p' p in class β.



*. What if $L = L'$? $CF(L, L) = H^*(L)$ $\ni d$

(L, ∇) (L, ∇) $\quad \cup \alpha$



$d\alpha = \sum_{\beta, \alpha'} n_\beta(\alpha, \alpha') q^{\omega(\beta)} \cdot \alpha'$ hol$_{\partial\beta} \nabla$.

Def. $HF(L_1, L_2) :=$ d-cohom. of $CF(L_1, L_2)$

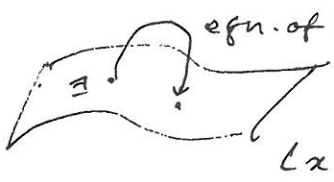
Fact. $HF(L, L') \cong HF(L, \varphi_H(L'))$, $\forall \varphi_H$: Hamil. iso (flow along eqn of motion)

Coming back to MS: $X_\Sigma \longleftrightarrow W$.
mirror.

$\varphi^* e^{i\gamma} = z \in \mathrm{Crit}(W) \longleftrightarrow$ cancellation of disk counting ($d=0$) $\longleftrightarrow HF((L, \nabla), (L, \nabla)) \neq 0$ $\quad (L_x, e^{i\gamma})$

$\Rightarrow HF(L_x, \varphi_H(L_x)) \neq 0$ $\forall$ Ham. isot. φ_H

$\Rightarrow L_x$ cannot be displaced from itself by Ham. flow.



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Homological mirror symm. $X_\Sigma \longleftrightarrow W : (\mathbb{C}^*)^n \to \mathbb{C}$

[Kontsevich 94"]. $Fuk(X_\Sigma) \cong MF(W)$ ($\mathbb{Z}_2$-gr. (free) $\mathbb{C}[z_1^{\pm}, \cdots, z_n^{\pm}]$-mod

obj: Lag
mor: $hom(L, L') = HF(L, L')$

obj: M, d s.t. $d^2 = (W-\lambda)\cdot id$.
i.e. $M_0 \underset{Q}{\overset{P}{\rightleftarrows}} M_1$ s.t.
$PQ = (W-\lambda)\cdot id_{M_0}$
$QP = (W-\lambda)\cdot id_{M_1}$

sympl

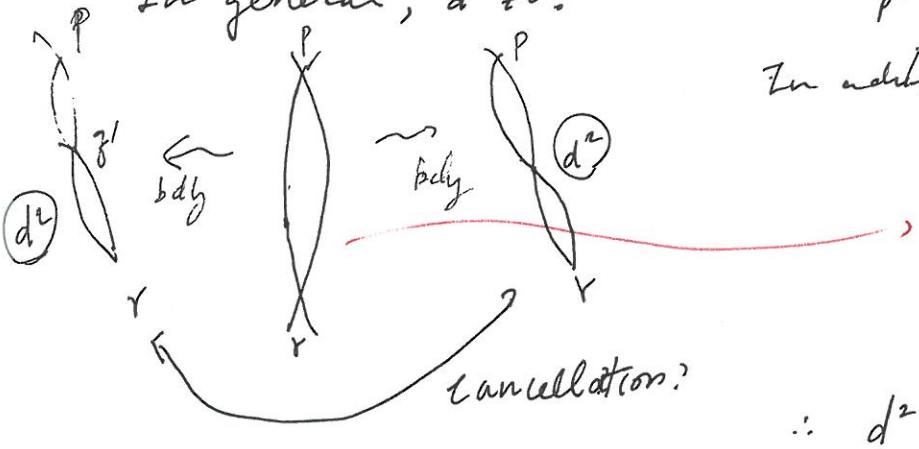
cpx

Q). Which MF corresp. to $L \in Fuk(X_\Sigma)$?

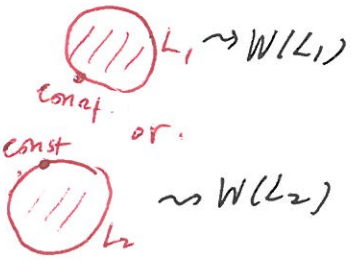
* Floer theory revisit; $CF(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} \mathbb{K}\langle p \rangle$

$$dp = \sum_{p'} \#\left(\frac{\beta}{p \quad p'}\right) q^{\omega(\beta)} \cdot p'$$

In general, $d^2 \neq 0$!

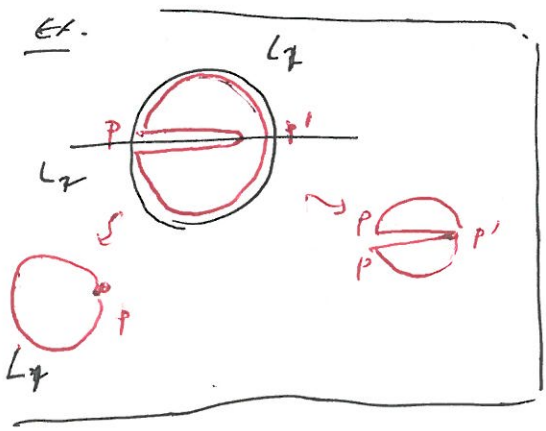


In addition

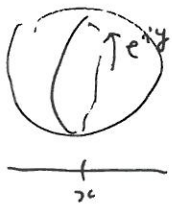


$\therefore d^2 = W(L_1) - W(L_2)$ — ★.
number number.

Ex.



Let L_1 vary around in the family $(L_x, \nabla = e^{\sqrt{-1}y})$



Set $L = L_2$ & fix.
$\lambda := W(L)$

★ $\Longleftrightarrow$ $d^2 = W(z) - \lambda$ MF!

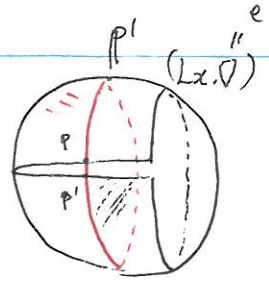
rmk. $CF((L_x, \nabla = e^{\sqrt{-1}y}), L)$ admits $\mathbb{Z}_2$-grading so that deg $d = 1$

$\Rightarrow CF_0 \underset{d}{\overset{d}{\rightleftarrows}} CF_1$. & $d^2 = (W-\lambda)\cdot id$

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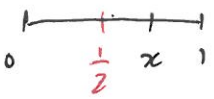
ex.



$\longleftrightarrow \quad W = z + \frac{q}{z}$

equator L $\longleftrightarrow$ MF ? $CF((Lx,0),L) =: M$

$M_0 = \langle p \rangle$, $M_1 = \langle p' \rangle$



$p \dashrightarrow p'$: $q^{x-\frac{1}{2}} e^{iy} - 1 = \frac{1}{\sqrt{q}} \cdot z - 1$

$p' \longrightarrow p$: $q^{\frac{1}{2}} - q^{(-x)} \cdot e^{-iy} = \sqrt{q} - \frac{q}{z}$

$\left(\frac{1}{\sqrt{q}} z - 1\right) \cdot \left(\sqrt{q} - \frac{q}{z}\right) = z + \frac{q}{z} - 2\sqrt{q}$

*. In general, it's hard to find such a part of L_x $\quad$ critv. of W.

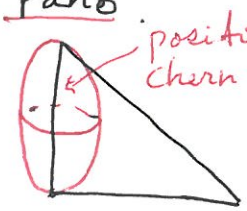
[Cho-H.-Lau] construct a mirror W from a toric fiber L w/ $\mathbb{C}^*$-flat connections $\Rightarrow$ explicit computation of mirror MF

• HMS for toric Fano

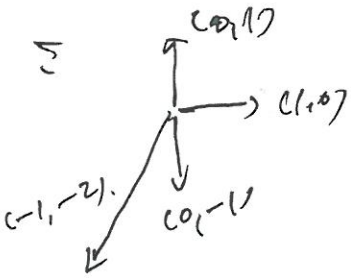
For non Fano?

need to count (hard to analyze)

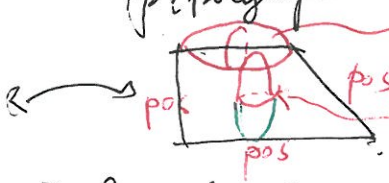




ex. $\mathbb{F}_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$; semi-Fano, still no neg. spheres.



P : polytope



[Chan-Lau] computed $\#\left(\mu=2\right)$

by comparing it w/ GW.

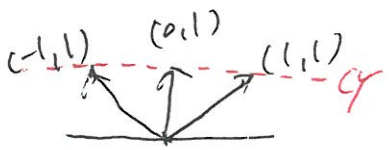
Sphere counting.

What about toric CY?

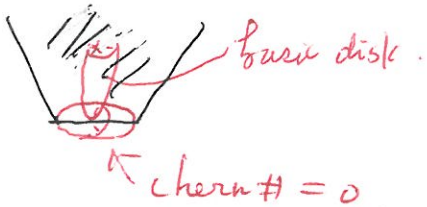
ex. $K_{\mathbb{P}^1}$

$c_1 = 0$. $\rightsquigarrow$ $\exists$ Chern # = 0 spheres.

Σ



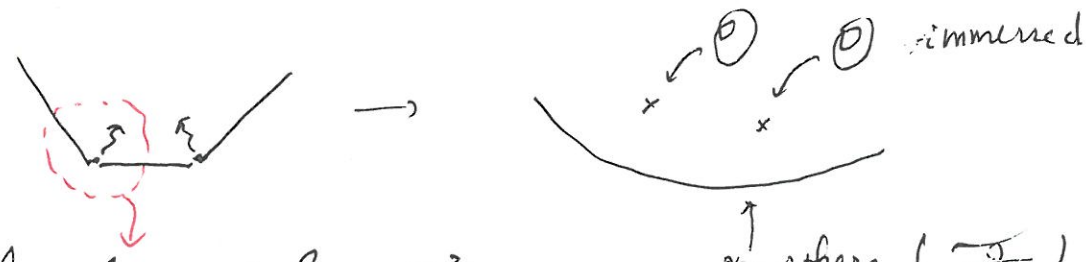
$\longleftrightarrow$



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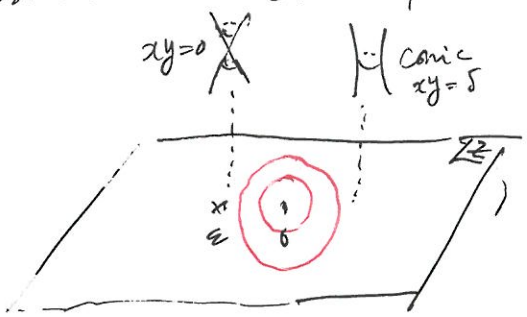
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Idea : use "non-toric" toric fib'n on X.



local model : $\mathbb{C}^2$.

consider a conic fib'n $\mathbb{C}^2 \to \mathbb{C}$
$(x,y) \mapsto xy+\varepsilon \quad \varepsilon \neq 0$
$\parallel$
z

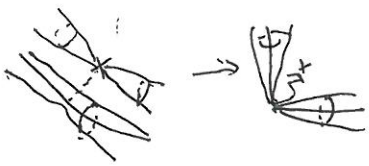


concentric circle about 0
$\times$
circle on conic fiber $\{t\}$
$\Downarrow$
T^2.

more precisely,

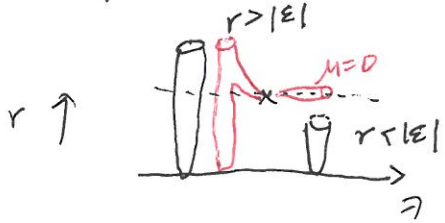
$T_{\lambda,r} := \{ |u|^2 - |v|^2 = \lambda, |x| = r \}$ give a fiber & λ, r vary

Note ; $\exists$ a singular fiber $T_{0,\varepsilon}$. ⊙ $\simeq S^2$.



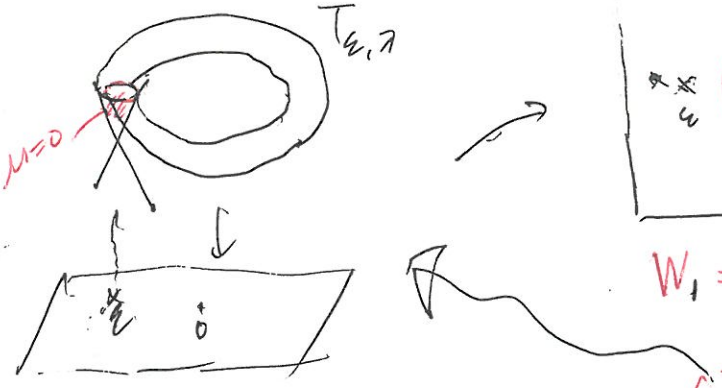
rmk. $\varepsilon \to 0$, get back toric fib'n.

[Rels. disks bounding $T_{\lambda,r}$]



for $T_{\lambda,r}$ w/ $r > |\varepsilon|$, $\exists$ two different classes of disks.





"wall-crossing"
($T_{\lambda,\varepsilon}$ bounds $u=0$ disks !)

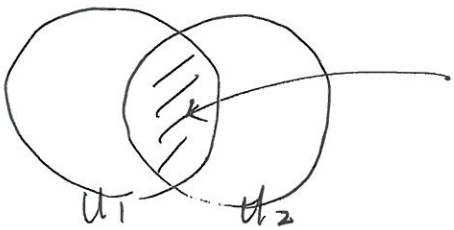
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To match two different W's, need coord. change.

i.e. the mirror consists of two charts U_1 & U_2

$U_1 \cong (\mathbb{C}^*)^2 \ni (z_1, z_2)$ $\quad W = z_2$

$U_2 \cong (\mathbb{C}^*)^2 \ni (z_1', z_2')$ $\quad W = z_2' + z_1' z_2'$

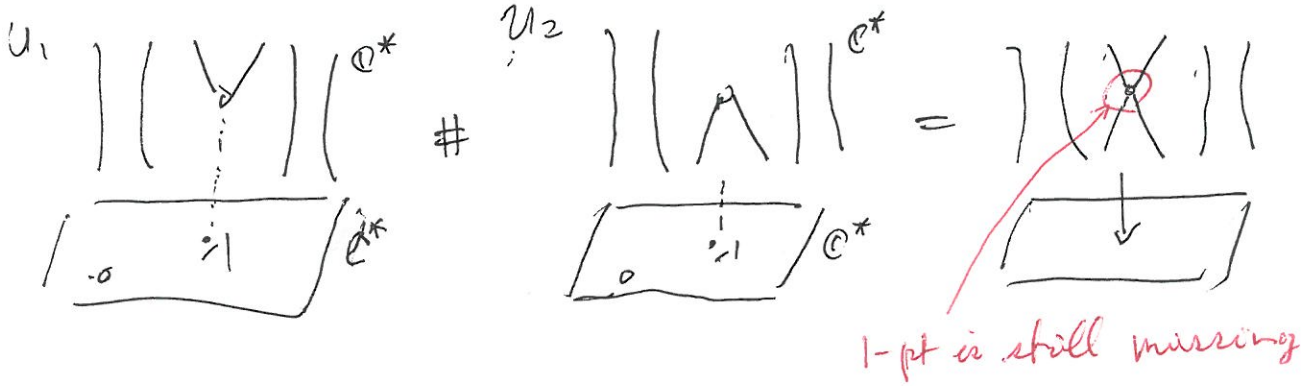


gluing : $z_1^0 = z_1'$

$z_2^0 = z_2'(1 + z_1')$

W_1 $\quad W_2$

Gluing picture



1-pt is still missing

rmk. missing point can be filled up by studying T

Wall crossing technique can be applied to Grassmannians, (& cluster systems)
as well as toric cf.
[Chan-Lau-Leung] [H.-Kim-Lau]

MooKeuk

2019/12/29 (일) 김현규 talk (제 15회 고등과학원 기하학 겨울학교)

title : Goldman symplectic structure on space of surface group representations, and skein algebra

references : Goldman (1984, 1986), Turaev (1991)

Goldman: "The symplectic nature of fundamental groups of surfaces"
"Invariant functions on Lie groups and Hamiltonian flows of surface group representations"

Turaev: "Skein quantization of Poisson algebras of loops on surfaces"

one of my research topics : Quantization of Teichmüller space of a non-compact Riemann surface, and its relationship with Kauffman bracket skein algebra.

오늘 주제랑은 다름

Teichmüller space / cluster variety : Poisson / symplectic geometry

~> quantization

related

Skein algebra : 3d topology → Geometric Topology 의 일종

(많은 motivation은 physics)

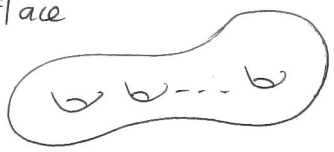
I. Topology (3d topology 말고. geometry를 하기 위한 공간.)

general object to study : moduli space (modulus ↔ parameter) of geometric structures on a manifold

좋은점? - 물리: phase space
- get examples of nice (symplectic) manifold / variety

오늘은 이 중 specific example 들만.

today: S = closed oriented surface



genus $g > 1$ ↑ assume.

G = a Lie group (with a mild condition) ← manifold 이면서 group

(or, an "algebraic group") ← variety 이면서 group.

main hero : $\boxed{\mathrm{Hom}(\pi_1(S), G)/G}$

$= \{$ group homomorphisms $\phi : \pi_1(S) \to G \}$ / conjugation by an element of G

($\pi_1(S)$: fundamental group of S)

(즉 $\phi \sim \phi' \overset{\text{def.}}{\iff} \exists g \in G$ s.t. $g \cdot \phi([\gamma]) \cdot g^{-1} = \phi'([\gamma])$ for all $[\gamma] \in \pi_1(S)$.)

"representation variety" 라고도 부른다. (e.g. when $G = GL(n, \mathbb{R})$ ⇝ set of all n-dim'l (real) representations of G up to isomorphism)

---

* Goldman (1984) showed ∃ a natural symplectic structure (non-degenerate closed 2-form) on ("smooth part" of) $\mathrm{Hom}(\pi_1(S), G)/G$.

Examples : ① $G = PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\pm Id$

$\rightsquigarrow$ Teichmüller space of S, with Weil-Petersson Kähler form

② $G = \mathbb{R} \rightsquigarrow H^1(S;\mathbb{R})$, with cup-product symplectic structure

③ $G = U(1) \rightsquigarrow$ Jacobi variety of a Riemann surface Σ
(circle) homeomorphic to S (, with Kähler form)

④ $G = U(n) \rightsquigarrow$ space of certain vector bundles on Σ

⋮

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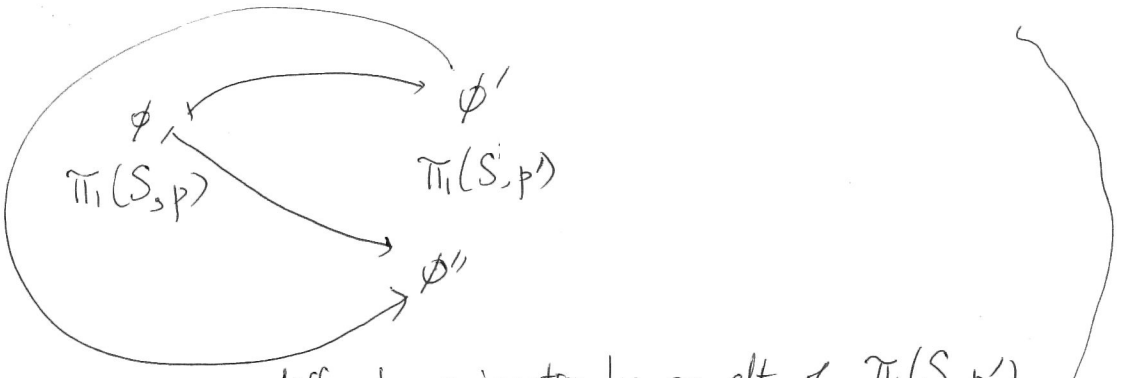
* Let's do it.

* Question : Choice of basepoint? $\pi_1(S,p)$? $\pi_1(S,p')$?

$\exists$ basepoint change isomorphism (not unique)

: gives a map $\phi \rightsquigarrow \phi'$

$(\phi : \pi_1(S,p) \to G)$ $(\phi' : \pi_1(S,p') \to G)$



differ by conjugation by an elt of $\pi_1(S,p')$

$\therefore$ the assignment $\phi \mapsto \phi'$ is well-def. up to conjugation in G

$\therefore$ ambiguity of choice of basept is absorbed by conjugation in G.

* Question : Is $\mathrm{Hom}(\pi_1(S), G)/G$ a manifold? topology?

S : genus g surface

$\rightsquigarrow \pi_1(S) \cong \langle A_1, B_1, \cdots, A_g, B_g \mid A_1B_1A_1^{-1}B_1^{-1} \cdots A_gB_gA_g^{-1}B_g^{-1} = 1 \rangle$

$\therefore \mathrm{Hom}(\pi_1(S), G) \underset{\text{as sets}}{\overset{1:1}{\longleftrightarrow}} \left\{ 2g\text{-tuples } (A_1, B_1, \cdots, A_g, B_g) \in G \times \cdots \times G \;\middle|\; A_1B_1A_1^{-1}B_1^{-1} \cdots A_gB_gA_g^{-1}B_g^{-1} = 1 \right\}$

↑ "polynomial" equations (in entries)

smooth (manifold이면서 group)

* When G is a Lie group : the set on the right-hand-side induces a topology on $\mathrm{Hom}(\pi_1(S), G)$.

(subspace topology from product topology on $G \times \cdots \times G$)

$\rightsquigarrow$ 그의 manifold 같이 생김

* When G is an algebraic group (algebraic variety 이면서 group)

(locally, zero set of polynomials)

: $\mathrm{Hom}(\pi_1(S), G)$ is an algebraic variety (locally, polynomial 들의 zero set 이다.)

* Note : $\mathrm{Hom}(\pi_1(S), G)$ is a singular variety.

quotient $\mathrm{Hom}(\pi_1(S), G)/G$ is even more singular.

$\rightsquigarrow$ manifold가 모두는 아님.

어떤 특정한 subset만 취해야 manifold 됨. 오늘은 skip...

("smooth part")

(그동 $\mathrm{Hom}(\pi_1(S), G)$ 혹은 $\mathrm{Hom}(\pi_1(S), G)/G$ 가 manifold라고 가정.)

II. Geometry

이용? tangent space?

need condition on G.

Def An orthogonal structure on a real Lie group G is a non-degenerate symmetric bilinear form

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},$$

where $\mathfrak{g}$ = Lie algebra of G, ($\cong T_1 G$)

that is invariant under $Ad(G)$, i.e.

$$B(Ad(g)X, Ad(g)Y) = B(X,Y), \quad \forall g \in G, \ \forall X, Y \in \mathfrak{g}$$

("Adjoint action")

Assumption : G has an orthogonal structure.

ex) semisimple group, with Killing form.

ex) reductive group ($Ad(G)$ semisimple)

ex) 〃 〃 of matrices : $B(X,Y) = tr(XY)$ (trace)

e.g. $GL(n,\mathbb{K})$, $SL(n,\mathbb{K})$, $Sp(n,\mathbb{K})$, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$

$O(p,q)$, $U(p,q)$, ...

~> 오늘은 시간상 거의 $GL(n,\mathbb{R})$만.

Question from audience : what is an example of G that does not have an orthogonal structure?
-> Goldman's paper (1984) says it seems not to have been characterized.

* local structure at $\mathrm{Hom}(\pi_1(S), G)$?

Let $\phi \in \mathrm{Hom}(\pi_1(S), G)$.

Let $\{\phi_t\}_t$ be a differentiable path in $\mathrm{Hom}(\pi_1(S), G)$ s.t. $\phi_0 = \phi$.

$$\phi_t(\gamma) = \exp\big(t \cdot \underbrace{u(\gamma)}_{\mathfrak{g}} + \underbrace{O(t^2)}_{\text{higher terms in } t}\big) \cdot \phi(\gamma) \quad (\text{for small } t\text{'s}) \qquad \forall \gamma \in \pi_1(S)$$

for some function $u : \pi_1(S) \to \mathfrak{g}$.

From $\phi_t(\gamma\eta) = \phi_t(\gamma) \cdot \phi_t(\eta)$ $\quad \forall \gamma, \eta \in \pi_1(S)$

↘ t 미분값 at $t=0$.

$\rightsquigarrow$ get $\boxed{u(\gamma\eta) = u(\gamma) + \mathrm{Ad}(\phi(\gamma))\, u(\eta)}$ (*) (exercise)

e.g. for matrix groups.

