The following are the video link and slides for Dan Freed’s Lecture 1.

https://livewarwickac-my.sharepoint.com/:v:/g/personal/u1972347_live_warwick_ac_uk/EdbA7QUx0ndDrPWkImhkPgkBQpi6e2nEOi1l7cnpRxacfQ?e=Qof5Lg

Topics in Field Theory and Topological Phases of Matter

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August 16–20, 2021

Based on joint work with Mike Hopkins and Constantin Teleman

Lecture topics

Lecture 1: Moduli problems in quantum theory

Lecture 2: Symmetry in field theory

Lecture 3: Classification of invertible field theories

Lecture 4: The 2-dimensional Ising model

Lecture 5: Boundary field theories

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Encouragement: I will cover many topics in an effort to paint a picture about recent interactions of geometry, topology, and physics. In doing so I draw on lots of different mathematics—I don't expect you'll have background for it all—so I encourage you to find details in references.

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Encouragement′: Please interrupt and ask questions!

These blackboards are available in real time at:

https://www.dropbox.com/sh/xxo9s06o9z7nqwl/AADy5EKRQGOBBHOvvNuzLrF-a?dl=0

Moduli problems