↓

뜻은의미? "group 1-cocycle" 이다.

Def For $\phi \in \mathrm{Hom}(\pi_1(S), G)$, denote by $\mathfrak{g}_{\mathrm{Ad}\phi}$ the $\pi_1(S)$-module obtained by composition $\pi_1(S) \xrightarrow{\phi} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$

(i.e. the symbol $\mathfrak{g}_{\mathrm{Ad}\phi}$ stands for the following representation of the group $\pi_1(S)$:

underlying vector space is: $\mathfrak{g}$

$\pi_1(S)$-action is : $\gamma . X := \mathrm{Ad}(\phi(\gamma))X$, $\forall \gamma \in \pi_1(S), \forall X \in \mathfrak{g}$.

(ϕ는 고정).)

Def $Z^1(\pi_1(S); \mathfrak{g}_{Ad\phi}) =$ the space of 1-cocycles of the group $\pi_1(S)$ with values in the $\pi_1(S)$-module $\mathfrak{g}_{Ad\phi}$

$$= \{ u : \pi_1(S) \to \mathfrak{g}_{Ad\phi} \mid \text{satisfying } (*) \}$$

(Remark : 임의의 group과 그 group의 module에 대해서 정의 가능.
"group cohomology" 검색..)

Lem. The (Zariski) tangent space to $\mathrm{Hom}(\pi_1(S), G)$ at ϕ is $Z^1(\pi_1(S); \mathfrak{g}_{Ad\phi})$.

(즉, $T_\phi \mathrm{Hom}(\pi_1(S), G) = Z^1(\pi_1(S); \mathfrak{g}_{Ad\phi})$)

↑ 첫시간.

---

* Now, consider the action of G on $\mathrm{Hom}(\pi_1(S), G)$ by conjugation.

⇝ study the tangent space of the orbit $G\phi$?

(이걸 구한 뒤, $T_\phi \mathrm{Hom}(\pi_1(S), G)$를 이걸로 자르면 된다.)

Let $\phi \in \mathrm{Hom}(\pi_1(S), G)$.

Let $\{g_t\}_t$ be a differentiable path in G s.t. $g_0 = g$.

⇝ $g_t = \exp(t \cdot u_0 + O(t^2))$ for some $u_0 \in \mathfrak{g}$. (for small t's)

Define $\phi_t(\gamma) = g_t^{-1} \cdot \phi(\gamma) \cdot g_t$, $\forall \gamma \in \pi_1(S)$, $\forall$ small t.

$\rightsquigarrow \{\phi_t\}_t$ is a differentiable path in $\mathrm{Hom}(\pi_1(S), G)$

$\rightsquigarrow$ the cocycle $u : \pi_1(S) \to \mathfrak{g}_{\mathrm{Ad}\phi}$ corresponding to this path $\{\phi_t\}_t$

is $\quad u(\gamma) = \mathrm{Ad}(\phi(\gamma))\, u_0 - u_0$.

(exercise, e.g. for matrix groups)

$\rightsquigarrow$ these u's are "1-coboundaries" $\in B^1(\pi_1(S); \mathfrak{g}_{\mathrm{Ad}\phi})$

Lem. $T_{[\phi]}\left(\mathrm{Hom}(\pi_1(S), G)/G\right) = Z^1(\pi_1(S); \mathfrak{g}_{\mathrm{Ad}\phi}) / B^1(\pi_1(S); \mathfrak{g}_{\mathrm{Ad}\phi})$

$= H^1(\pi_1(S); \mathfrak{g}_{\mathrm{Ad}\phi})$

(1st group cohomology

(of group $\pi_1(S)$, with coefficients in the $\pi_1(S)$-module $\mathfrak{g}_{\mathrm{Ad}\phi}$.)

By "universal coefficient theorem" of cohomology,

$$H^1(\pi_1(S); \mathfrak{g}_{\mathrm{Ad}\phi}) \cong \mathrm{Hom}_{\mathbb{Z}}\left(H_1(\pi_1(S)), \mathfrak{g}_{\mathrm{Ad}\phi}\right)$$

(group cohomology) — $H_1(\pi_1(S))$: 1st group homology; abelianization of group $\pi_1(S)$

$$\cong \mathrm{Hom}_{\mathbb{Z}}\left(H_1(S), \mathfrak{g}_{\mathrm{Ad}\phi}\right) \quad (\because \text{대수위상})$$

$H_1(S)$: singular homology

$$\cong H^1(S; \mathfrak{g}_{\mathrm{Ad}\phi})$$

(singular cohomology)

Consider

$$H^1(S;\mathfrak{g}_{Ad\rho}) \times H^1(S;\mathfrak{g}_{Ad\rho}) \to H^2(S;\mathbb{R})$$

given by cup-product of cohomology (검색!)

with coefficient pairing induced by $B:\mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$.

Note : $H^2(S;\mathbb{R}) \cong \mathbb{R}$ (e.g. by Poincaré duality)

Thm [Goldman]

This bilinear map

$$T_{[\rho]}(\mathrm{Hom}(\pi_1(S),G)/G) \times T_{[\rho]}(\mathrm{Hom}(\pi_1(S),G)/G) \to \mathbb{R}$$

is skew-symmetric, non-degenerate, and is "closed".

- skew-symmetric (∵ cup-product is skew-symmetric, B is symmetric)
- non-degenerate (∵ Poincaré duality for S, non-degeneracy of B)
- "closed" (as a differential 2-form on $\mathrm{Hom}(\pi_1(S),G)/G$) (proof 어렵다)

⇝ Goldman symplectic form on $\mathrm{Hom}(\pi_1(S),G)/G$.

---

ex) $G = PSL(2,\mathbb{R})$ 일때

Teichmüller space of S 의 Weil-Petersson form 과 "일치".

Question from audience : B 를 다른것 쓰면 어떻게 되나? -> Goldman's paper에는 별 얘기 없음.
-> 이거 질문했던 분은 제게 연락 주시면 같이 고민해보죠..

Ⅲ. Algebra of functions (on $\mathrm{Hom}(\pi_1(S), G)/G$)

Let's find nice fcns on $\mathrm{Hom}(\pi_1(S), G)$ invariant under G.

Def A C^1 fcn $f: G \to \mathbb{R}$ is an *invariant* fcn if

it is invariant under inner automorphisms

$$f(ghg^{-1}) = f(h), \quad \forall g, h \in G.$$

---

ex) G : a matrix (Lie) group

$$f(\sigma) = \mathrm{tr}(\sigma)$$

↑ matrix trace

Def $f: G \to \mathbb{R}$ an invariant fcn

$B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ an orthogonal structure.

The *variation fcn* F of f (w.r.t. B) is the unique map

$F: G \to \mathfrak{g}$ s.t. $\forall X \in \mathfrak{g}, \forall \sigma \in G,$

$$B(F(\sigma), X) = \frac{d}{dt}\Big|_{t=0} f(\sigma \cdot e^{tX}).$$

(f의 X방향 미분 at σ)

---

ex) $G = GL(n, \mathbb{R})$

$B(X, Y) = \mathrm{tr}(XY)$, Then $F: G \to \mathfrak{g} = \mathrm{Mat}_{n\times n}(\mathbb{R})$ is inclusion.

(i.e. $F(x) = x$) ↑ exercise!

$f(\sigma) = \mathrm{tr}(\sigma)$

For an invariant fcn $f: G \to \mathbb{R}$

and a loop $\alpha \in \pi_1(S)$,

define a function

$$f_\alpha : \mathrm{Hom}(\pi_1(S), G) \to \mathbb{R}$$

as $f_\alpha(\phi) := f(\phi(\alpha))$, $\forall \phi \in \mathrm{Hom}(\pi_1(S), G)$

(f: fixed, α: fixed)

Then f_α is invariant under the G action by conjugation.

$\rightsquigarrow$ induces a fcn $f_\alpha : \mathrm{Hom}(\pi_1(S), G)/G \to \mathbb{R}$.

geometrically natural, smooth fcns.

$\rightsquigarrow$ let's ask Poisson brackets among them.

(일반적으로 : (M, ω) symplectic manifold $\rightsquigarrow I_\omega : \Omega^1(M) \xrightarrow{\sim} \mathfrak{X}(M)$

$f, g \in C^\infty(M) \rightsquigarrow df, dg \in \Omega^1(M)$

$\rightsquigarrow I_\omega(df), I_\omega(dg) \in \mathfrak{X}(M)$.

$\rightsquigarrow \{f, g\} := \omega(I_\omega(df), I_\omega(dg)) \in C^\infty(M)$.

$= (Hf)(g)$

(Hf: Hamiltonian vector field for f.)

Poisson bracket)

Thm (Goldman)

α, β : two oriented closed curves in S
that are immersions with transverse double points
(한번에 두 점에서만 만난다. 겹치는 것 X)

$f, f' : G \to \mathbb{R}$ any two invariant fcns.

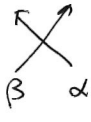
$\rightsquigarrow$ variations $F, F' : G \to \mathfrak{g}$.

Then the Poisson bracket $\{f_\alpha, f'_\beta\}$ is given by : $\forall [\phi] \in \mathrm{Hom}(\pi_1(S), G)/G$,

$$\{f_\alpha, f'_\beta\}([\phi]) = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) \cdot B\big(F(\phi(\alpha_p)), F'(\phi(\beta_p))\big),$$

where $\alpha \# \beta = \{\text{intersections of } \alpha, \beta\}$

$p \in \alpha \# \beta$: $\varepsilon(p; \alpha, \beta) = \pm 1$ is the oriented intersection number of α, β at p.

( : +1,  : −1)

$\alpha_p, \beta_p \in \pi_1(S, p)$

우변의 ϕ는 any $\phi : \pi_1(S,p) \to G$ representing $[\phi]$.

Cor. If α, β are disjoint, then f_α, f'_β Poisson-commute.

Thm (Goldman) For $G = GL(n, \mathbb{R})$, $B(X,Y) = \mathrm{tr}(XY)$, $f(\sigma) = \mathrm{tr}(\sigma)$

$$\{f_\alpha, f_\beta\} = \sum_{p \in \alpha \# \beta} \varepsilon(p; \alpha, \beta) \cdot f_{\alpha_p \beta_p}$$

↖ product loop in $\pi_1(S,p)$

[exercise]

(Cor. $\mathrm{Span}_{\mathbb{Z}}\{f_\alpha\}_{\alpha\in\pi_1(S)}$ is closed under Poisson bracket)

(for $G=GL(n,\mathbb{R})$) (Note: GL(n,R)의 경우, 곱하기에는 닫혀있지 않음. Turaev가 했듯이, 인위적으로 symmetric algebra 를 generate할 수는 있음)

Ⅳ. Quantization (Turaev's version)

Def. A Poisson bracket on a commutative algebra A_0 is a bilinear map

$$A_0\times A_0\to A_0$$
$$(a,b)\mapsto\{a,b\}$$

s.t. ① $\{\cdot,\cdot\}$ is a Lie bracket (i.e. skew-symmetric & satisfies Jacobi identity)

② $\{ab,c\}=a\{b,c\}+\{a,c\}b$ (Leibniz rule)

$(A_0,\{\cdot,\cdot\})$ is a Poisson algebra.

---

ex) Symplectic mfld $\rightsquigarrow (C^\infty(M),\underbrace{\{\cdot,\cdot\}}_{\text{아까 정의}})$ is a Poisson algebra

---

Algebraic quantization of a Poisson algebra

Let A be a non-commutative algebra over $\Bbbk[h]$ (polynomial ring) ($\Bbbk$ = field, or $\mathbb{Z}$)

(assume A is a free $\Bbbk[h]$-module)

Assume ① A/hA is commutative

② $\forall a,b\in A,\quad [a,b]\in hA$.

Then $\{a+hA,\ b+hA\}:=h^{-1}[a,b]+hA$ makes A/hA a Poisson algebra.

Given a (commutative) Poisson algebra, finding such A is called "quantization" problem.

V. Knots and skeins (3d topology)

Skein algebra $\mathcal{A}(S \times [0,1])$ of surface S.

↑ 3d mfld

$\mathcal{L} = \{$ isotopy classes of oriented links L in $S \times [0,1] \}$

↑ (bunch of embedded circles)

$\mathcal{A}(S \times [0,1])$ = free $\mathbb{K}[h]$-module with basis $\mathcal{L}$,

mod out by the "skein relations"

$$L_+ - L_- - hL_0 ,$$

where

(L_+, L_0, L_-) ranges over all "Conway triples"

of links which are identical to each other except at a small ball, where they look like



(note: knot theory 에서 온 relation)

$L \in \mathcal{L} \mapsto [L] \in \mathcal{A}(S \times [0,1])$ "skein"

"skein relation": $[L_+] - [L_-] = h[L_0]$ (Conway skein relation)

product is defined as

$$[L][L'] := [L \cup L']$$

("superposition" product)

↑ isotope so that $L' \subseteq S \times [0, \frac{1}{2}]$

↑ isotope so that $L \subseteq S \times [\frac{1}{2}, 1]$

(강의에서는 이 둘을 바꿔서 썼는데, 이러나 저러나 상관은 없음.)

"Thm" (Turaev) The (Conway) skein algebra $\mathcal{A}(S \times [0,1])$ is a quantization of the Poisson algebra "generated by" the fcns f_α, for $G = GL(n, \mathbb{R})$.

(note: 진정한 algebra of functions의 quantization은 SL2 경우밖에 안 되어 있다고 생각함.)

LEGENDRIANS AND CONSTRUCTIBLE SHEAVES

BYUNG HEE AN

1. Overview

Let (W, ω) be a symplectic manifold consisting of

(1) an even-dimensional smooth manifold W and
(2) a closed, non-degenerate two-form ω on W.

Example 1.1. A typical example of symplectic manifolds is a *cotangent bundle* $(T^*X, \omega = \sum_i dp_i \wedge dq_i)$, where X is a smooth manifold and q_i's are local coordinates of X.

(1) if $X = \mathbb{R}^1$, then $(T^*\mathbb{R}^1, dx \wedge dy)$;
(2) if $X = \mathbb{R}^2$, then $(T^*\mathbb{R}^2, dp_1 \wedge dx + dp_2 \wedge dz)$.

A submanifold $L^n \subset W^{2n}$ is called *Lagrangian* if the symplectic form ω vanishes on L.

Example 1.2. The zero-section $T^*_X X \subset T^*X$ is a Lagrangian submanifold.

In symplectic geometry, one of main interest is to understand symplectic manifolds in terms of Lagrangians. For examples,

- Lagrangian submanifolds,
- their intersections,
- and so on.

All these information can be repackaged as "Fukaya categories".

Assumption 1.3. From now on, we assume that $W = T^*X$ and either $X = \mathbb{R}^1$ or $\mathbb{R}^2$.

It is important to notice that the symplectic form is *exact*

$$\omega = d\alpha, \qquad \alpha = xdy \quad \text{or} \quad p_1 dx + p_2 dz.$$

The *contact boundary* or *infinite boundary* of T^*X is the collection of all "rays" emitting from the origin in each cotangent fiber, denoted by $T^\infty X$. Then one can identify $T^\infty X$ with the unit co-sphere bundle of X.

$$T^\infty X \subset T^*X, \qquad T^\infty \mathbb{R}^n \simeq S^{n-1} \times \mathbb{R}^n$$

The pair $(T^\infty X, \alpha)$ defines a *contact manifold*, which is an odd-dimensional smooth manifold equipped with a contact form α such that

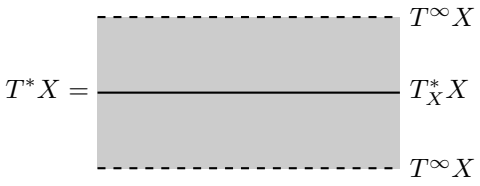
$$\alpha \wedge (d\alpha)^n \neq 0$$

everywhere. Then the odd-dimensional counterpart of Lagrangian submanifolds in a contact manifold is the *Legendrian submanifold* $\Lambda \subset T^\infty X$ such that

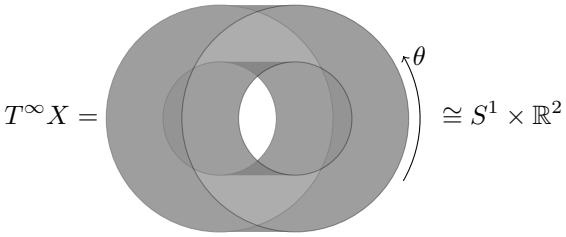
Disclaimer: All definitions, examples and claims in this note work at least for the cases considered here. In order to consider the generalized cases, all definitions, examples and claims should be modified or carefully checked.

(1) Λ is $(n-1)$-dimensional when X is n-dimensional,
(2) the contact form α vanishes on Λ.

Example 1.4. If $X = \mathbb{R}$, then $T^\infty X = \{\epsilon \mid \epsilon = \pm 1\} \times \mathbb{R}$ and $\alpha = \epsilon dx$.



If $X = \mathbb{R}^2$, then $T^\infty X = \{\theta \in S^1\} \times \mathbb{R}^2$ and $\alpha = \cos\theta dx + \sin\theta dz$.



There is a natural projection map $\pi : T^*X \to X$, which restricts to $T^\infty X \to X$ such that

$$\pi : T^\infty\mathbb{R} \simeq S^0 \times \mathbb{R} \to \mathbb{R}, \qquad (\epsilon, x) \mapsto x,$$
$$\pi : T^\infty\mathbb{R}^2 \simeq S^1 \times \mathbb{R}^2 \to \mathbb{R}^2, \qquad (\theta, (x, z)) \mapsto (x, z).$$

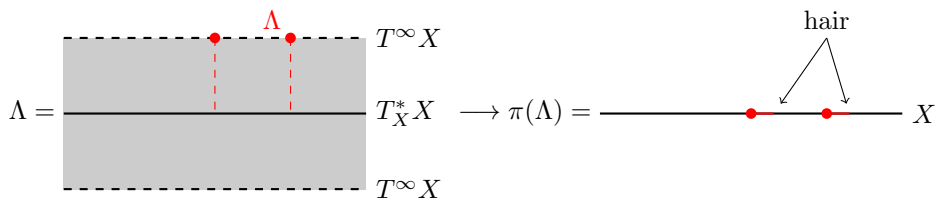
Notice that at each smooth point of $\pi(\Lambda) \subset X$, there are two choices of the missing coordinate ϵ or θ since the coordinate θ is perpendicular to the projection due to the Legendrian condition

$$0 = \cos\theta dx + \sin\theta dz = (\cos\theta, \sin\theta) \cdot (dx, dz).$$

Conversely, any (piecewise smooth) codimension 1 subspace Y of X together with a choice of perpendicular covectors at each point, one can realize Y as the projection of a Legendrian subspace of $T^\infty X$. We call such a choice of covectors a *hair*. Therefore we have the following correspondence

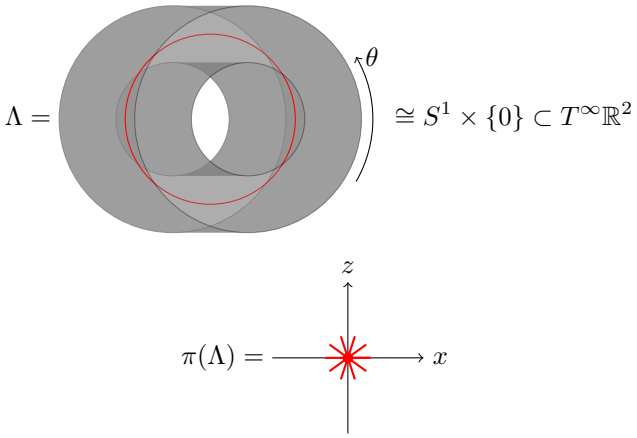
$$\{\text{Legendrians} \subset T^\infty X\} \longleftrightarrow \{\text{Diagrams of codim 1 with hairs} \subset X\}.$$

Example 1.5. (1)

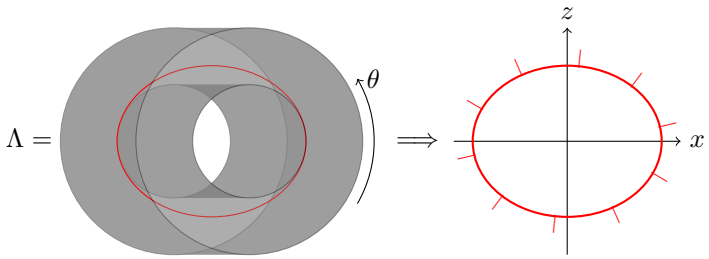


The hair points the *positive* direction since Λ is contained in the positive part of $T^\infty X$.

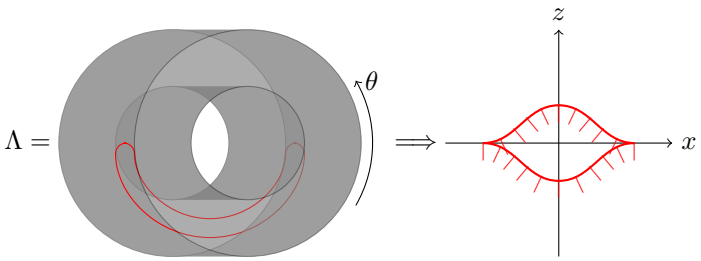
(2) Λ is a circle



(3) Λ is an ellipse



(4) $\pi(\Lambda)$ may have cusps.



Theorem 1.6 (Nadler-Zaslow, Ganatra-Pardon-Shende)**.** *There are equivalences (of DG categories)*

$$NZ : Sh(X) \to Fuk^\epsilon(T^*X),$$
$$NZ_\Lambda : Sh_\Lambda(X) \to Fuk^\epsilon(T^*X, \Lambda),$$
$$GPS : Sh_\Lambda(X)^c \to W(T^*X, \Lambda),$$

where

(1) $Sh_\Lambda(X)$ *is the category of constructible sheaves on* X *whose microlocal support is controlled by* $\Lambda \subset T^\infty X$,
(2) $Fuk^\epsilon(T^*X, \Lambda)$ *is the (infinitesimally wrapped) Fukaya category whose Lagrangians are asymptotic to* Λ *at infinity, and*
(3) $W(T^*X, \Lambda)$ *is the (subcategory of) wrapped Fukaya category whose Lagrangians are away from* Λ *at infinity.*

From now on, we will see how $Sh_\Lambda(X)$ is defined and computed by using the projection Λ.

2. Category of constructible sheaves

2.1. **Sheaves.**

> A *sheaf* is a tool for systematically tracking locally defined data attached to the open sets of a topological space.
>
> — Wikipedia

Roughly speaking, a sheaf $\mathcal{F}$ is an assignment of any (algebraic) object[1] to each open subset of X

$$\mathcal{F} = \{(U \mapsto \mathcal{F}(U)) \mid U \overset{\text{open}}{\subset} X\}$$

such that

(1) for each $U \subset V$, there exists a morphism $\mathcal{F}_{UV} : \mathcal{F}(V) \to \mathcal{F}(U)$,
(2) for each $U \subset V \subset W$,
$$\mathcal{F}_{VW} \circ \mathcal{F}_{UV} = \mathcal{F}_{UW},$$
(3) for each $U = \bigcup_i U_i$,
$$\mathcal{F}(U) \simeq \lim \left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j) \right).$$

In other words, $\mathcal{F}(U)$ is completely determined by $\mathcal{F}(U_i)$'s. For example, if $U = U_1 \cup U_2$, then $\mathcal{F}(U)$ is the fiber product of two homomorphisms $\mathcal{F}(U_i) \to \mathcal{F}(U_1 \cap U_2)$.

Example 2.1. Each of the following defines a sheaf.

(1) For each open and simply-connected $U \subset X$, we set $\mathcal{F}(U) = \mathbb{K}$. Then for each disjoint union $U = \coprod_i U_i$,
$$\mathcal{F}(U) = \prod_i \mathbb{K}$$
and $\mathcal{F}$ is called the *constant sheaf*.
(2) Let $I = [0, 1] \subset \mathbb{R}$. For each open and simply-connected U, define
$$\mathcal{F}(U) = \begin{cases} \mathbb{K} & U \cap I \neq \emptyset; \\ 0 & U \cap I = \emptyset. \end{cases}$$
(3) Let $J = (0, 1) \subset \mathbb{R}$. For each open and simply-connected U, define
$$\mathcal{F}(U) = \begin{cases} \mathbb{K} & U \subset J; \\ 0 & \text{otherwise.} \end{cases}$$
(4) Let $Z = [0, 1) \subset \mathbb{R}$. For each open and simply-connected U, define
$$\mathcal{F}(U) = \begin{cases} \mathbb{K} & U \subset (-\infty, 1) \text{ and } U \cap Z \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

[1] In this talk, we assign a vector space or a chain complex.

(5) Let $* \in X$ be a chosen point. For each open and simply-connected U, define

$$\mathcal{F}(U) = \begin{cases} \mathbb{K} & * \in U; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}$ is called the *skyscraper sheaf*.

For each $x \in X$, a *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at x is defined to be the colimit (or direct limit)

$$\mathcal{F}_x = \operatorname*{colim}_{x \in U} \mathcal{F}(U) = \varinjlim_{x \in U} \mathcal{F}(U).$$

Roughly speaking, a stalk measures the contribution of the chosen point in the sheaf $\mathcal{F}$.

Example 2.2. The stalk at each point of the sheaves in the previous examples is as follows:
(1) For the constant sheaf, $\mathcal{F}_x = \mathbb{K}$ for any $x \in X$.
(2)–(4),

$$\mathcal{F}_x = \begin{cases} \mathbb{K} & x \in I, J \text{ or } Z; \\ 0 & \text{otherwise.} \end{cases}$$

(5), the only nontrivial stalk is at $x = *$

$$\mathcal{F}_x = \begin{cases} \mathbb{K} & x = *; \\ 0 & \text{otherwise.} \end{cases}$$

2.2. **Stratifications and Constructible sheaves.** A *stratification* or *cell structure* on X is the increasing filtration S of closed subsets

$$S = X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that

(1) $S_i \coloneqq X_i \setminus X_{i-1}$ is a (possibly empty) i-dimensional manifold, called the ith *stratum*,
(2) for each $x \in S_i$, there exists a neighborhood U of x such that

$$U \cong \mathbb{R}^i \times \operatorname{Cone}(L_x),$$

where L_x is the *link*, a compact $(n - i - 1)$ stratified space.

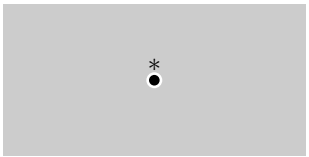
Example 2.3. Any regular or simplicial complex structure possibly with open cells defines a stratification.

(1)

$X_0 = \{0, 1\} \subset X_1 = \mathbb{R}^1$,

$S_0 = \{0, 1\}$, $S_1 = (-\infty, 0) \amalg (0, 1) \amalg (1, \infty)$

(2)

$X_0 = \{*\} \subset X_1 = \{*\} \subset X_2 = \mathbb{R}^2$, 

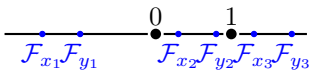
$S_0 = \{*\}$, $S_1 = \emptyset$, $S_2 = \mathbb{R}^2 \setminus \{*\}$

For a stratification S of X, a sheaf $\mathcal{F}$ on X is *S-constructible* if for each i and connected component $s \in S_i = X_i \setminus X_{i-1}$, the restriction $\mathcal{F}|_s$ is a constant sheaf.

In particular, if $\mathcal{F}$ is S-constructible, then the stalk $\mathcal{F}_x$ is the same at each point x in one component of the stratum.

$$[x] = [y] \in \pi_0(s), s \in S_i \Longrightarrow \mathcal{F}_x \cong \mathcal{F}_y.$$

Example 2.4. In the previous examples, one can see that $\mathcal{F}_x = \mathcal{F}_y$ for all $x, y < 0$, $x, y \in (0, 1)$ or $x, y > 1$. Therefore all sheafs of the previous examples are constructible for some stratification S.



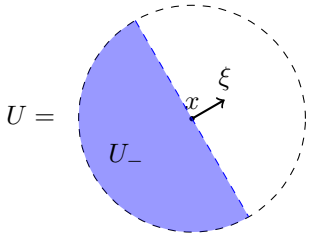
$$\mathcal{F}_{x_i} \cong \mathcal{F}_{y_i}$$

Assumption 2.5. For simplicity, we will assume that closures of all connected component in each stratum are homeomorphic to the closed disks.[2]

2.3. **Singular supports.** On the other hand, as seen in the previous examples, even stalks at top dimensional strata coincide, stalks at points in the lower strata may be different. Indeed, one can measure which *directions* give us a non-trivial propagation of stalks.

Let $\mathcal{F}$ be an S-constructible sheaf on X for some stratification S. The *singular support* $SS(\mathcal{F})$ of $\mathcal{F}$ is the closed subspace of T^*X defined as follows:

- for each $(x, \xi) \in T^*X$ with $\xi \neq 0$, we say that $(x, \xi) \notin SS(\mathcal{F})$ if there exists a neighborhood $U \subset X$ of x such that the restriction map
$$\mathcal{F}(U) \to \mathcal{F}(U_-)$$
is isomorphism (on homologies), where U_- the back half-space with respect to ξ at x.
- for each $(x, 0) \in T^*_X X$, we say that $(x, 0) \notin SS(\mathcal{F})$ if there exists a neighborhood $U \subset X$ such that the restriction $\mathcal{F}|_U$ is 0.[3]



Example 2.6. (1) For a constant sheaf, neither positive nor negative direction give us a non-trivial propagation of stalks since stalks are the same everywhere. Therefore the singular support is the zero-section

$$SS(\mathcal{F}) = \{(x, 0) \mid x \in X\}.$$

(2) At $x = 0$, the back half-space has the stalk 0 but any neighborhood at $x = 0$ has $\mathbb{K}$ as the value of $\mathcal{F}$, i.e., $\mathcal{F}_x = \mathbb{K}$. Therefore the positive direction has a non-trivial propagation of stalks.

$$\mathbb{K} \cong \mathcal{F}((-\epsilon, \epsilon)) \ncong \mathcal{F}((-\epsilon, 0)) \cong 0.$$

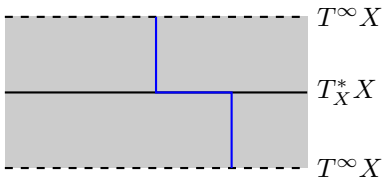
[2]Indeed, for each stratification S, the above assumption can be achived by subdividing strata.
[3]This is the same as the definition of the *support* of the sheaf.

However, the negative direction gives us a trivial propagation since

$$\mathbb{K} \cong \mathcal{F}((-\epsilon, \epsilon)) \overset{\cong}{\mathcal{F}} ((0, \epsilon)) \cong \mathbb{K}.$$

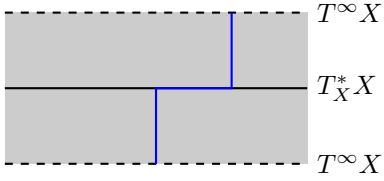
Similarly, at $x = 1$, the only negative direction gives us a non-trivial propagation of stalks. Therefore

$$SS(\mathcal{F}) = \{(0, \xi) \mid \xi > 0\} \cup [0, 1] \cup \{(1, \xi) \mid \xi < 0\}$$

$=$ 

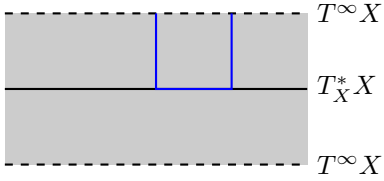
(3) Similarly, we have

$$SS(\mathcal{F}) = \{(0, \xi) \mid \xi < 0\} \cup [0, 1] \cup \{(1, \xi) \mid \xi > 0\}$$

$=$ 

(4)

$$SS(\mathcal{F}) = \{(0, \xi) \mid \xi > 0\} \cup [0, 1] \cup \{(1, \xi) \mid \xi > 0\}$$

$=$ 

(5) At $x \neq *$, there are no directions giving us a non-trivial propagation of stalks as the constant sheaf. However at $x = *$, every direction gives us a non-trivial propagation of stalks. Therefore the singular support of the skyscraper sheaf

$$SS(\mathcal{F}) = \{(*, \xi) \mid \xi \in T^*_* X\}$$

is the cotangent fiber at the designated point $*$.

Proposition 2.7. *The singular support $SS(\mathcal{F})$ is a conical Lagrangian subspace of T^*X.*

Here, the conical subspace means a space equipped with $\mathbb{R}_{>0}$ action. Obviously, $SS(\mathcal{F})$ is conical since it is closed under the $\mathbb{R}_{>0}$ action defined by

$$t \cdot (x, \xi) \coloneqq (x, t\xi)$$

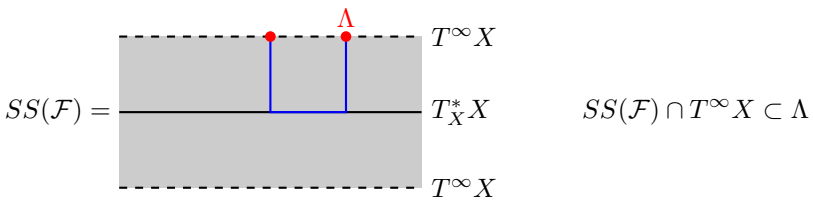
for any $t > 0$ and $\xi \in T^*_x X$.

In particular, the intersection of $SS(\mathcal{F})$ and the contact manifold $T^\infty X$ will be a Legendrian subspace.

We say that the singular support of a constructible sheaf $\mathcal{F}$ is *controlled by* a Legendrian Λ if

$$SS(\mathcal{F}) \cap T^\infty X \subset \Lambda.$$

Example 2.8. Let $\Lambda = \{+1\} \times \{0, 1\} \subset T^\infty\mathbb{R} \cong \{\pm 1\} \times \mathbb{R}$. Then among (2)–(4) in the previous examples, the singular support only for (4) is controlled by Λ.



Now we can define $Sh_\Lambda(X)$ as the (dg-category consisting of) constructible sheaves on X, whose singular supports are controlled by Λ.

Indeed, the correspondence of Nadler-Zaslow assigns each sheaf $\mathcal{F}$ to a Lagrangian obtained by smoothing its singular support $SS(\mathcal{F})$.

3. Computations of shaves

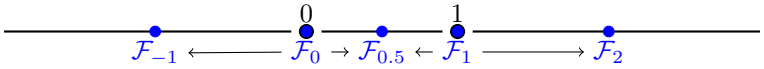
Let Λ be a Legendrian in $T^\infty X$. Then its projection $\pi(\Lambda) \subset X$ gives us a stratification. Under the certain assumption such as a general position of Λ, we will have a good enough stratification of X.

3.1. 0-dimensional example.

Let $\Lambda = \{+1\} \times \{0, 1\} \subset T^\infty\mathbb{R}$ as before. Then any sheaf $\mathcal{F}$ on $\mathbb{R}$ whose singular support is controlled by Λ will be characterized by stalks

$$\mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_{0.5}, \mathcal{F}_1, \quad \text{and} \quad \mathcal{F}_2.$$

On the other hand, since any neighborhoods of 0 and 1 have restrictions onto open sets contained in $(-\infty, 0), (0, 1)$ and $(0, 1), (1, \infty)$, respectively, there are natural maps



Now, since Λ is contained in the *positive* side, there are no nontrivial propagations of stalks along negative directions. It means that two morphisms

$$\mathcal{F}_0 \to \mathcal{F}_{0.5} \quad \text{and} \quad \mathcal{F}_1 \to \mathcal{F}_2$$

must be isomorphism.[4] Therefore the sheaf $\mathcal{F}$ is the same as the following information

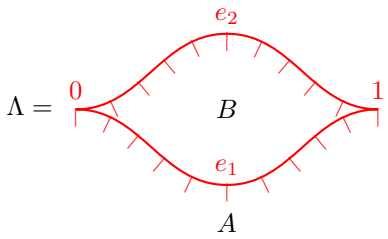
$$\mathcal{F}_{-1} \leftarrow \mathcal{F}_{0.5} \leftarrow \mathcal{F}_2,$$

which is a representation of a *quiver*

$$\bullet \leftarrow \bullet \leftarrow \bullet.$$

[4]The reader should not be confused the directions. The negative direction contribute the restriction $(-0.5, 0.5) \to (0, 0.5)$, where the latter is the *back* half-space with respect to the negative direction.

3.2. **1-dimensional example.** Let $\Lambda \subset T^\infty \mathbb{R}^2$ be the Legendrian unknot depicted below



whose hairs are given in the below directions.

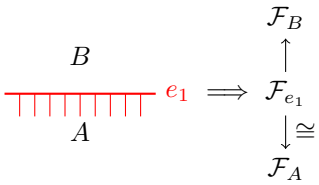
Then it gives us a stratification

$$X_0 = \{0, 1\} \subset X_1 = \Lambda \subset X_2 = \mathbb{R}^2,$$

where 0 and 1 are the left and right cusps, respectively. Then

$$S_0 = \{0, 1\}, \qquad S_1 = e_1 \amalg e_2, \qquad S_2 = A \amalg B$$

As before, near the edge e_1, we have three stalks, denoted by $\mathcal{F}_A, \mathcal{F}_{e_1}$ and $\mathcal{F}_B$, and maps



where the isomorphism comes from the fact that the above direction is *not* in the singular support.

Similarly, at the edge e_2, we have

$$\mathcal{F}_B \xleftarrow{\cong} \mathcal{F}_{e_2} \to \mathcal{F}_A.$$

On the other hand, at each cusp, we have

$$(\mathcal{F}_A \to \mathcal{F}_B \to \mathcal{F}_A) \cong \mathrm{Id}\,^5$$

and therefore the sheaf $\mathcal{F}$ is determined by

$$\mathcal{F}_A \to \mathcal{F}_B \to \mathcal{F}_A$$

whose composition is the identity (in homology).

Since we are working over a field $\mathbb{K}$, we have a decomposition $\mathcal{F}_B = \mathcal{F}_A \oplus \mathcal{F}'_B$ so that

$$\mathcal{F}_A \to \mathcal{F}_B \to \mathcal{F}_A = (\mathcal{F}_A \to \mathcal{F}_A \to \mathcal{F}_A) \oplus (0 \to \mathcal{F}'_B \to 0).$$

In many cases, we assume that the stalk outside the compact region in where Λ is contained is zero, i.e., $\mathcal{F}_A = 0$. Then this is the same as the choice of $\mathcal{F}'_B$ and therefore any constructible sheaf whose singular support is controlled by the Legendrian unknot is the same as a choice of a vector space, or equivalently, a representation of the A_1-quiver $\bullet$.

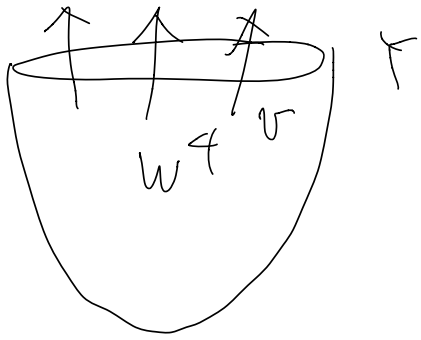
[5]This can be obtained by analyzing the singular support but we omit the detail.

On symplectic fillings of small Seifert 3-manifolds.

(Joint work with Jongil Park)

- What is symplectic filling?

(W^4, ω) $\begin{cases} W^4 : \text{compact 4-manifold with } \partial W^4 = Y \\ \omega : \text{symplectic 2-form on } W^4 \end{cases}$



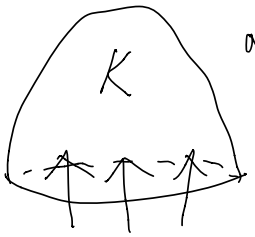
$\exists$ Liouville vector field v (i.e. $L_v \omega = \omega$)

$\xi = \ker(i_v \omega)$ gives a contact structure on Y.

We call (W, ω)

a (convex) symplectic filling of (Y, ξ).

Rmk ① If ω is inward, we call (K, ω)



a concave symplectic filling of (Y, ξ).

② Every compact, oriented, contact 3-manifold admits infinitely many concave fillings.

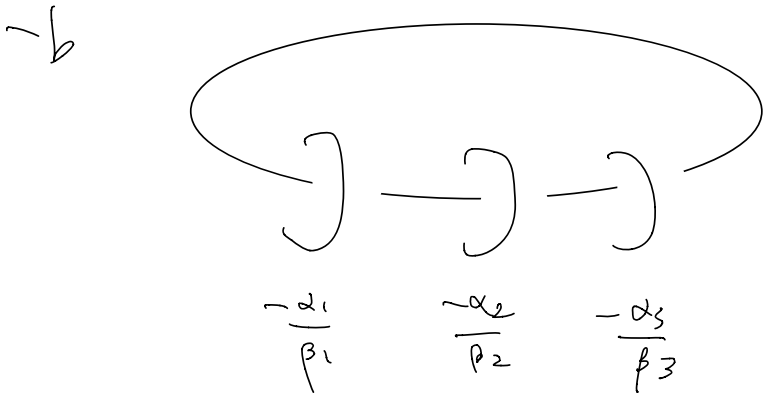
In contrast, not every compact oriented contact 3-manifold admits a convex filling.

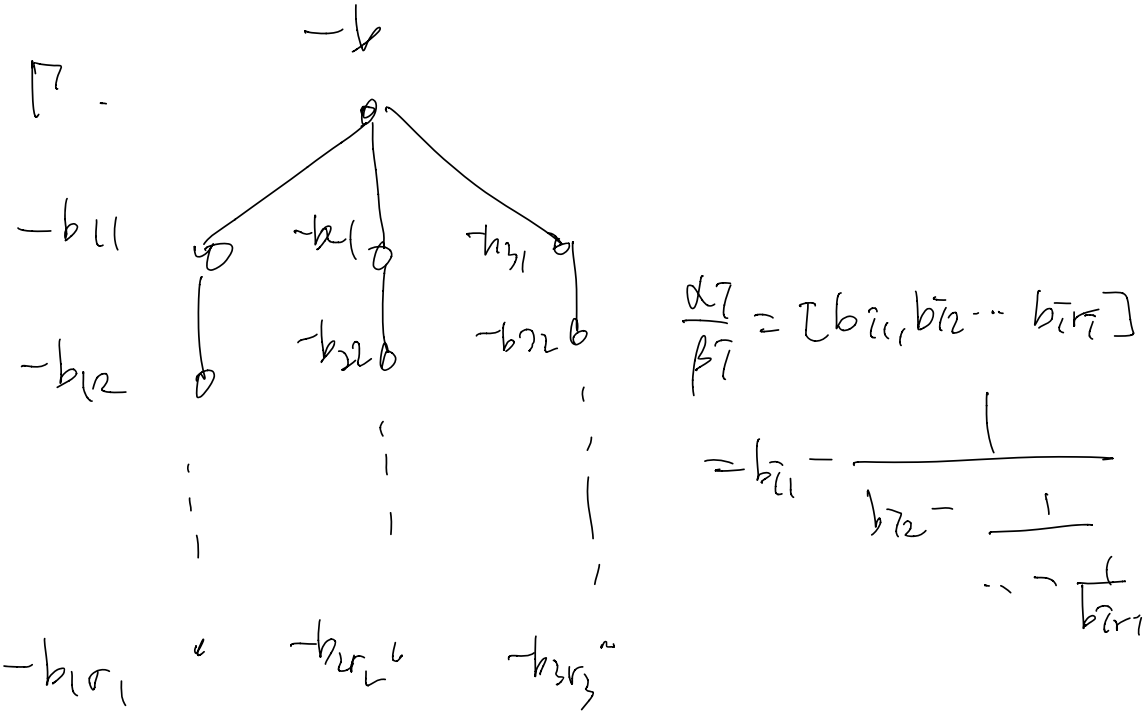
③ With a concave filling K & a convex filling W of Y, we get a closed symp 4 manifold

$$X := W \cup_Y K.$$

Small Seifert 3-manifold Y

S^1-fibration over a Riemann surface with at most 3 singular fibers.



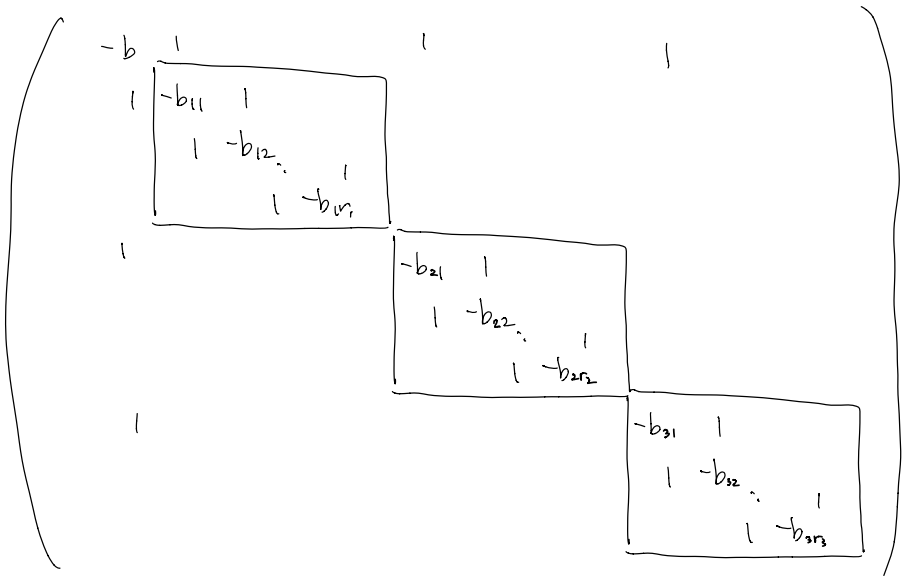


Contact structure of Y

- Intersection matrix of Γ is negative definite.
- Each vertex is symplectic 2-sphere
- Each edge represents an ω-orthogonal intersection.

$\longrightarrow$ symplectic structure of plumed 4-manifold X induces a contact structure on Y. (Gay-Stipsicz)

* Intersection matrix of Γ



Classification of symplectic fillings of Y.

We consider "minimal" case.

Strategy.

For a given symplectic filling W of Y.

glue it with concave cap K

to get a closed symplectic 4-manifold $X = W \cup K$

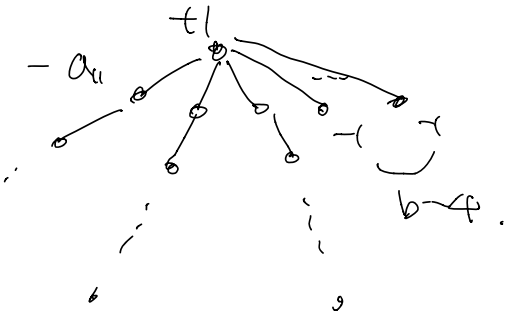
And then show X is a rational symplectic 4-manifold.
(i.e, $X \cong \mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$).

$\Rightarrow$ Classifying symplectic fillings = find all embeddings of K.

What is K? : Also plumed 4-manifold.

If $b \geq 4$.

$\longrightarrow$ $K =$

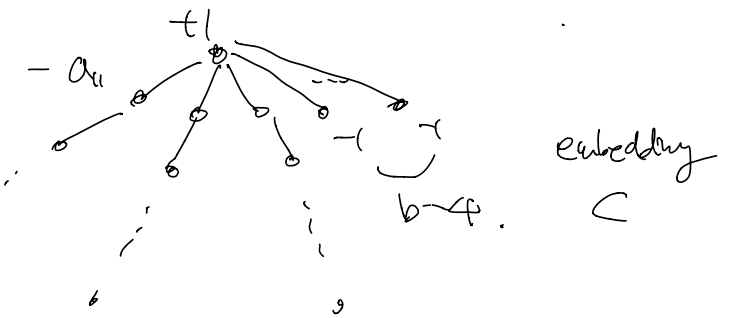


- X is rational.

In K, we have symplectic $(+1)$-sphere.

Theorem (Mcduff)

Let (X, ω) be a closed symp 4-manifold
If X contains a symplectically embedded 2-sphere C
with $[C]^2 = 1$, then X becomes a complex projective plane
after blowing down (-1) curves away from C.



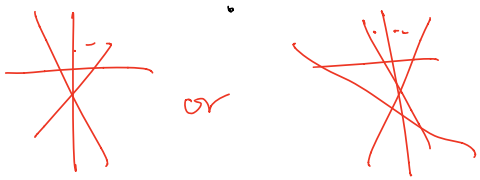
embedding C

$X \cong \mathbb{CP}^2 \# N\overline{\mathbb{CP}}^2$

blow downs

$\mathbb{CP}^2$

$\subset$ embedding.



Thm (Lisca, Blanchet-Ono, Starkston)
The image of K under blow downs is
a neighborhood of b symplectic spheres.

Furthermore, with the conditions $(b \geq 4)$.
b symplectic spheres are isotopic to
b many distinct lines in $\mathbb{CP}^2$ through symplectic spheres

$\rightsquigarrow$ Essential data of W is

homological data of K in $X \cong \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$.

Once we get classification result,
the following natural question arise.

Q: Is there any surgery description of fillings?

Rational blowdown

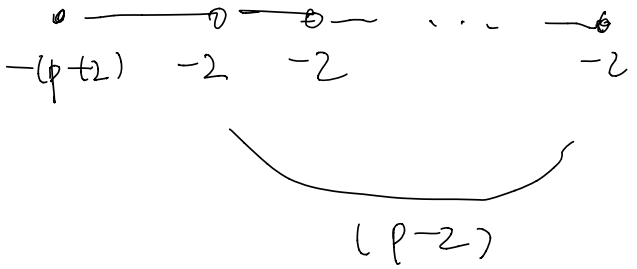
Def. X is a smooth 4-manifold.
$C \subset X$ negative definite plumed 4-manifold
s.t $\partial C \cong \partial B$ with $H_*(B; \mathbb{Q}) \cong H_*(D^4; \mathbb{Q})$.
Operation of replacing C with B is called

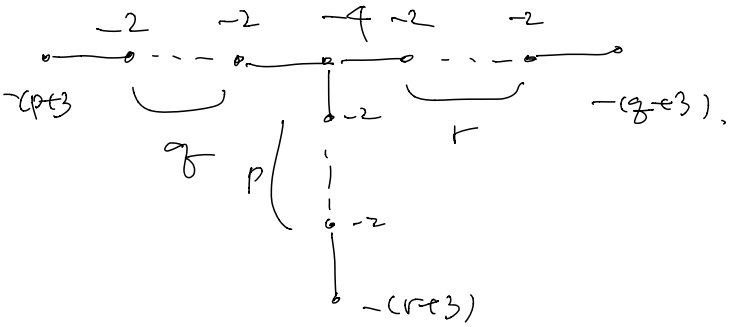
Rational blowdown surgery.

Rmk i) Rational blowdown can be done in symplectic category:
If X is symplectic & $C \subset X$ symplectically embedded,
$(X \setminus C) \cup B$ also admit symplectic structure
Actually, B is another symplectic filling of $\partial C \cong \partial B$.

ii). Rational blowdown surgery does not affect
symplectic structure near boundary.
: If there is an embedding of $C \subset W$
We can obtain another symplectic filling
$W' := (W \setminus C) \cup B$.

example)





thm. (Kapl-Ozbagci , C.-park)

Every minimal symplectic filling W of (X,ξ) $\begin{pmatrix} b\geq 4 \\ k=3 \end{pmatrix}$.

is realized by the corresponding curve configuration.

Furthermore it is obtained by a sequence of rational blowdowns from the plumed 4-manifold X according to Γ.

Homogeneous dynamics and application to the lattice counting problem

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2019-12-30

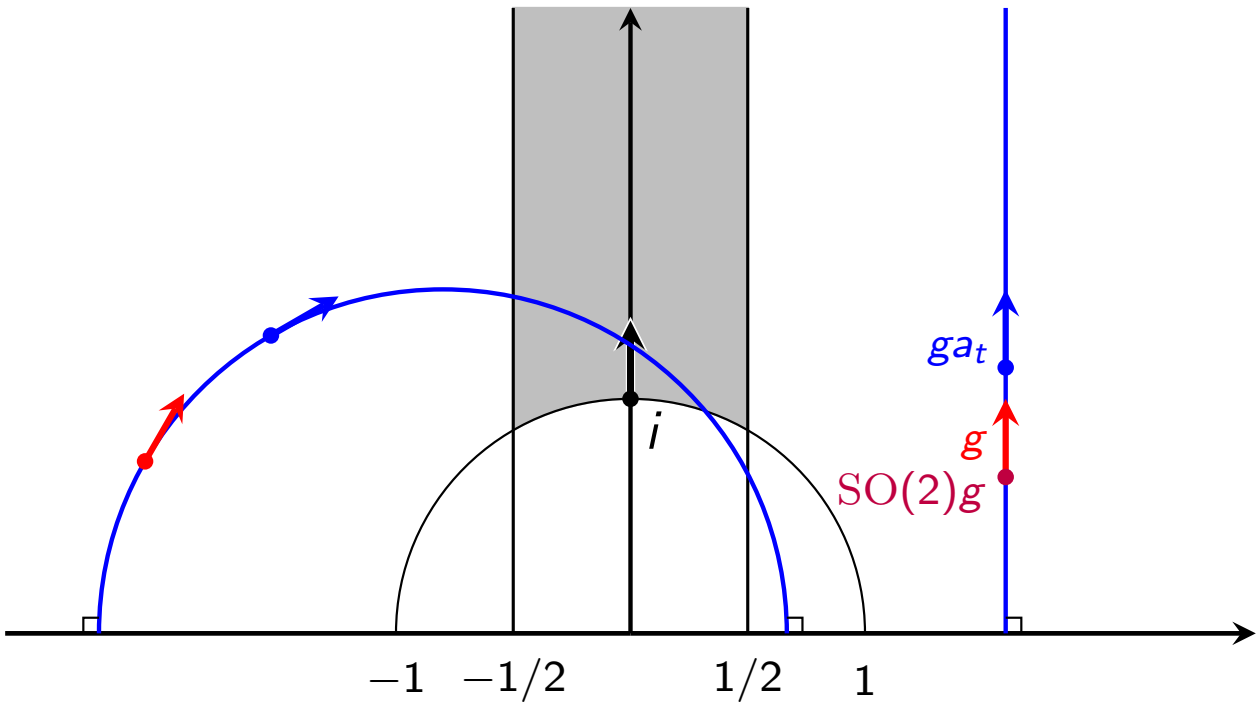
The homogeneous space $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$

- $\mathbb{H}^2 = \{z = x + iy : y > 0\} \subseteq \mathbb{C}$: the hyperbolic plane
  metric : $d_{hyp} = d_{euc}/y$
- $\mathrm{SL}_2(\mathbb{R}) \curvearrowright \mathbb{H}^2$ by isometries:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+c}{bz+d}$$ Mobius transformation.

- $\mathrm{Stab}_G(i) = \mathrm{SO}(2) =: K$ maximal cpt subgp of G
  $\Rightarrow \mathrm{SO}(2) \setminus \mathrm{SL}_2(\mathbb{R}) \simeq \mathbb{H}^2$, $\mathrm{SO}(2) \simeq \mathbb{S}^1 \simeq T^1_i(\mathbb{H}^2)$
  $\Rightarrow \mathrm{SL}_2(\mathbb{R}) \simeq T^1(\mathbb{H}^2)$
  $\Rightarrow \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}) \simeq T^1(\mathbb{H}^2/\mathrm{SL}_2(\mathbb{Z}))$

Geodesic Flow



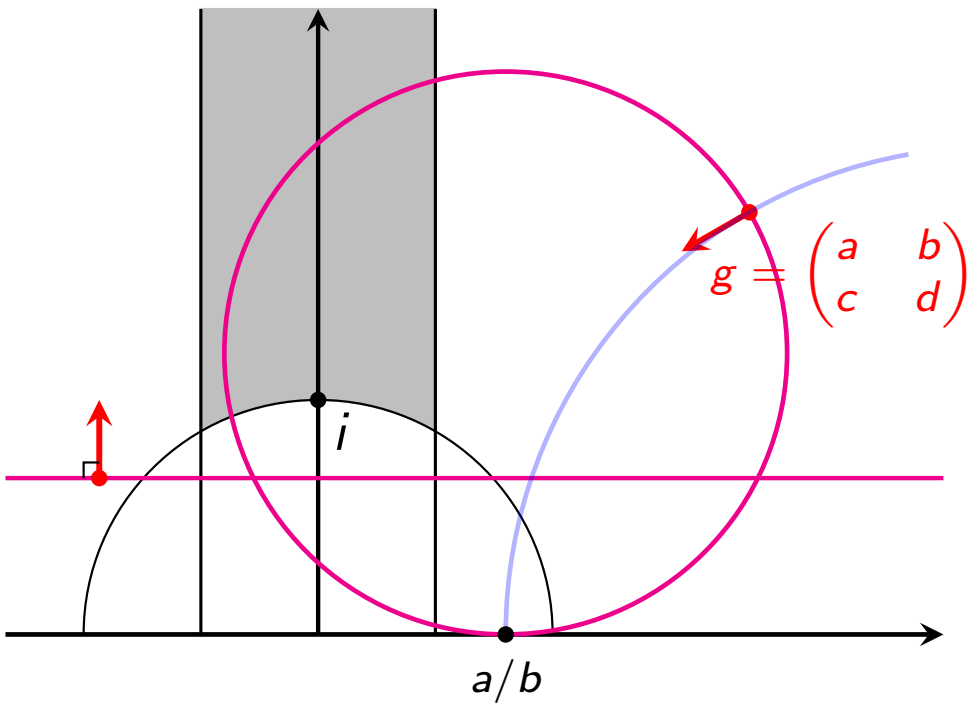
- geodesic flow $A = \{a_t = \mathrm{diag}(e^{t/2}, e^{-t/2})\}$
- $\{\text{geodesics in } \mathbb{H}^2\} \leftrightarrow \{KgA \subseteq K \setminus G : g \in \mathrm{SL}_2(\mathbb{R})\}$
  $(K = \mathrm{SO}(2))$

Unipotent flow (Horocycle flow)

unipotent flow $\left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} \leq \mathrm{SL}_2(\mathbb{R})$

$\leftrightarrow$ horocycle in $\mathbb{H}^2 = \begin{cases} \text{a circle tangent to the real line} \\ \text{or a horizontal line.} \end{cases}$

$$\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}(2)\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$$



- orbit $\{\pi(gu_t) : t \in \mathbb{R}\}$ in $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$
- $\{\pi(gu_t) : t \in \mathbb{R}\}$: closed
  $\Leftrightarrow \exists t_0 \in \mathbb{R}$ s.t. $gu_{t_0} \in \mathrm{SL}_2(\mathbb{Z})$

  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & t_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

  $\Leftrightarrow a/b$: rational or ∞.
- a/b : irrational ?

Ratner's theorems

- G : Lie group e.g. $G = \mathrm{SL}_n(\mathbb{R})$
- Γ : lattice subgp of G e.g. $\Gamma = \mathrm{SL}_n(\mathbb{Z})$
  (discrete, $\mathrm{vol}(G/\Gamma) < \infty$)
- $H \leq G$: closed subgp generated by unipotents

$$\text{e.g. } U = \left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} \leq \mathrm{SL}_2(\mathbb{R})$$

$$\text{or } H = \mathrm{SO}(p,q) \leq \mathrm{SL}_n(\mathbb{R}),\ p+q=n$$

$$(\text{cf. } \mathrm{SO}^\circ(n-1,1) = \mathrm{Isom}^+(\mathbb{H}^{n-1}))$$

- $\pi : G \to G/\Gamma$: projection

Orbit Closure Theorem $\exists L$: closed subgp s. t.

(1) $\overline{\pi(H)} = \pi(L)$,

(2) $\pi(L)$ brings a finite U-invariant measure μ_L.

Ratner's theorems

Orbit Closure Theorem

$\exists L$: closed subgp s. t.

(1) $\overline{\pi(H)} = \pi(L)$,

(2) $\pi(L)$ brings a finite U-invariant measure μ_L.

Equidistribution Theorem

$f \in C_0^{bdd}(G/\Gamma)$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\pi(u_t))dt = \int_{\pi(L)} f d\mu_L.$$

Measure Classification Theorem

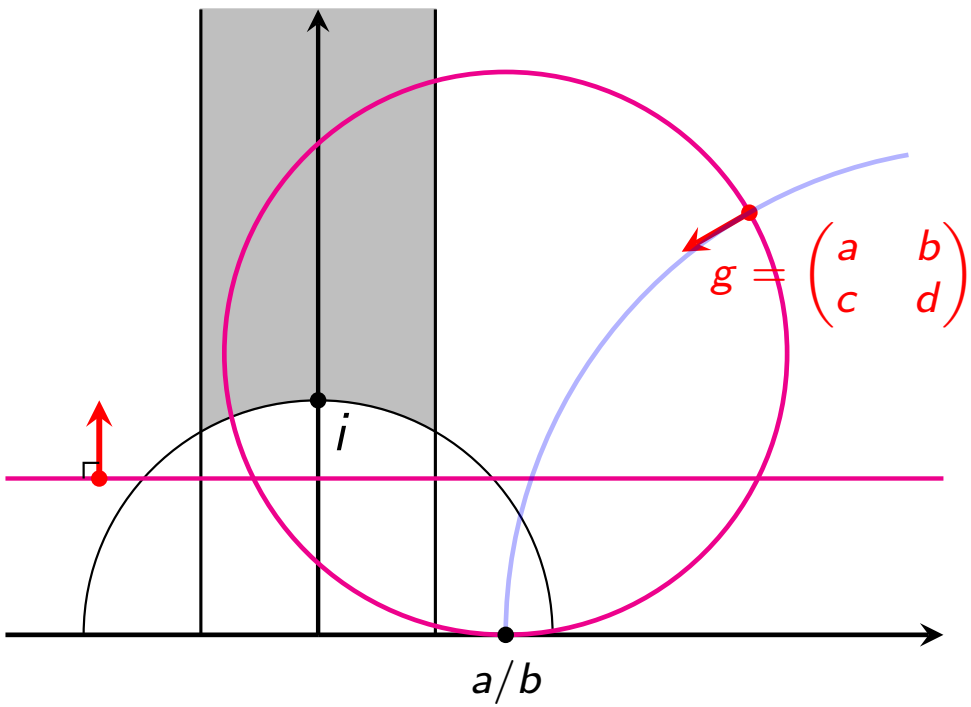
Any U-invariant ergodic measure is of the form μ_L, for some $L \leq G$.

Example : hyperbolic plane $\mathbb{H}^2$

unipotent flow $\left\{ u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} \leq \mathrm{SL}_2(\mathbb{R})$

$\leftrightarrow$ horocycle in $\mathbb{H}^2$.

$$\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$$



If a/b : irrational,

- $\overline{\{\pi(gu_t) : t \in \mathbb{R}\}} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$.
- $f \in C_0^{bdd}(\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z}))$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\pi(gu_t))dt = \int_{\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})} f \, dg.$$

Oppenheim conjecture

- $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z}) \simeq \{$ unimodular lattices in $\mathbb{R}^n\}$

  $$g\mathrm{SL}_n(\mathbb{Z}) \mapsto g\mathbb{Z}^n$$

- $\mathrm{vol}(\mathbb{R}^n/g\mathbb{Z}^n) = |\det g| = 1$

Conjecture (Oppenheim)

q : quadratic form on $\mathbb{R}^n$. e.g. $q(x,y,z) = x^2 + \sqrt{2}y^2 - z^2$

Is there a nonzero $v \in \mathbb{Z}^n$ such that

$$(0 <)\ |q(v)| < \varepsilon \text{ for any } \varepsilon > 0?$$

Theorem (Margulis, 1987)

$n \geq 3$, q : nondegenerate quadratic form on $\mathbb{R}^n$ s.t.

- q : isotropic ($q \sim 2x_1x_n + x_2^2 + - \cdots$),
- q : irrational ($q \neq \lambda q_0$, $\lambda \in \mathbb{R}$, q_0 : defined over $\mathbb{Q}$).

Then $q(\mathbb{Z}^n)$ is dense in $\mathbb{R}$.

Margulis' idea

Step 1 (Dimension Reduction) $\exists W \leq \mathbb{R}^n$ s.t.

$$\begin{cases} q|_W : \text{nondegenerate, isotropic, irrational} \\ \mathrm{rk}(\mathbb{Z}^n \cap W) = 3 \end{cases}$$

$\Rightarrow$ may assume : $n = 3$.

Step 2 $\mathrm{SO}(q) = \{g \in \mathrm{SL}_3(\mathbb{R}) : q(gv) = q(v),\ \forall v \in \mathbb{R}^3\}$
$\mathrm{SL}_3(\mathbb{Z}) = \{g \in \mathrm{SL}_3(\mathbb{R}) : g\mathbb{Z}^3 = \mathbb{Z}^3\}$

$\Rightarrow\ q(\mathbb{Z}^3) = q(\mathrm{SO}(q)\mathrm{SL}_3(\mathbb{Z})\mathbb{Z}^3)$

$\Rightarrow \overline{q(\mathbb{Z}^3)} = \mathbb{R}$ if $\overline{\pi(\mathrm{SO}(q))} = \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$,
where $\pi : \mathrm{SL}_3(\mathbb{R}) \to \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$.

Step 3

$$\begin{cases} \text{Ratner theorem} : \exists L \text{ s.t } \overline{\pi(\mathrm{SO}(q))} = \pi(L) \\ \text{maximality of } \mathrm{SO}(2,1) \text{ in } \mathrm{SL}_3(\mathbb{R}) \end{cases}$$

$\Rightarrow$ either $\overline{\pi(\mathrm{SO}(q))} = \pi(\mathrm{SO}(q))$ or $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$.

Margulis' idea

Step 1 (Dimension Reduction) $\exists W \leq \mathbb{R}^n$ s.t.

$$\begin{cases} q|_W \text{ : nondegenerate, isotropic, irrational} \\ \mathrm{rk}(\mathbb{Z}^n \cap W) = 3 \end{cases}$$

$\Rightarrow$ may assume : $n = 3$.

Step 2 $\mathrm{SO}(q) = \{g \in \mathrm{SL}_3(\mathbb{R}) : q(gv) = q(v),\ \forall v \in \mathbb{R}^3\}$
$\mathrm{SL}_3(\mathbb{Z}) = \{g \in \mathrm{SL}_3(\mathbb{R}) : g\mathbb{Z}^3 = \mathbb{Z}^3\}$

$\Rightarrow\ q(\mathbb{Z}^3) = q(\mathrm{SO}(q)\mathrm{SL}_3(\mathbb{Z})\mathbb{Z}^3)$

$\Rightarrow \overline{q(\mathbb{Z}^3)} = \mathbb{R}$ if $\overline{\pi(\mathrm{SO}(q))} = \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$,
where $\pi : \mathrm{SL}_3(\mathbb{R}) \to \mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$.

Step 3

$$\begin{cases} \text{dynamical property of a unipotent flow} \\ \text{maximality of } \mathrm{SO}(2,1) \text{ in } \mathrm{SL}_3(\mathbb{R}) \end{cases}$$

$\Rightarrow$ either $\overline{\pi(\mathrm{SO}(q))} = \pi(\mathrm{SO}(q))$ or $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$.

Quantitative Oppenheim conjecture

Theorem

q : nondeg. iso. irr. q.f. of sign $(p, q) \neq (2, 1), (2, 2)$
$\Omega \subset \mathbb{R}^n$: bounded convex set. e.g. $\Omega = B_1(0)$
$(a, b) \subset \mathbb{R}$: bounded interval

$$N^q_{(a,b)}(T) := \#\{\vec{v} \in \mathbb{Z}^n \cap B_T(0) : q(\vec{v}) \in (a, b)\} \sim \lambda_{q,\Omega}(b - a)T^{n-2}. \quad (*)$$

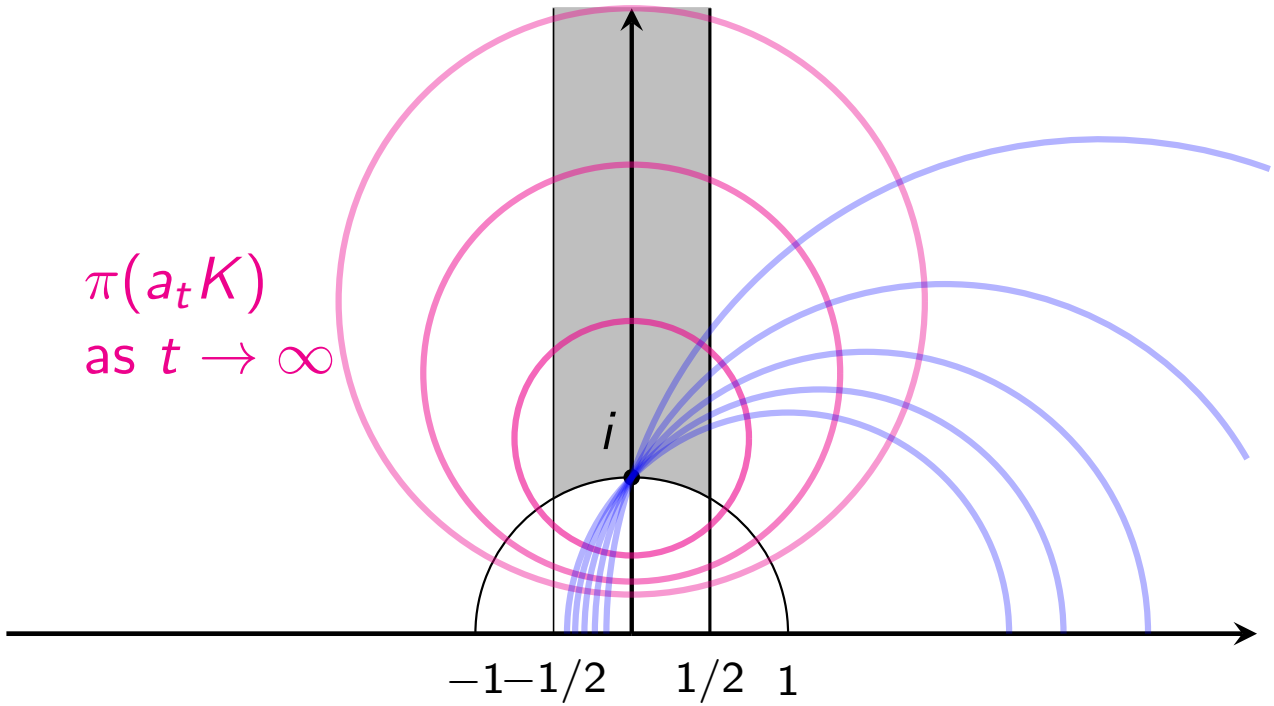
[Dani-Margulis'93] Lower bound
[Eskin-Margulis-Mozes'98] Upper bound

Theorem [E-M-M'98]

When sign(q)=$(2, 1)$ or $(2, 2)$, for given $\Omega \subset \mathbb{R}^n$, $(a, b) \subset \mathbb{R}$, (*) holds for almost every q.

Idea : Hyperbolic Plane $\mathbb{H}^2$ (sign(q)=(2,1))

- $K \simeq \mathrm{SO}(2)$: maximal cpt subgp of $\mathrm{SO}(q) \simeq \mathrm{SL}_2(\mathbb{R})$
  $\{a_t = \mathrm{diag}(e^{-t}, 1, \ldots, 1, e^t)\}$: diagonal flow of $\mathrm{SO}(q)$
- Want : use Equidistribution property for orbits of $a_t K$ as $t \to \infty$.



- $a_t K$-orbit $\leftrightarrow$ circle in $\mathbb{H}^2$ of radius $2t$
  $\Rightarrow$ measure of $a_t K \to$ Lebesgue measure of $\mathbb{S}^1$

Homogeneous spaces of $\mathrm{SO}(q)$, $\mathrm{rk}(q)=4$

- isotropic quadratic form q of rank 4 is equivalent to either

$$q \sim x^2 + y^2 + z^2 - w^2 \quad \text{or} \quad q \sim 2xy + 2zw$$

- $\mathrm{SO}(x^2 + y^2 + z^2 - w^2) = \mathrm{SO}(3,1) \simeq \mathrm{SL}_2(\mathbb{C})$
  the group of orientation-preserving isometries of $\mathbb{H}^3$.
- $\exists$ local isom $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}(2,2)$

$\mathrm{SO}(2,1) \simeq \mathrm{SL}_2(\mathbb{R})$	$\mathrm{SO}(3,1) \simeq \mathrm{SL}_2(\mathbb{C})$	$\mathrm{SO}(2,2) \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$
$K = \mathrm{SO}(2)$	$\mathrm{SO}(3)$	$\mathrm{SO}(2) \times \mathrm{SO}(2)$
$K \setminus G \simeq \mathbb{H}^2$	$\mathbb{H}^3$	$\mathbb{H}^2 \times \mathbb{H}^2$
$a_t K$-orbit $\simeq S^1_{2t}$	S^2_{2t}	$S^1_t \times S^1_t$

The Quantitative Oppenheim conjecture for S-arithmetic quadratic forms

field $\mathbb{Q}_p$

p : an odd prime

- $\mathbb{Q}_p$= the completion field of $\mathbb{Q}$ w.r.t $|p^n u|_p = p^{-n}$.

$$x = \sum_{i \geq i_0} a_i p^i = (..., 0, 0, a_{i_0}, a_{i_0+1}, \ldots, a_k, \ldots),$$

where $a_i \in \{0, 1, \ldots, p-1\}$.

Topology

- $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$

$$= \left\{ \sum_{i \geq 0} a_i p^i \right\} \text{ compact open set}$$

- basis : $\{x \in \mathbb{Q}_p : |x - x_0| \leq p^{-n}\} = x_0 + p^n \mathbb{Z}_p$
- Haar measure of $\mathbb{Q}_p$: $\mu_p(x_0 + p^n \mathbb{Z}_p) = p^{-n}$

S-arithmetic

- $S = \{\infty, p_1, \ldots, p_s\}$, p_i : odd prime
  e.g. $S = \{\infty, p\}$
- $\mathbb{Q}_S = \mathbb{R} \times \prod \mathbb{Q}_p = \mathbb{R} \times \mathbb{Q}_p \ \leftrightarrow \mathbb{R}$
- $\mathbb{Z}_S = \{(z, z) \in \mathbb{Q}^2 : z = m/p^k,\ m, k \in \mathbb{Z}\ \}$
  : the set of S-integers in $\mathbb{Q}_S \ \leftrightarrow \mathbb{Z}$
- Λ : S-lattice = Free $\mathbb{Z}_S$-module in $\mathbb{Q}_S^n$ of rank n, i.e.,
  $\exists \overrightarrow{\mathrm{v}}_1, \ldots, \overrightarrow{\mathrm{v}}_n$ s.t. (a) $\Lambda = \mathbb{Z}_S \overrightarrow{\mathrm{v}}_1 \oplus \cdots \oplus \mathbb{Z}_S \overrightarrow{\mathrm{v}}_n$;
  (b) $\mathbb{Q}_S^n$ is generated by $\overrightarrow{\mathrm{v}}_1, \ldots, \overrightarrow{\mathrm{v}}_n$.
- $q = \{q^0/\mathbb{R}^n, q^p/\mathbb{Q}_p^n\}$ S-adic quadratic form

  q : rational if $q = (\lambda_0 q', \lambda_p q') = \lambda q'$,
  where q' : quadratic form $/\mathbb{Q}$, $\lambda = (\lambda_0, \lambda_p) \in \mathbb{Q}_S$.

- $I(a, b) = (a_0, b_0) \times (a_p + p^{b_p}\mathbb{Z}_p) \subset \mathbb{Q}_S$ S-interval

S-arithmetic Quantitative Oppenheim conjecture

For $T = (T_0, T_p)$, let

$$N^q_{I(a,b)}(T) = \left\{ \vec{\mathrm{v}} \in \mathbb{Z}^n_S : q(\vec{\mathrm{v}}) \in I(a,b) \text{ and } \begin{array}{l} \|\vec{\mathrm{v}}\|_\infty < T_0; \\ \|\vec{\mathrm{v}}\|_p < T_p. \end{array} \right\}.$$

Theorem (H.-Lim-Mallahi Karai (2017))

q : isotropic irrational quadratic form of rank ≥ 4 s.t.

each $q^v \approx xy - zw$.

Then as $T_v \to \infty$ for $\forall\ v = \infty,\ p$,

$$|N^q_{I(a,b)}(T)| \sim \lambda(q)\mu(I(a,b))(\prod_v T_v)^{n-2}.$$

Theorem (H. preprint)

For almost all isotropic quadratic forms q of rank 3 or 4, as $T_v \to \infty$ for each v,

$$N^q_{I(a,b)}(T) \sim \lambda_q\mu(I(a,b))(\prod_v T_v)^{n-2}.$$

$\mathrm{SO}(q)(\mathbb{Q}_p)$, with $\mathrm{rk}(q) = 3$

- $\exists$ 7 isotropic equi. classes of q. f.
- $\mathrm{SO}(q)$ is unique, i.e., $\mathrm{SO}(q) \simeq \mathrm{SO}(q')$ even if $q \nsim q'$.
- $\mathrm{SO}(2xz - y^2)(\mathbb{Q}_p) \simeq \mathrm{SL}_2(\mathbb{Q}_p)$,
  (choose) maximal cpt subgp $K = \mathrm{SL}_2(\mathbb{Z}_p)$.
- $K \setminus \mathrm{SL}_2(\mathbb{Q}_p)$?

The $(p+1)$-regular tree $\mathcal{T}_p$

- Consider free $\mathbb{Z}_p$-modules L of rank 2 in $\mathbb{Q}_p^2$
- $L_1 \sim L_2$ if $L_2 = p^k L_1$, i.e. $\exists v_1, v_2 \in \mathbb{Q}_p^2$ s.t.

$$L_1 = \mathbb{Z}_p v_1 \oplus \mathbb{Z}_p v_2 \quad \text{and} \quad L_2 = \mathbb{Z}_p(p^k v_1) \oplus \mathbb{Z}_p(p^k v_2).$$

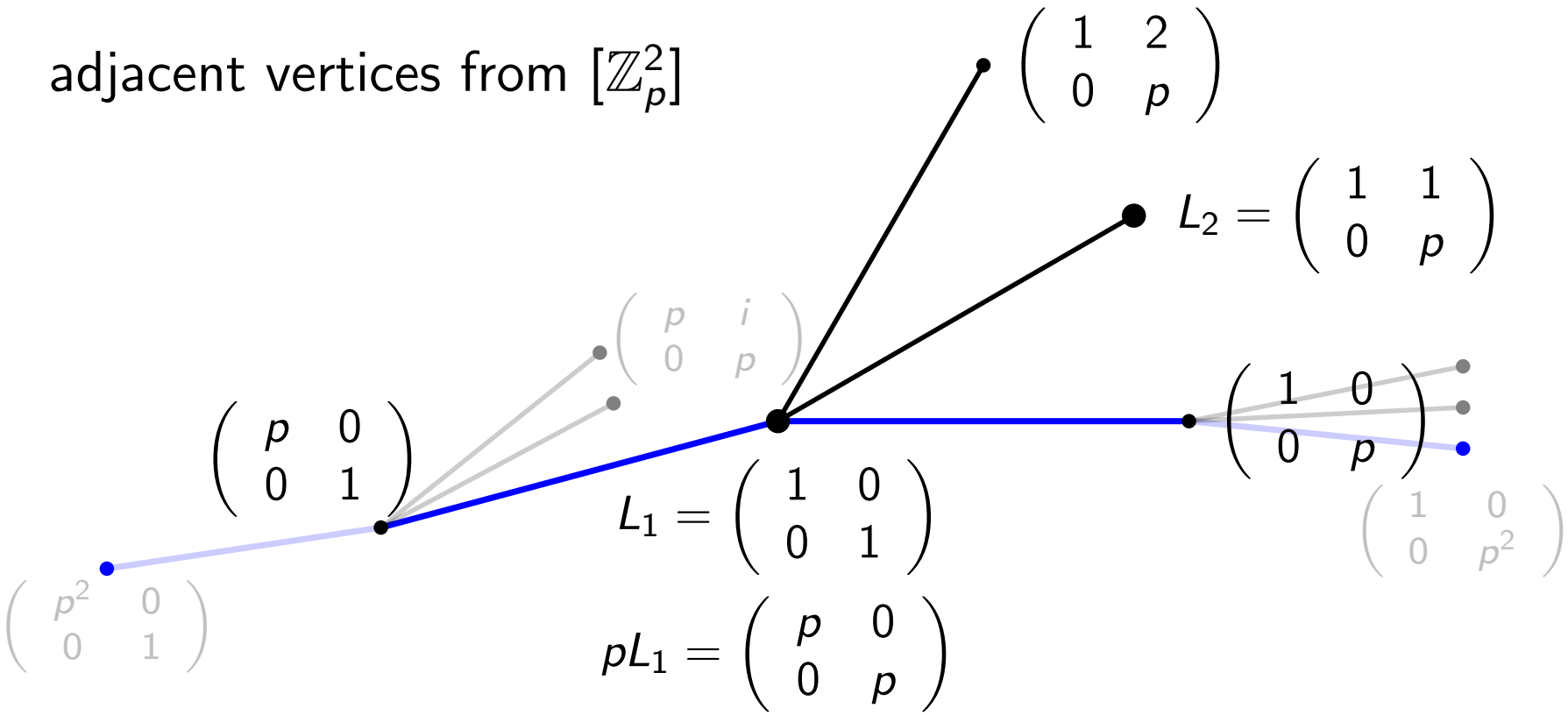
- $\mathcal{V}(\mathcal{T}_p) = \{[L] \mid L : \text{ free } \mathbb{Z}_p\text{-modules of rank 2 in } \mathbb{Q}_p^2 \}$,
  $\mathcal{E}(\mathcal{T}_p) \ni (x_1, x_2)$ if $\exists L_1,\ L_2$ s.t. $L_1 \in x_1,\ L_2 \in x_2$ and

$$pL_1 \subsetneq L_2 \subsetneq L_1.$$

Example : $\mathcal{T}_3$

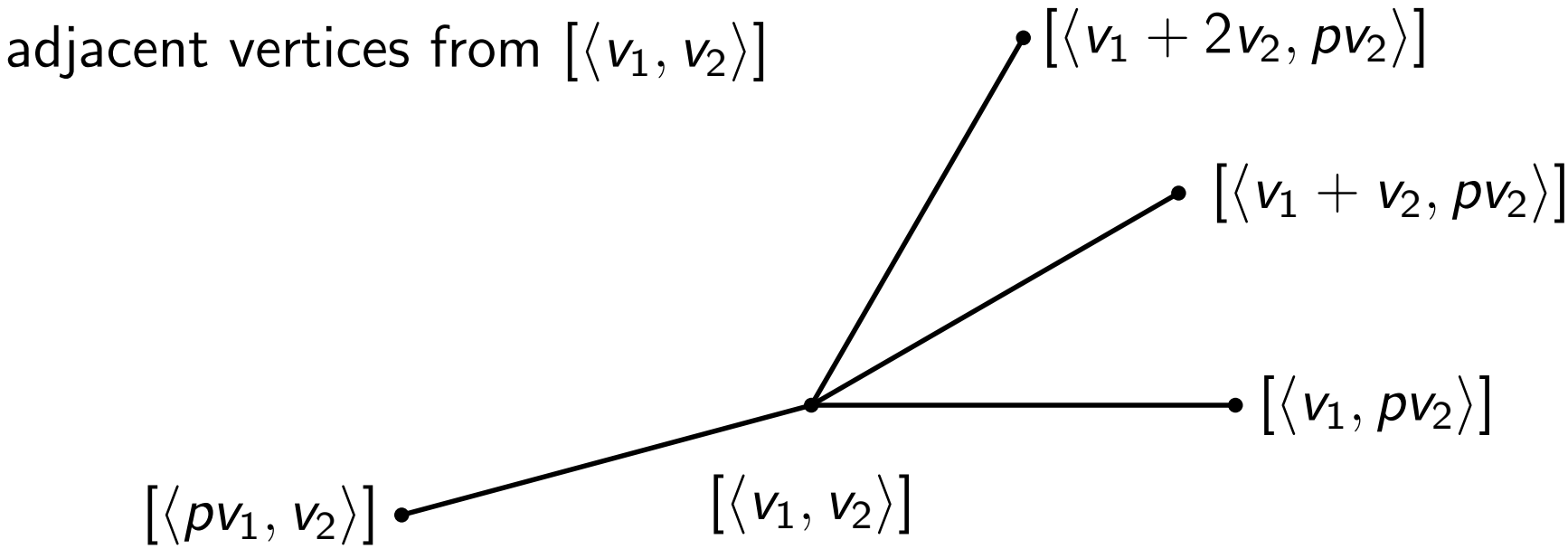
Notation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow [L = \mathbb{Z}_p(a,b) \oplus \mathbb{Z}_p(c,d)]$

adjacent vertices from $[\mathbb{Z}_p^2]$



e.g. $pL_1 \subsetneq L_2 = \langle (1,1),(0,p) \rangle \subsetneq L_1 = \langle (1,0),(0,1) \rangle$

- $L_2 \subseteq L_1$: obvious and $(0,1) \notin L_2$
- $(p,0) = p(1,1) - (0,p) \Rightarrow pL_1 \subseteq L_2$ and $(1,1) \notin pL_1$



- $\mathrm{GL}_2(\mathbb{Z}_p) \setminus \mathrm{GL}_2(\mathbb{Q}_p) \simeq \mathcal{T}_p$ by $[\langle v_1, v_2\rangle].g = [\langle v_1 g, v_2 g\rangle]$.
- $\mathrm{SL}_2(\mathbb{Z}_p) \setminus \mathrm{SL}_2(\mathbb{Q}_p) \simeq \{x \in \mathcal{V}(\mathcal{T}_p) : d(x, [\mathbb{Z}_p^2]) = \text{even}\}$
- $a_t = \mathrm{diag}(p^t, p^{-t})$, $K = \mathrm{SL}_2(\mathbb{Z}_p)$
  $a_t K$-orbit $\leftrightarrow \{v \in \mathcal{T}_p : d(v, [\mathbb{Z}_p^2]) = 2t\} =: S_{2t}^1$
  $\Rightarrow$ measure on $a_t K \to$ counting measure on S_{2t}^1

$\mathrm{GL}_3(\mathbb{Z}_p) \setminus \mathrm{GL}_3(\mathbb{Q}_p) \simeq \mathcal{B}_{3,p}$ (Building)

- L : free $\mathbb{Z}_p$-module of rank 3 in $\mathbb{Q}_p^3$
- $L_1 \sim L_2$ if $L_2 = p^k L_1$, i.e. $\exists v_1, v_2, v_3 \in \mathbb{Q}_p^3$ s.t.

$$L_1 = \langle v_1, v_2, v_3 \rangle \quad \text{and} \quad L_2 = \langle p^k v_1, p^k v_2, p^k v_3 \rangle.$$

- $\mathcal{V}(\mathcal{B}_{3,p}) = \{ [L] |\ L :\ \text{free } \mathbb{Z}_p\text{-modules of rank 3 in } \mathbb{Q}_p^3 \}$
- $\mathcal{E}(\mathcal{B}_{3,p}) \ni (x_1, x_2)$ if $\exists L_1 \in x_1,\ L_2 \in x_2$ s.t.

$$pL_1 \subsetneq L_2 \subsetneq L_1.$$

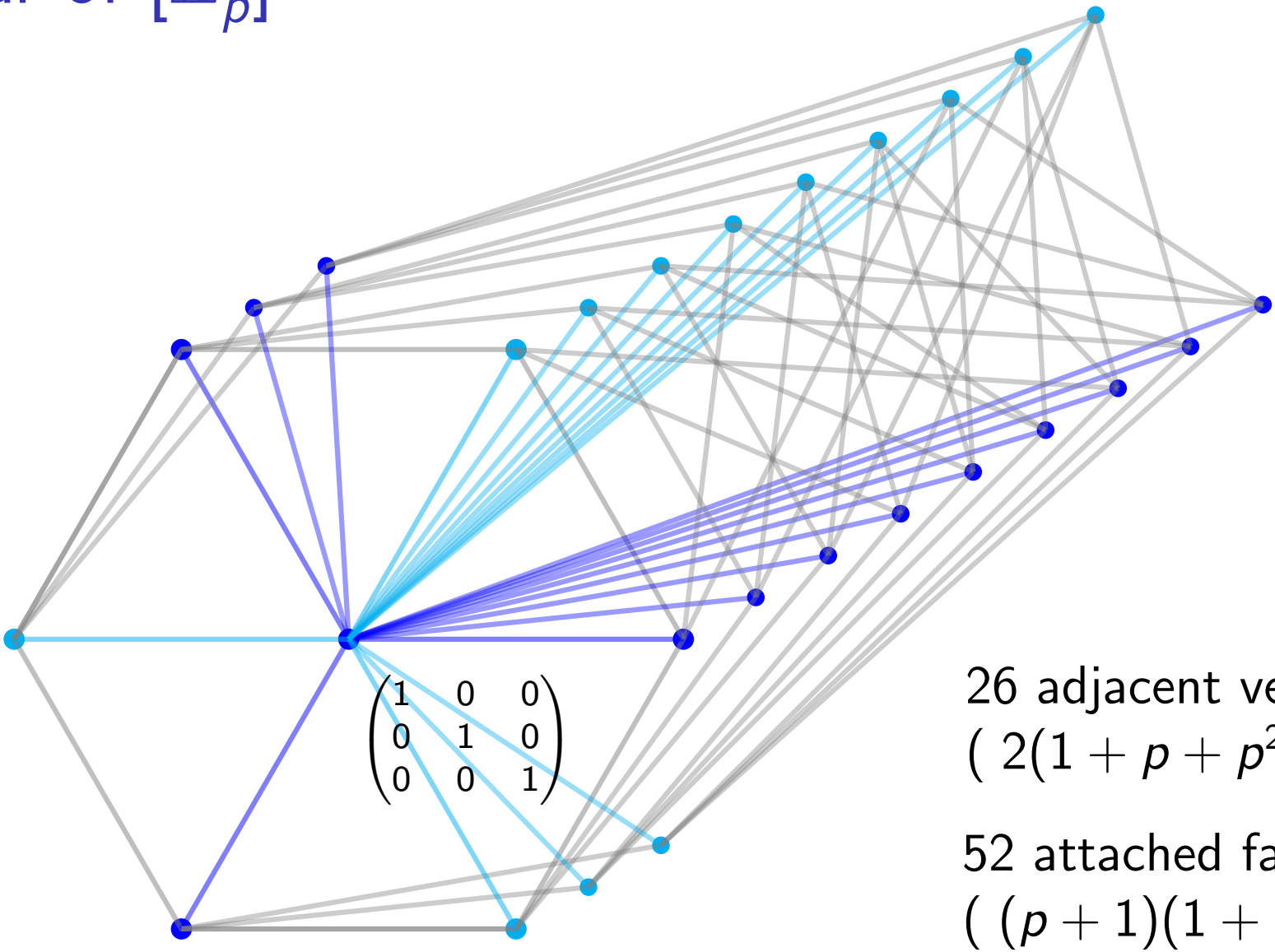
- $\mathcal{F}(\mathcal{B}_{3,p}) \ni (x_1, x_2, x_3)$ if $\exists L_1 \in x_1,\ L_2 \in x_2,\ L_3 \in x_3$ s.t.

$$pL_1 \subsetneq L_2 \subsetneq L_3 \subsetneq L_1.$$

e.g.

$$p\langle e_1, e_2, e_3 \rangle \subsetneq \langle e_1, pe_2, pe_3 \rangle \subsetneq \langle e_1, e_2, pe_3 \rangle \subsetneq \langle e_1, e_2, e_3 \rangle$$

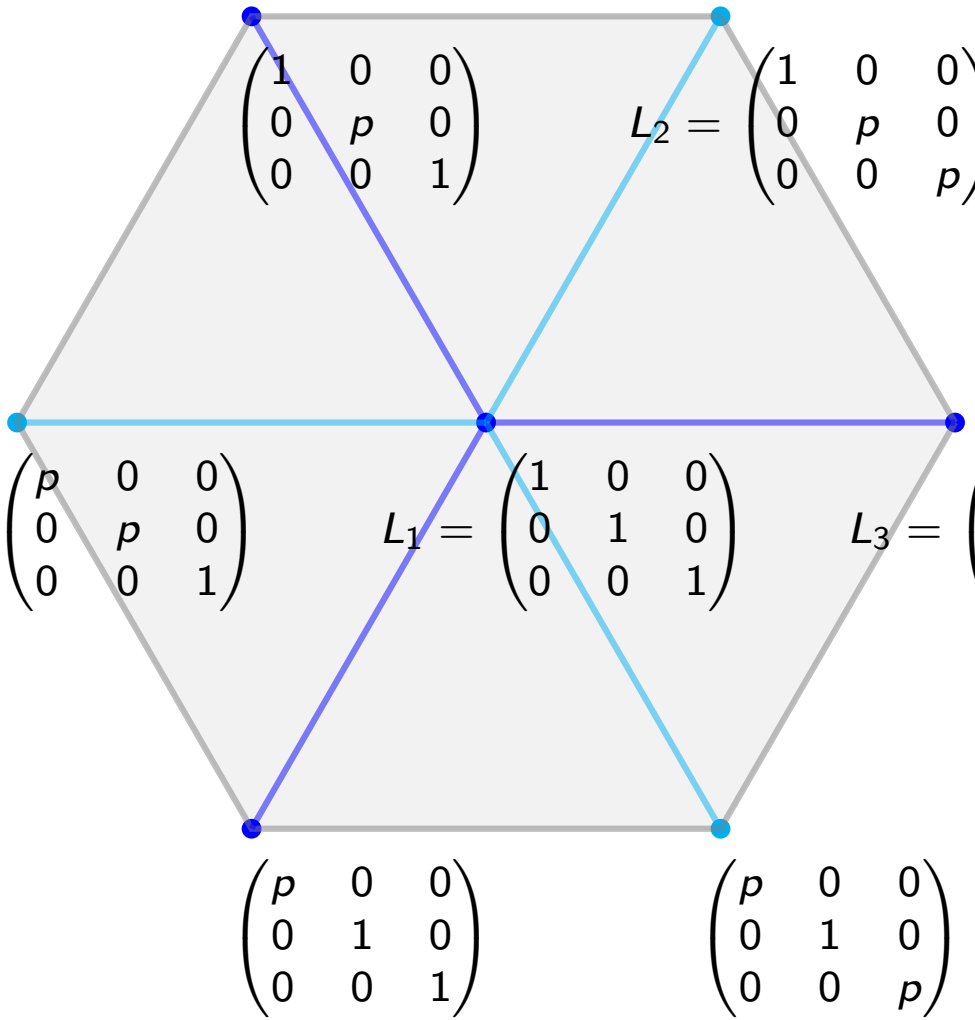
Star of $[\mathbb{Z}_p^3]$



26 adjacent vertices
($2(1 + p + p^2)$)

52 attached faces
($(p + 1)(1 + p + p^2)$)

projection of the Star (in $\mathbb{R}^2$)



- For each vertex, there are two types of adjacent vertices

  e.g. $L_1 = \langle e_1, e_2, e_3 \rangle$

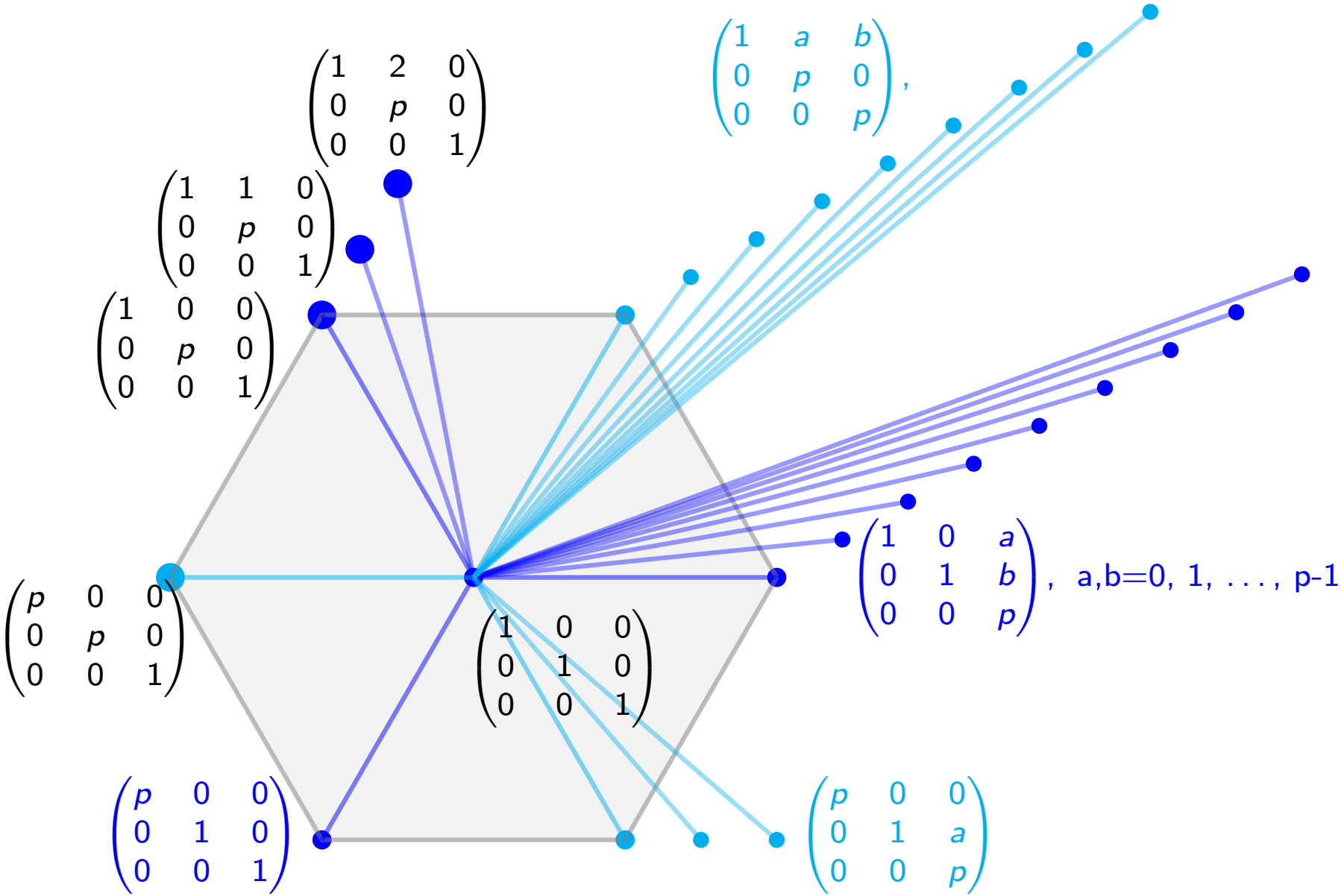
  $L_2 = \langle e_1, pe_2, pe_3 \rangle$

  $L_3 = \langle e_1, e_2, pe_3 \rangle$

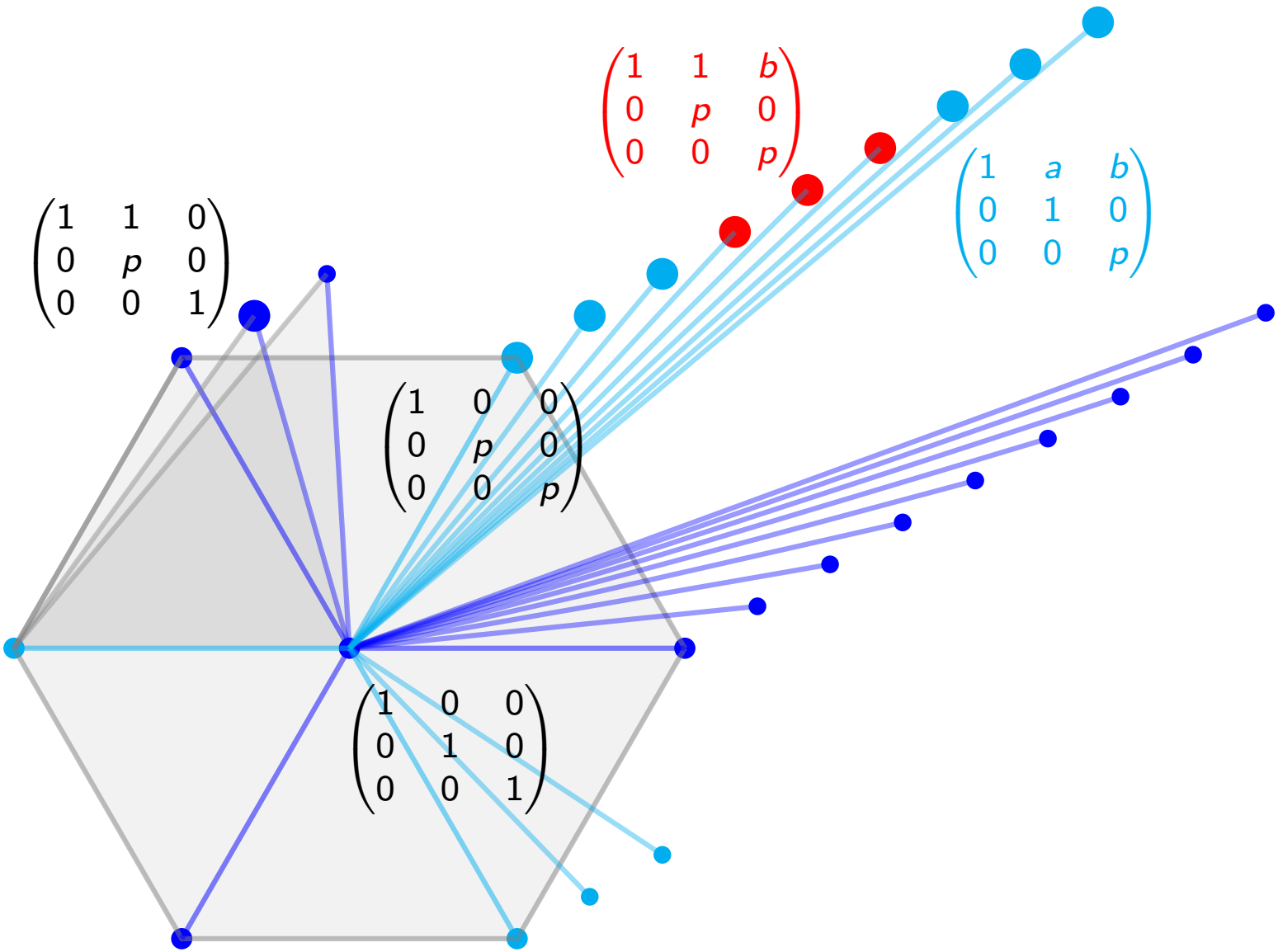
$\Rightarrow [L_1 : L_2] = p^2,$

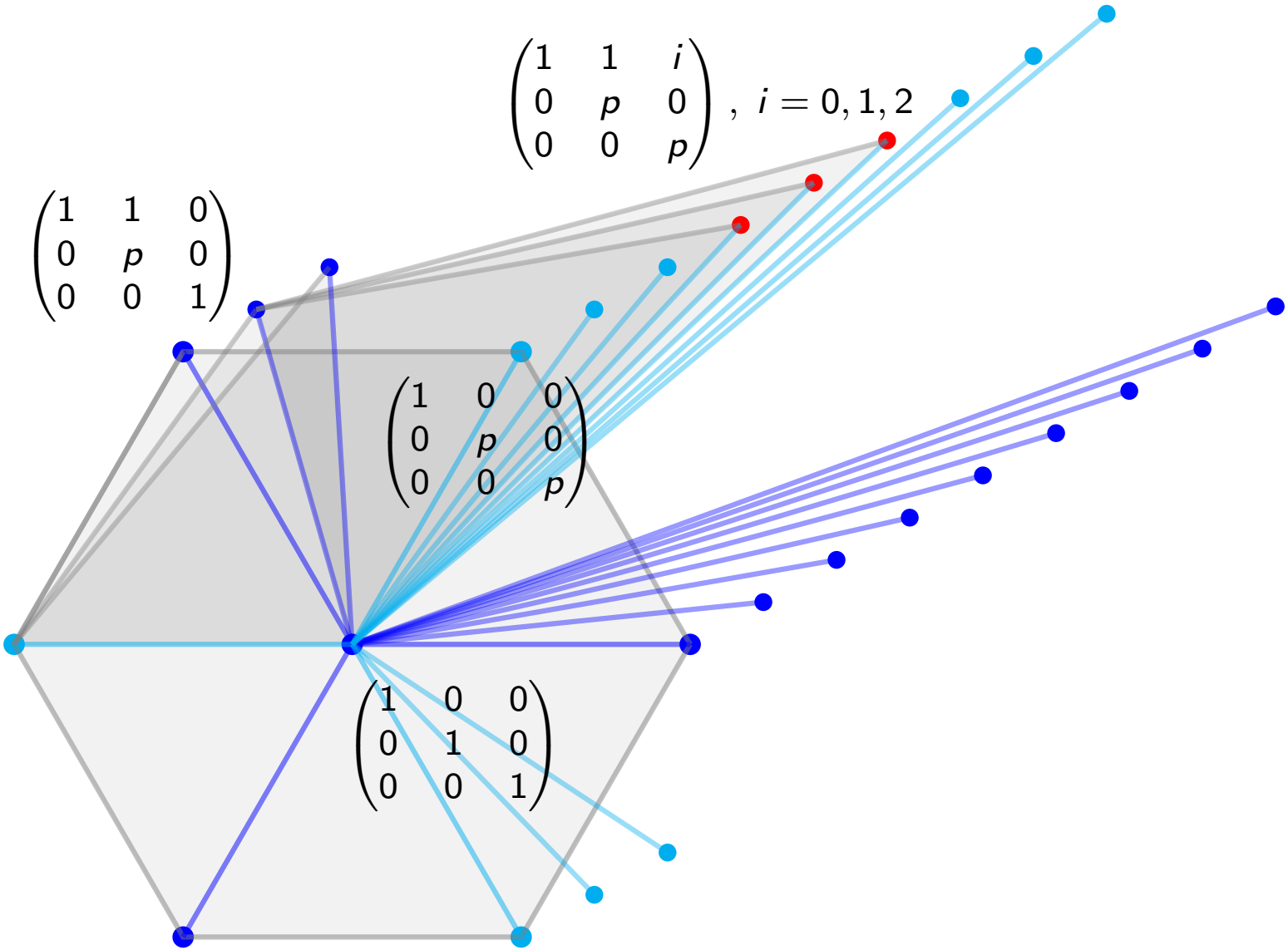
$[L_1 : L_3] = p.$

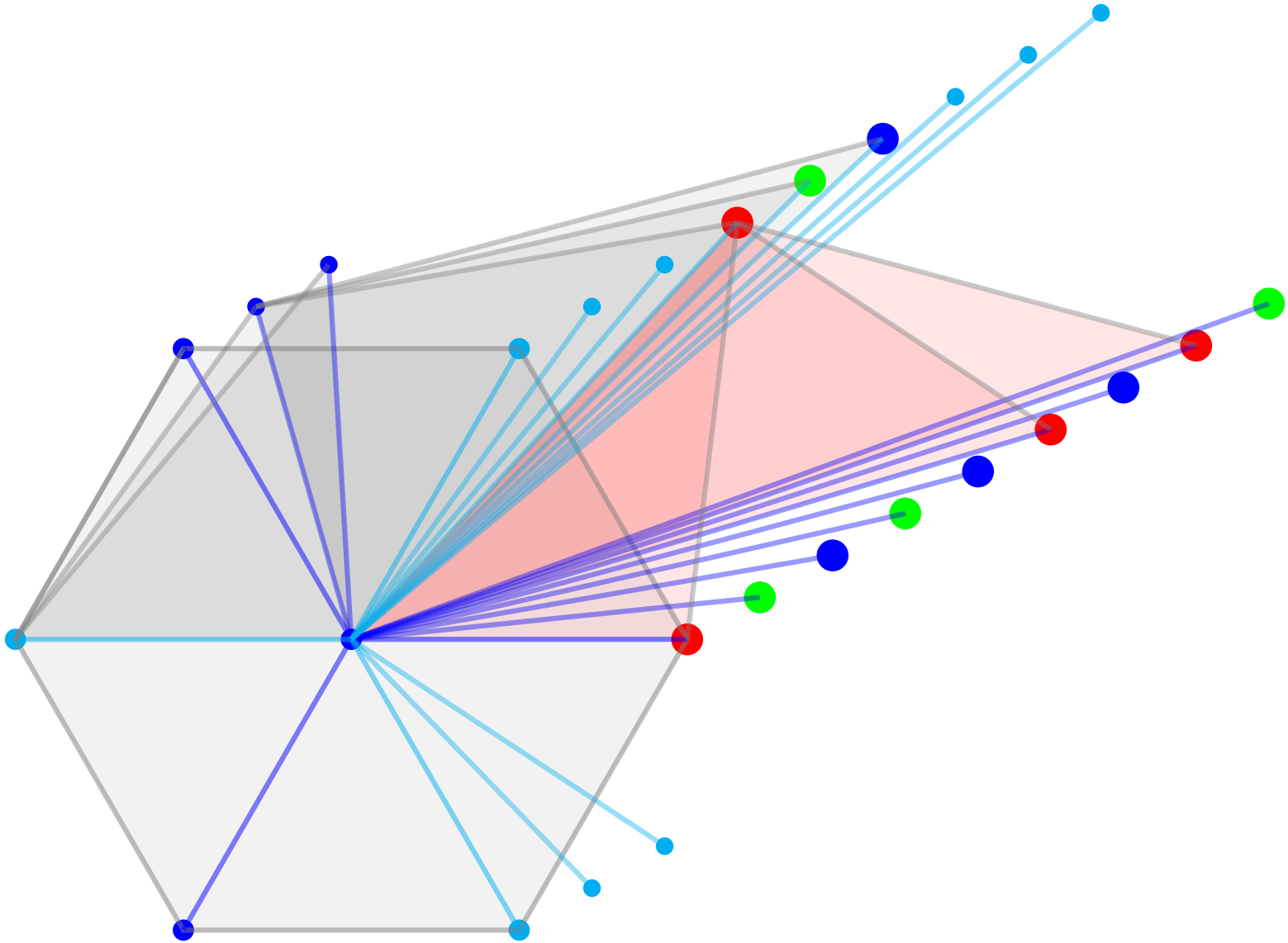
adjacent vertices (w.r.t. the same proj images)

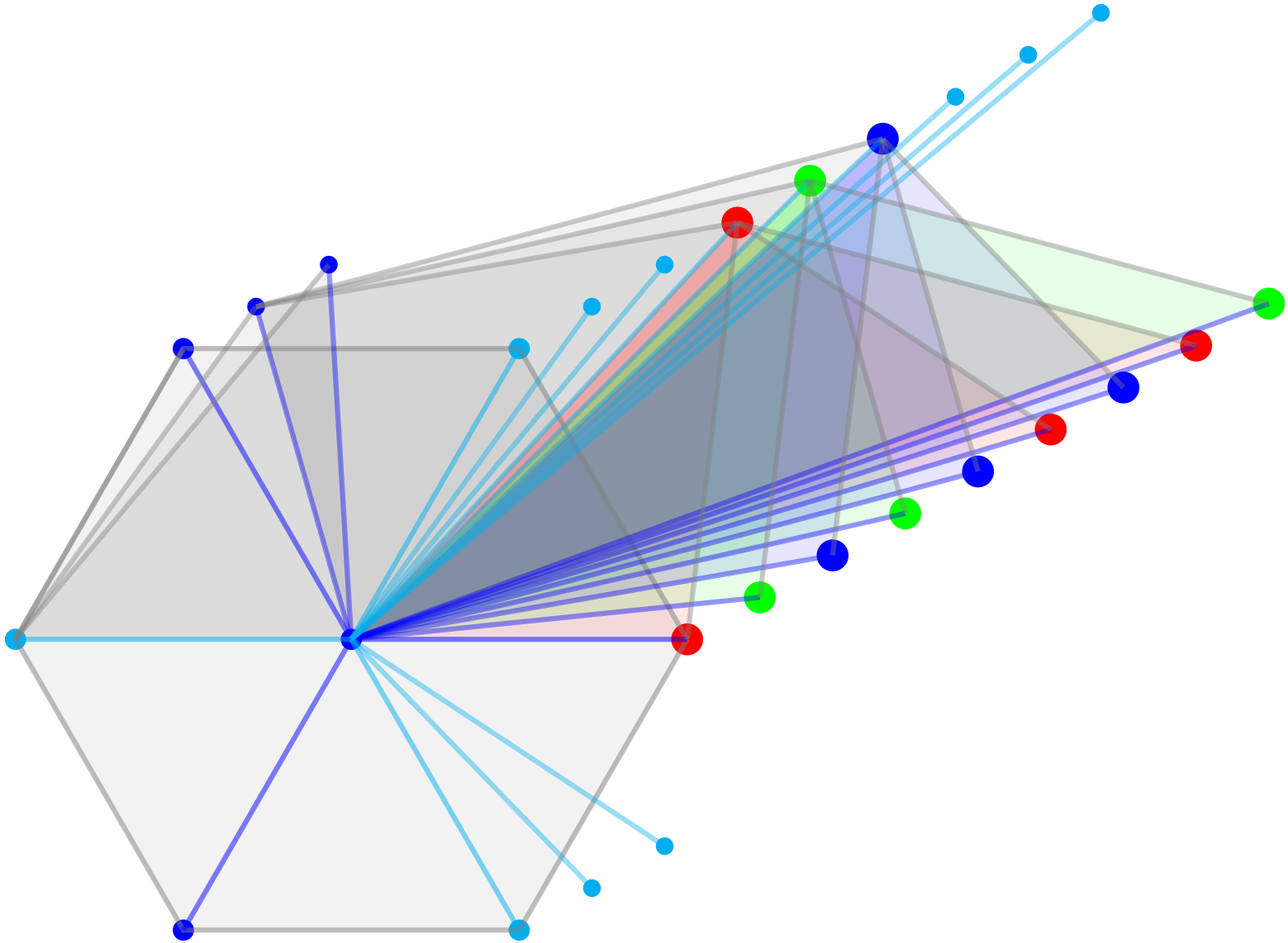


faces containing $\langle e_1, e_2, e_3 \rangle$ & $\langle pe_1, pe_2, e_3 \rangle$

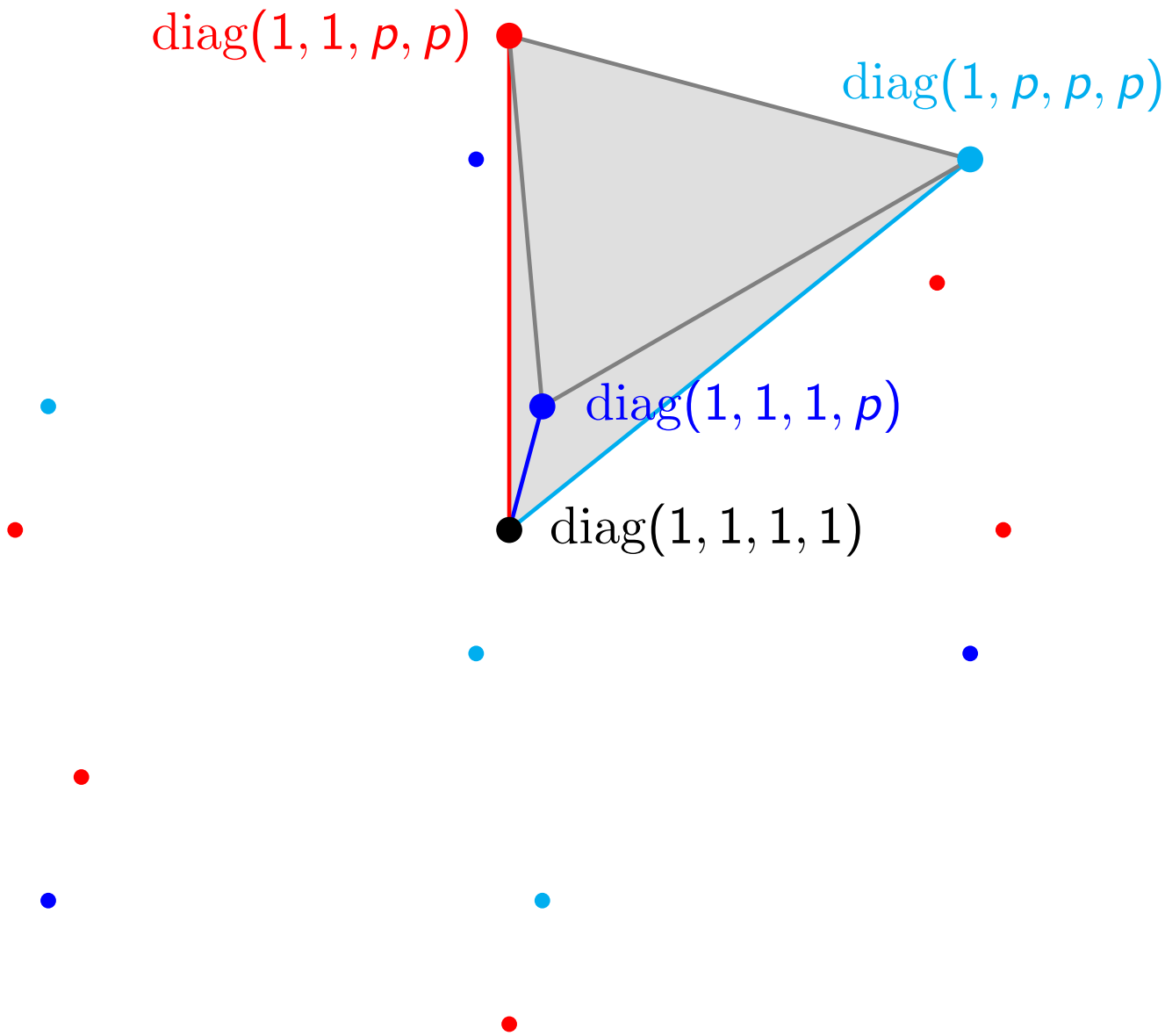



$\begin{pmatrix} 1 & 1 & i \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$, $i = 0, 1, 2$
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

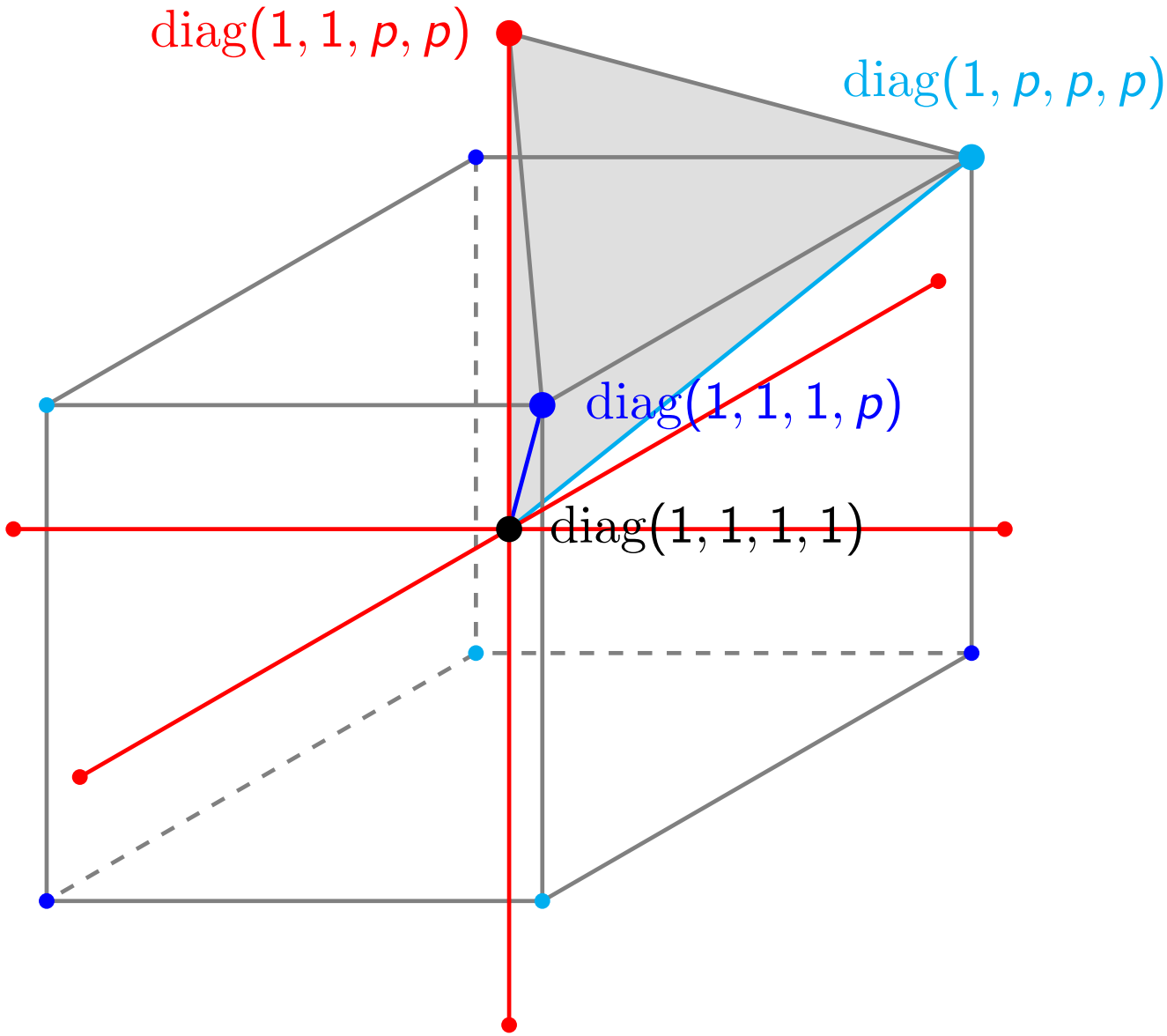




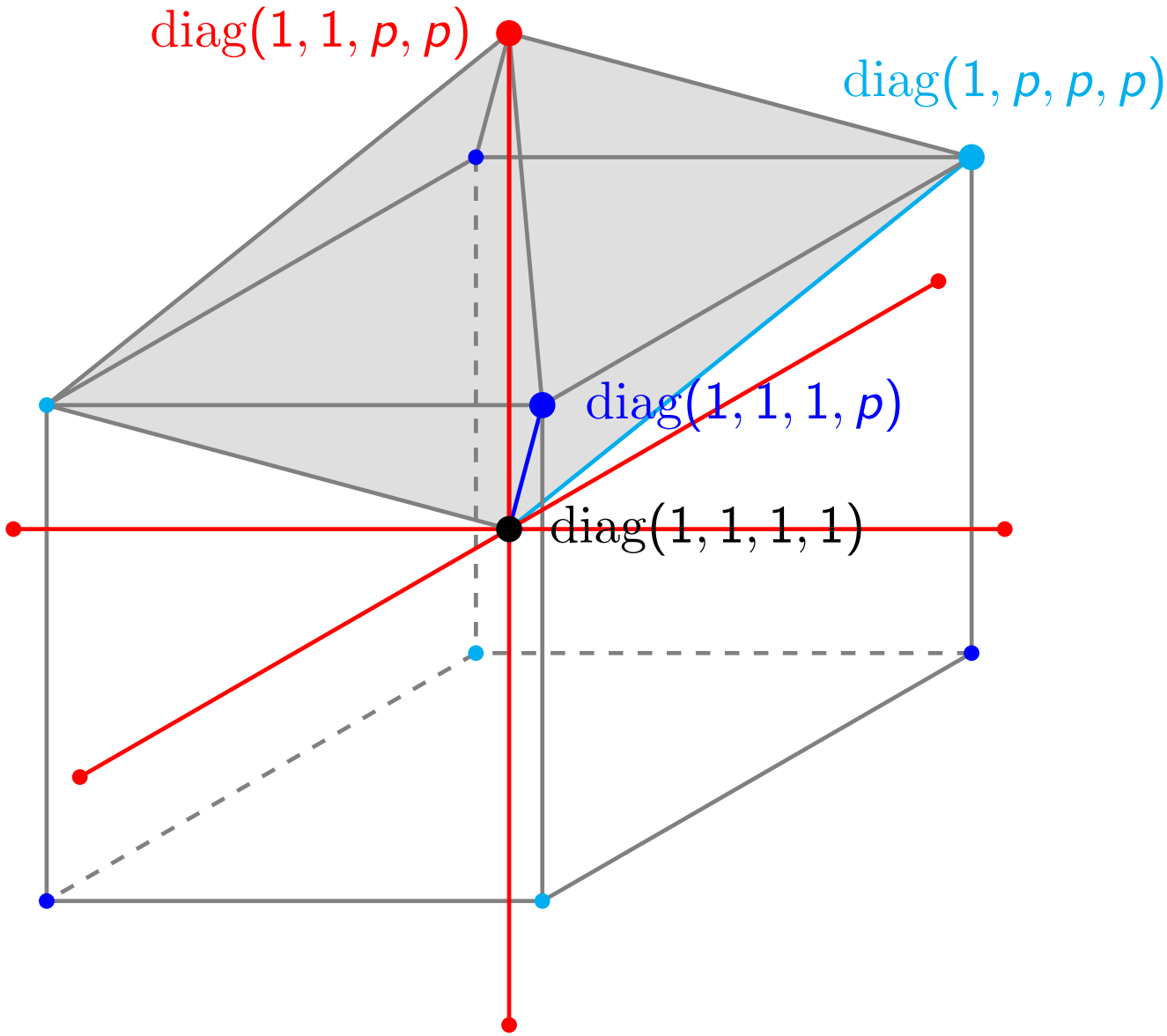
projection of the Star of $\mathrm{GL}_4(\mathbb{Z}_p) \setminus \mathrm{GL}_4(\mathbb{Q}_p)$



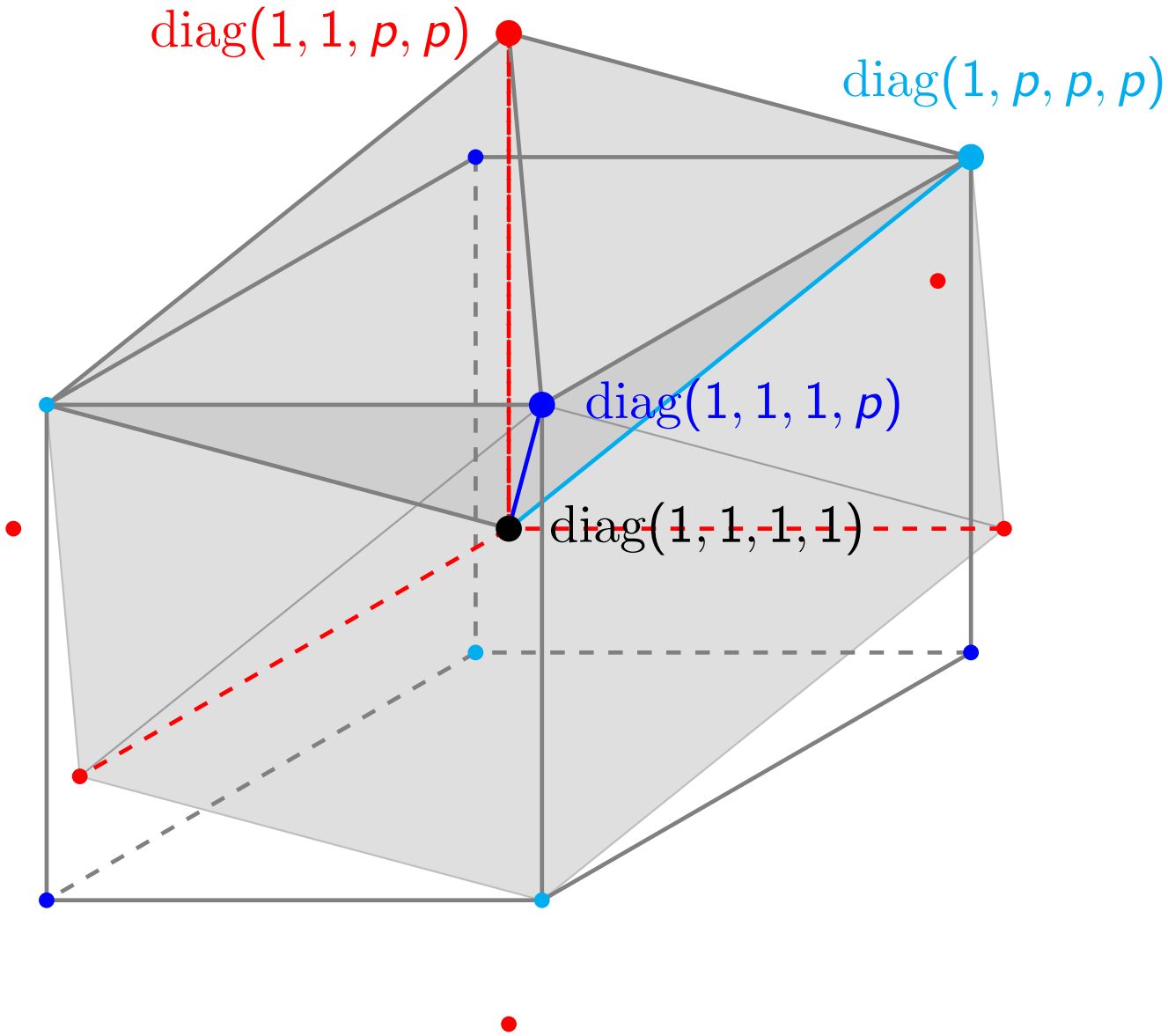
projection of the Star of $\mathrm{GL}_4(\mathbb{Z}_p) \setminus \mathrm{GL}_4(\mathbb{Q}_p)$



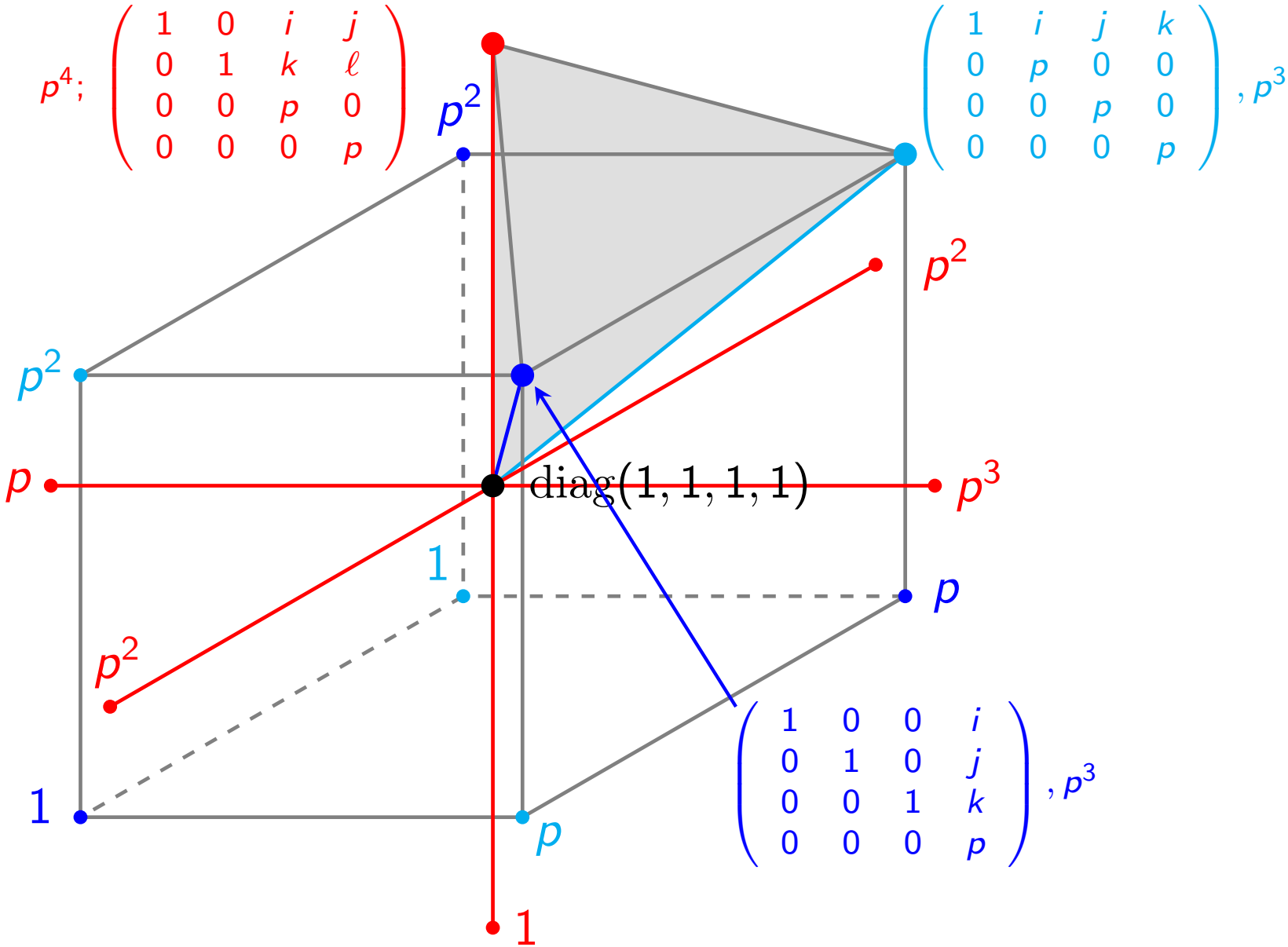
projection of the Star of $\mathrm{GL}_4(\mathbb{Z}_p) \setminus \mathrm{GL}_4(\mathbb{Q}_p)$



projection of the Star of $\mathrm{GL}_4(\mathbb{Z}_p) \setminus \mathrm{GL}_4(\mathbb{Q}_p)$



vertices of the Star of $\mathrm{GL}_4(\mathbb{Z}_p) \setminus \mathrm{GL}_4(\mathbb{Q}_p)$



$\mathrm{SO}(q)(\mathbb{Q}_p)$ with $\mathrm{rk}(q) = 4$

- All 8 equi. classes of q. f. of rank 4 are isotropic.

Lemma (H.)

q : isotropic quadratic form of $\mathbb{Q}_p^4$.

If $q \sim xy - zw$, then $\mathrm{SO}(q)$ is of $\mathbb{Q}_p$-rank one.

Corollary (H.)

If $q \sim xy - zw$, then $K \setminus \mathrm{SO}(q)$ is a tree.

rank=3	rank=4	
$\mathrm{SO}(q) \simeq \mathrm{SL}_2(\mathbb{Q}_p)$	$\mathrm{SO}(q)$	$\mathrm{SO}(q) \simeq \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_p)$
$K = \mathrm{SL}_2(\mathbb{Z}_p)$	$\mathrm{SO}(q)(\mathbb{Z}_p)$	$\mathrm{SL}_2(\mathbb{Z}_p) \times \mathrm{SL}_2(\mathbb{Z}_p)$
$K \setminus G \simeq \mathcal{T}_p$	Tree in $\mathcal{B}_4$	$\mathcal{T}_p \times \mathcal{T}_p$
$a_t K$-orbit $\simeq S^1_{2t}$	S^2_{2t}	$S^1_t \times S^1_t$

Thank you!!

Shape optimization problems for Steklov eigenvalue

Dong-Hwi Seo

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제15회 고등과학원 기하학 겨울학교, Dec. 30, 2019

Plan of the talk

1. Introduction
2. Previous results
3. Proof of the result

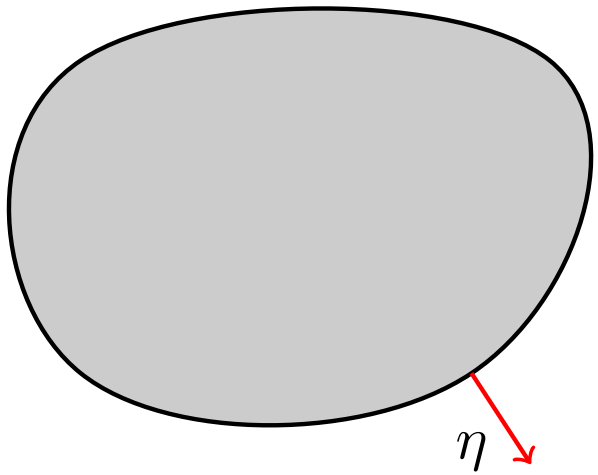
Introduction

Steklov eigenvalue problem

Let M^m : Riemannian manifold, $\Omega \subset M$: smooth domain. Finding all $\sigma \in \mathbb{R}$ for which there exists $u \in C^\infty(\Omega)$ which satisfies

$$\begin{cases} \Delta u &= 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} &= \sigma u & \text{on } \partial\Omega, \end{cases}$$

where η is the outward unit normal vector on $\partial\Omega$.



Steklov eigenvalue problem

The problem is equivalent to the following eigenvalue problem of Dirichlet-to-Neumann operator.
Given $u \in C^\infty(\partial\Omega)$, let $\hat{u}$ be the harmonic extension of u :

$$\begin{cases} \Delta\hat{u} &= 0 \quad \text{in } \Omega \\ \hat{u} &= u \quad \text{on } \partial\Omega \end{cases}.$$

Define a Dirichlet-to-Neumann operator

$$L : C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$$
$$u \mapsto \frac{\partial\hat{u}}{\partial\eta}.$$

Steklov eigenvalue problem

Then L is a nonnegative self-adjoint operator with discrete spectrum

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots \to \infty.$$

The first eigenvalue $\sigma_1(\Omega)$ is characterized as following.

$$\sigma_1(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla v|^2 dV}{\int_{\partial\Omega} v^2 ds} \middle| v \in H^1(\Omega) \setminus \{0\} \text{ and } \int_{\partial\Omega} v ds = 0 \right\}.$$

Shape optimization problem

Our shape optimization problem is stated as follows : Given a collection $\mathcal{F}$ of subsets in M, finding $\Omega_0 \in \mathcal{F}$ such that

$$\sigma_1(\Omega_0) = \max_{\Omega \in \mathcal{F}} \sigma_1(\Omega).$$

Previous results

Results on Euclidean spaces

- (Weinstock, 1954) For $M = \mathbb{R}^2$, Ω_0 is a disk among simply connected smooth sets with fixed boundary length.
- (Brock, 2001) For $M = \mathbb{R}^m$, Ω_0 is a ball among all smooth domains with fixed domain volume.
- (Bucur-Ferone-Nitsch-Trombetti, 2017) For $M = \mathbb{R}^m$, $m \geq 3$, Ω_0 is a ball among convex sets with fixed boundary volume.
- (Fraser-Schoen, 2019) For $M = \mathbb{R}^m, m \geq 3$, Ω_0 is not a ball among smooth contractible domains with fixed boundary volume.

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Results on non Euclidean spaces

- (Escobar, 1999) For $M = \mathbb{S}^2$ or $\mathbb{H}^2$, Ω_0 is a geodesic ball among all simply connected domains with fixed domain area.
- (Binoy-Santhanam, 2014) For noncompact rank one symmetric spaces M, Ω_0 is a geodesic ball among all smooth domains with fixed domain volume.

Note that noncompact rank one symmetric spaces(nCROSS) are $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$, and $\mathbb{O}H^2$.

Results on non Euclidean spaces

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Mixed Steklov-Dirichlet eigenvalue problem

Theorem (Hersch-Payne, 1968)

Let $\Omega \subset \mathbb{R}^2$ be a bounded doubly connected domain and $\partial\Omega = C_1 \cup C_2$, where C_1 and C_2 are the inner and outer boundary, respectively. Consider a mixed Steklov-Dirichlet eigenvalue problem

$$\begin{cases} \Delta u &= 0 & \text{in } \Omega \\ u &= 0 & \text{on } C_1 \\ \frac{\partial u}{\partial \eta} &= \sigma u & \text{on } C_2. \end{cases}$$

Then $\sigma_1(\Omega) \leq \sigma_1(A)$, where A is the annulus conformally equivalent to Ω and the perimeter of the outer boundary of A is equal to the perimeter of C_2.

Recent results on annular regions

Denote the closure of a set $A \subset M$ by $\mathrm{cl}(A)$.

Theorem (Santhanam-Verma, 2018)

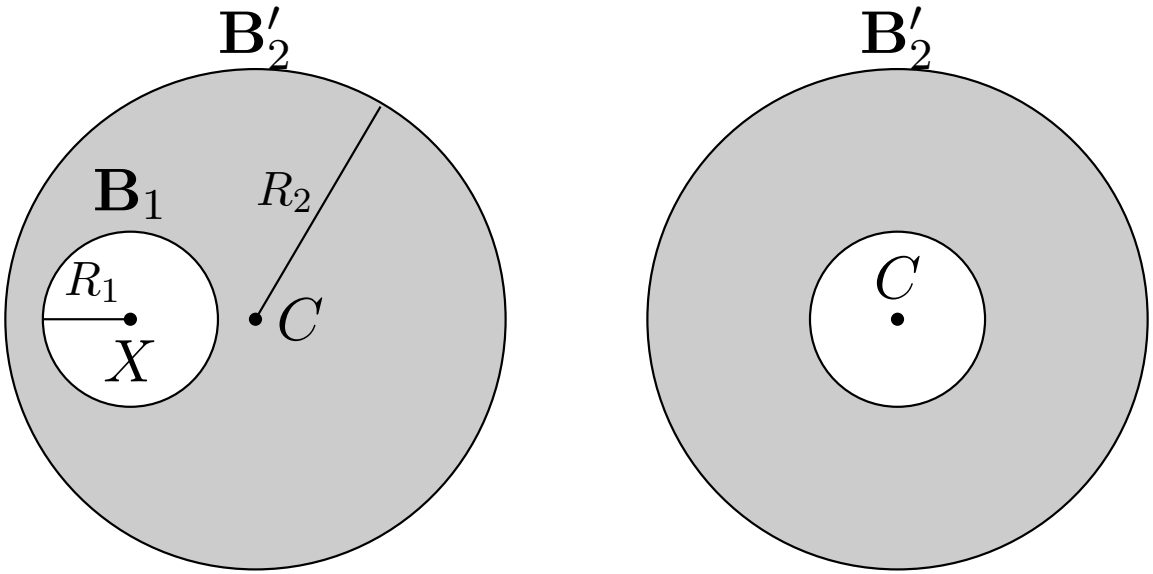
Let $\mathbf{B}_1, \mathbf{B}_2' \subset \mathbb{R}^m$ be balls of radii $R_1, R_2 > 0$, respectively, such that $\mathrm{cl}(\mathbf{B}_1) \subset \mathbf{B}_2'$. Then the first mixed Steklov-Dirichlet eigenvalue of the problem

$$\begin{cases} \Delta u &= 0 & \text{in } \mathbf{B}_2' \backslash \mathrm{cl}(\mathbf{B}_1) \\ u &= 0 & \text{on } \partial \mathbf{B}_1 \\ \frac{\partial u}{\partial \eta} &= \sigma u & \text{on } \partial \mathbf{B}_2'. \end{cases}$$

attains maximum if and only if $\mathbf{B}_1$ and $\mathbf{B}_2'$ are concentric.

Recent results on annular regions

X : center of $\mathbf{B}_1$, C : center of $\mathbf{B}_2'$



Recent results on annular regions(2)

Denote the injectivity radius of M by $inj(M)$.

Theorem (S., 2019)

Let M be a two-point homogeneous space. Let $\mathbf{B}_1$ and $\mathbf{B}_2'$ be geodesic balls of radii $R_1, R_2 > 0$, respectively, such that $\mathrm{cl}(\mathbf{B}_1) \subset \mathbf{B}_2'$ and $R_2 < inj(M)/2$. Then the first mixed Steklov-Dirichlet eigenvalue of the problem

$$\begin{cases} \Delta u &= 0 & \text{in } \mathbf{B}_2' \backslash \mathrm{cl}(\mathbf{B}_1) \\ u &= 0 & \text{on } \partial \mathbf{B}_1 \\ \frac{\partial u}{\partial \eta} &= \sigma u & \text{on } \partial \mathbf{B}_2' \end{cases}$$

attains maximum if and only if $\mathbf{B}_1$ and $\mathbf{B}_2'$ are concentric.

Two-point homogeneous space

Definition

A connected Riemannian manifold M is called two-point homogeneous space if $x_i, y_i \in M$ with $d(x_1, y_1) = d(x_2, y_2)$, there is an isometry g of M such that $g(x_1) = x_2$ and $g(y_1) = y_2$.

		CROSS	nCROSS
$\mathbb{K} = \mathbb{R}$	$\mathbb{R}^n$	$\mathbb{S}^n, \mathbb{R}P^n$	$\mathbb{R}H^n$
$\mathbb{K} = \mathbb{C}$	$\cdot$	$\mathbb{C}P^n$	$\mathbb{C}H^n$
$\mathbb{K} = \mathbb{H}$	$\cdot$	$\mathbb{H}P^n$	$\mathbb{H}H^n$
$\mathbb{K} = \mathbb{O}$	$\cdot$	$\mathbb{O}P^2$	$\mathbb{O}H^2$

TABLE – Two-point homogeneous spaces

Recent results on annular regions (3)

Theorem (Ftouhi, 2019)

Let $\mathbf{B}_1, \mathbf{B}_2' \subset \mathbb{R}^m$ *be balls of radii* R_1, R_2*, respectively, such that* $\mathrm{cl}(\mathbf{B}_1) \subset \mathbf{B}_2'$*. Then the first Steklov eigenvalue of the problem*

$$\begin{cases} \Delta u &= 0 & \text{in } \mathbf{B}_2' \backslash \mathrm{cl}(\mathbf{B}_1) \\ \frac{\partial u}{\partial \eta} &= \sigma u & \text{on } \partial \mathbf{B}_1 \cup \partial \mathbf{B}_2'. \end{cases}$$

attains maximum if and only if $\mathbf{B}_1$ *and* $\mathbf{B}_2'$ *are concentric.*

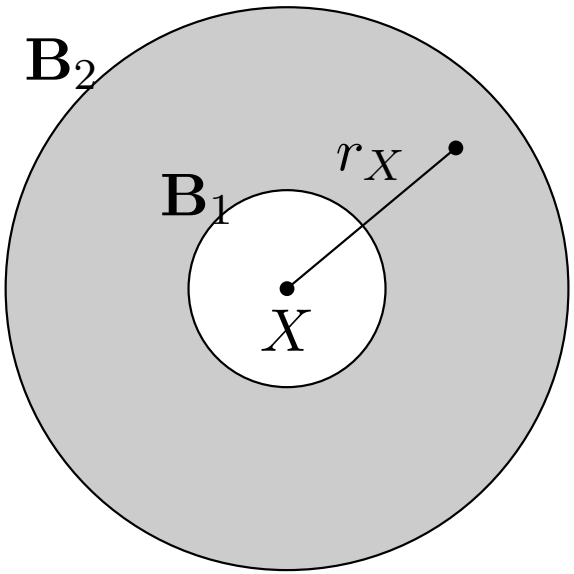
Proof of the result

Step 1 of the proof

Let $\mathbf{B}_2$ be the geodesic ball of radius R_2, concentric to $\mathbf{B}_1$.

Step 1

Finding the first eigenfunction in $\mathbf{B}_2 \setminus \mathrm{cl}(\mathbf{B}_1)$.



$a \circ r_X$ is the first eigenfunction, where

- $r_X(\cdot)$: distance from X in M,
- ω : (radial) density function of M,
- $a(r) := \displaystyle\int_{R_1}^{r} \frac{1}{\omega(t)} dt.$

Step 2 of the proof

Step 2

Comparing the variational characterizations of $\sigma_1(\mathbf{B}_2 \setminus \mathrm{cl}(\mathbf{B}_1))$ and $\sigma_1(\mathbf{B}_2' \setminus \mathrm{cl}(\mathbf{B}_1))$.

Since $a \circ r_X$ is a test function of the variational characterization of $\sigma_1(\mathbf{B}_2' \setminus \mathrm{cl}(\mathbf{B}_1))$,

$$\inf \left\{ \frac{\int_{\mathbf{B}_2' \setminus \mathrm{cl}(\mathbf{B}_1)} |\nabla v|^2 dV}{\int_{\partial \mathbf{B}_2'} v^2 ds} \middle| v \in H^1(\Omega) \setminus \{0\} \text{ and } v = 0 \text{ on } \partial \mathbf{B}_1 \right\},$$

we have

$$\sigma_1(\mathbf{B}_2' \setminus \mathrm{cl}(\mathbf{B}_1)) \leq \frac{\int_{\mathbf{B}_2' \setminus \mathrm{cl}(\mathbf{B}_1)} |\nabla (a \circ r_X)|^2 dV}{\int_{\partial \mathbf{B}_2'} (a \circ r_X)^2 dS_2'}.$$

Step 2 of the proof

Note that $a \circ r_X$ is the eigenfunction corresponding to $\sigma_1(\mathbf{B}_2 \setminus \mathrm{cl}(\mathbf{B}_1))$. Then to show $\sigma_1(\mathbf{B}_2' \setminus \mathrm{cl}(\mathbf{B}_1)) \leq \sigma_1(\mathbf{B}_2 \setminus \mathrm{cl}(\mathbf{B}_1))$, it is enough to show that

$$\int_{\mathbf{B}_2' \setminus \mathrm{cl}(\mathbf{B}_1)} |\nabla(a \circ r_X)|^2 dV \leq \int_{\mathbf{B}_2 \setminus \mathrm{cl}(\mathbf{B}_1)} |\nabla(a \circ r_X)|^2 dV,$$

and

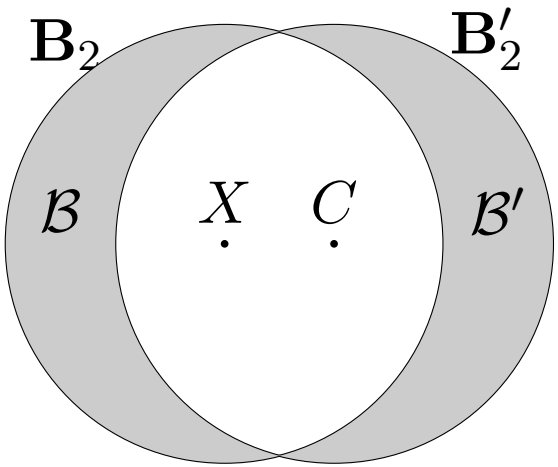
$$\int_{\partial \mathbf{B}_2} (a \circ r_X)^2 dS_2 \leq \int_{\partial \mathbf{B}_2'} (a \circ r_X)^2 dS_2',$$

where S_2 and S_2' are the measures on $\partial \mathbf{B}_2$ and $\partial \mathbf{B}_2'$ induced from the metric of M, respectively.

Step 3 of the proof

Step 3

$$\int_{\mathbf{B}_2'\setminus \mathrm{cl}(\mathbf{B}_1)} |\nabla(a\circ r_X)|^2 dV \leq \int_{\mathbf{B}_2\setminus \mathrm{cl}(\mathbf{B}_1)} |\nabla(a\circ r_X)|^2 dV,$$



(Idea of the proof) If M is Euclidean space or nCROSS, $|\nabla(a\circ r_X(\cdot))| = \frac{1}{\omega(r_X(\cdot))}$ has bigger value in $\mathcal{B}$ than those in $\mathcal{B}'$. □

Remark

If M is CROSS, the previous argument holds if $R_2 < inj(M)/4$. Because $\omega(r)$ is no longer increasing if $r > inj(M)/2$. Therefore we need more delicate argument on $\mathcal{B}$ and $\mathcal{B}'$ for Step 3.

Step 4

$$\int_{\partial \mathbf{B}_2} (a \circ r_X)^2 dS_2 \leq \int_{\partial \mathbf{B}_2'} (a \circ r_X)^2 dS_2',$$

Step 4 of the proof

Consider a function

$$\int_{\partial \mathbf{B}_2'} (a \circ r_{(\cdot)})^2 dS_2' : \mathbf{B}_2' \to \mathbb{R}$$

that assigns to $X \in \mathbf{B}_2'$

$$\int_{\partial \mathbf{B}_2'} (a \circ r_X)^2 dS_2'$$

and its gradient at each $X \in \mathbf{B}_2'$ is

$$\nabla \left(\int_{\partial \mathbf{B}_2'} (a \circ r_{(\cdot)})^2 dS_2' \right) (X) \in T_X M.$$

Main argument in Step 4

Proposition

$$\nabla\left(\int_{\partial\mathbf{B}_2'}(a\circ r_{(\cdot)})^2 dS_2'\right)(X)=\begin{cases}-g(X)\cdot\vec{v}_X(C) & \text{if } X\neq C,\\ 0 & \text{if } X=C,\end{cases}$$

where $g:\mathbf{B}_2'\setminus\{C\}\to\mathbb{R}^+$ *is a positive function and* $\vec{v}_X(C)$ *is the unit tangent vector at* X *toward* C*. Furthermore,*

$$\int_{\partial\mathbf{B}_2'}(a\circ r_C)^2 dS_2'\leq\int_{\partial\mathbf{B}_2'}(a\circ r_X)^2 dS_2',$$

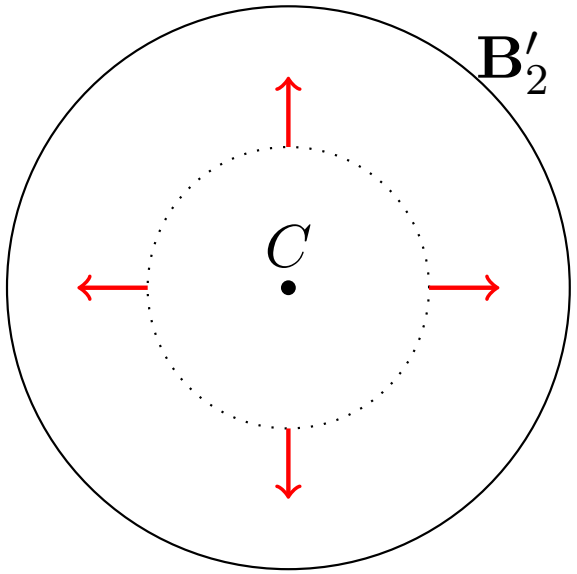
and equality holds if and only if $X=C$*.*

Main argument in Step 4

The identities imply that the gradient field of the function

$$\int_{\partial \mathbf{B}_2'} (a \circ r_{(\cdot)})^2 dS_2' : \mathbf{B}_2' \to \mathbb{R}$$

can be depicted by the picture below, so the inequality follows.



Main argument in Step 4

Note that $\mathbf{B}_2$ is a ball of radius R_2, centered at X. Therefore we have

$$\int_{\partial \mathbf{B}_2} (a \circ r_X)^2 dS_2 = \int_{\partial \mathbf{B}_2'} (a \circ r_C)^2 dS_2'.$$

Thus our proposition implies the inequality in Step 4,

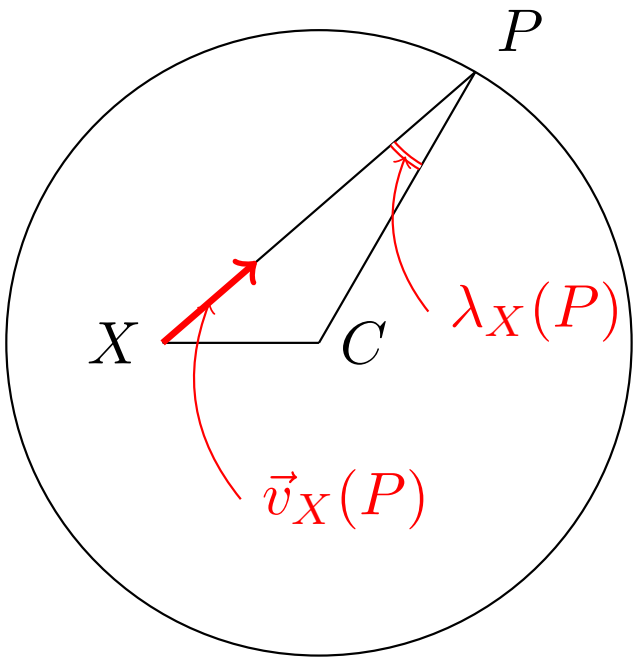
$$\int_{\partial \mathbf{B}_2} (a \circ r_X)^2 dS_2 \leq \int_{\partial \mathbf{B}_2'} (a \circ r_X)^2 dS_2'.$$

Remark

This proposition is the main difference between our proof and Santhanam-Verma's proof.

Main argument in Step 4

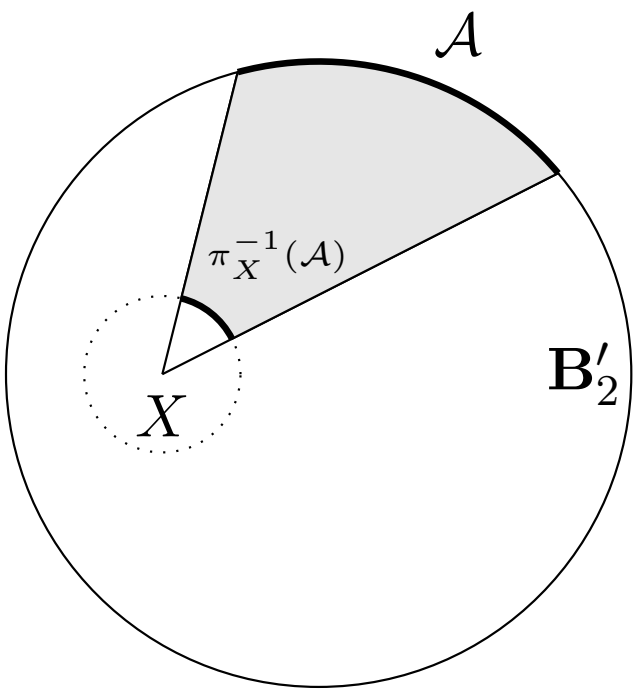
(Idea of the proof of Proposition)



Let

- $\lambda_X(P)$: angle at $P \in \partial \mathbf{B}_2'$ of the triangle (PXC),
- $\vec{v}_X(P)$: unit tangent vector at X toward P, and

Main argument in Step 4



- $\pi_X : \mathbb{S}^{m-1} \subset \mathbb{R}^m \to \partial\mathbf{B}_2'$ is the projection map.

Then we can show that

$$dS_2' = \frac{\omega(r_X)}{\cos\lambda_X} d\pi_{X\#}\mu, \qquad (*)$$

where μ : canonical measure on $\mathbb{S}^{m-1}$.

Main argument in Step 4

Then we have

$$
\begin{aligned}
&-\nabla\left(\int_{\partial\mathbf{B}_2'}(a\circ r_{(\cdot)})^2(P)dS_2'(P)\right)(X)\\
&=\int_{\partial\mathbf{B}_2'}\frac{2(a\circ r_X)}{\omega(r_X)}\cdot(-\nabla r_{(\cdot)}(P))(X)dS_2'(P)\\
&=\int_{\mathbb{S}^{m-1}}\left(2(a\circ r_X)\cdot\frac{\vec{v}_X}{\cos\lambda_X}\right)\circ\pi_X d\mu.
\end{aligned}
$$

For the second identity, we used the fact that $-\nabla r_{(\cdot)}(P)=\vec{v}_X(P)$ and $(*)$.

Main argument in Step 4

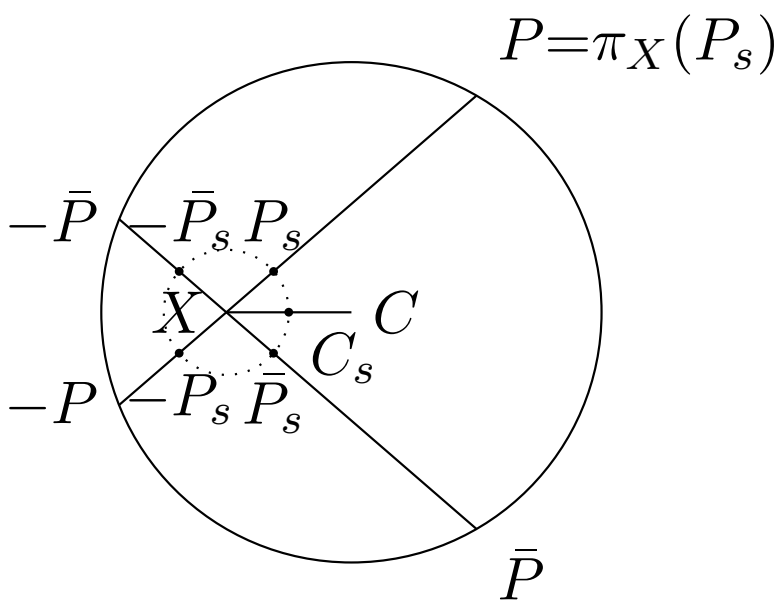
Note that a is an increasing function. Using trigonometric laws, we can conclude

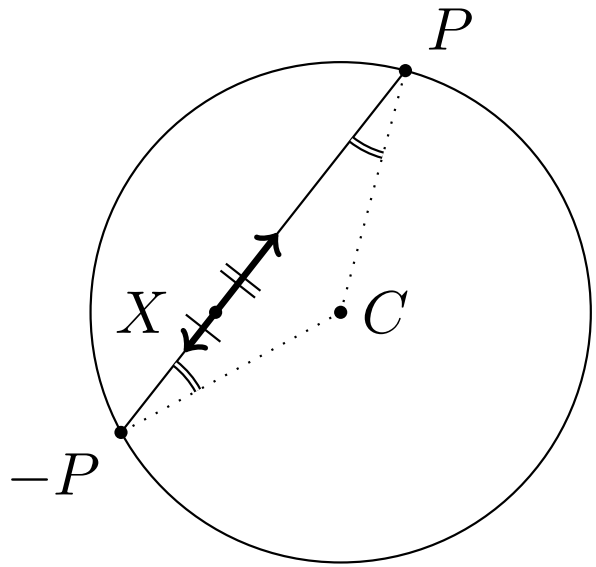
$$\int_{\mathbb{S}^{m-1}} \left(2(a \circ r_X) \cdot \frac{\vec{v}_X}{\cos \lambda_X} \right) \circ \pi_X d\mu$$

Main argument in Step 4

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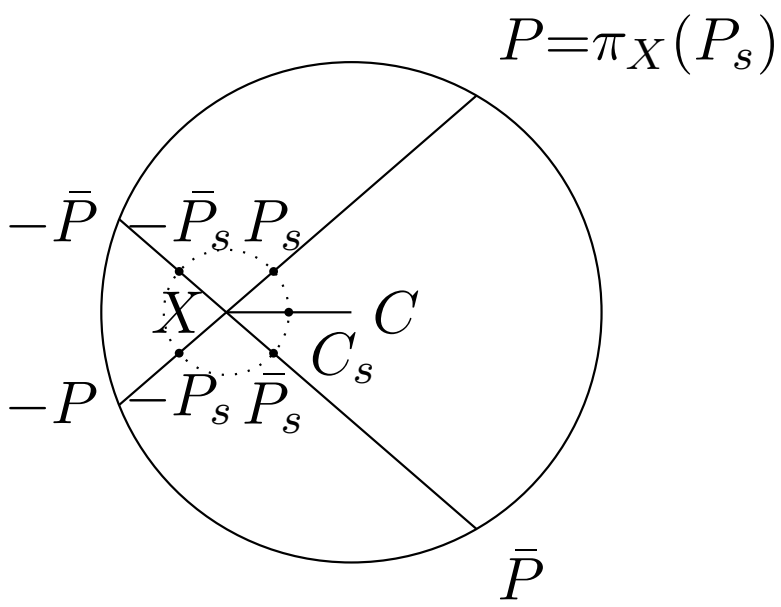


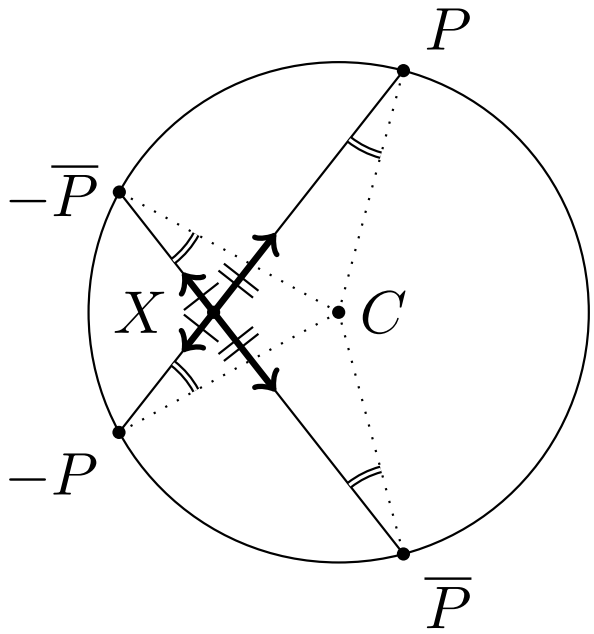


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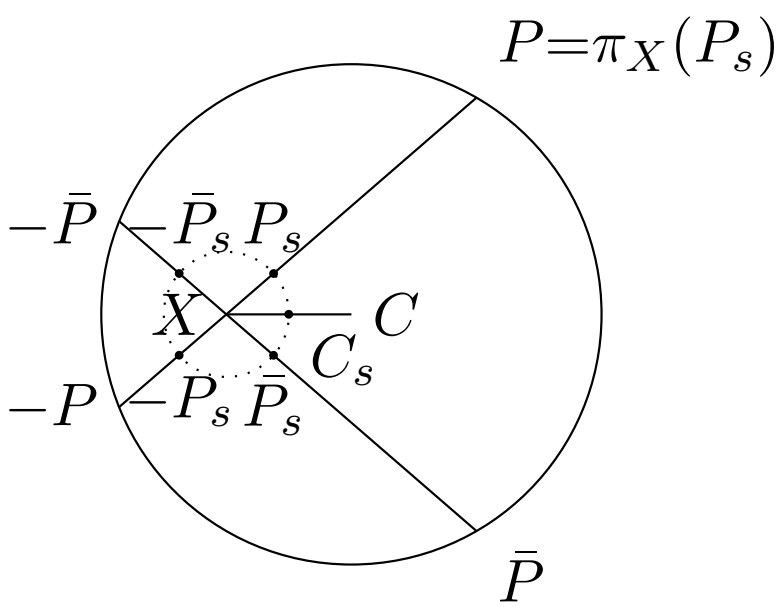


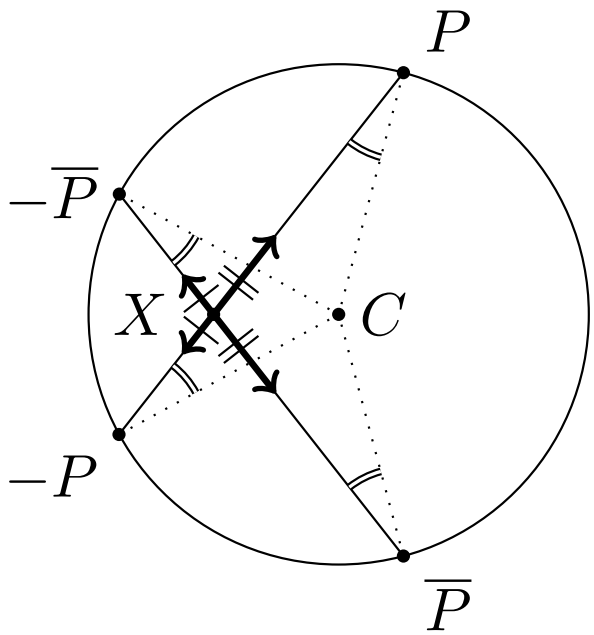
Main argument in Step 4

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$$\int_{\mathbb{S}^{m-1}} \left(2(a \circ r_X) \cdot \frac{\vec{v}_X}{\cos \lambda_X} \right) \circ \pi_X d\mu$$

is a vector starting from X toward C. □





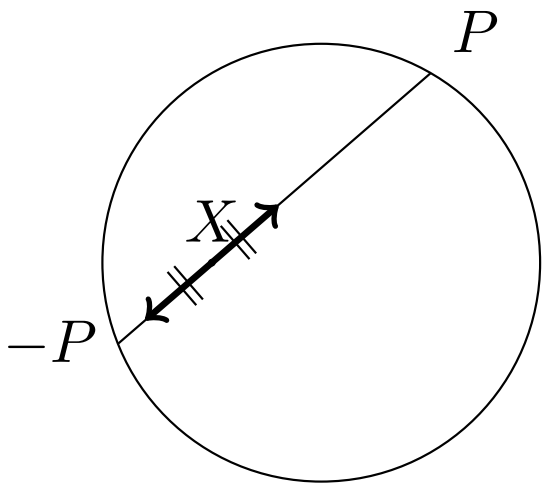
Newton's shell theorem

The proof of Proposition is motivated from the proof of following.

Corollary (Newton's shell theorem)

We have

$$\int_{\partial \mathbf{B}_2'} \frac{\vec{v}_X}{\omega(r_X)} dS_2' = 0.$$



(Idea of proof) Using (*), we have

$$\begin{aligned}(\text{LHS}) &= \int_{\mathbb{S}^{m-1}} \left(\frac{\vec{v}_X}{\cos \lambda_X} \right) \circ \pi_X d\mu \\ &= 0. \quad \square\end{aligned}$$

Thank you.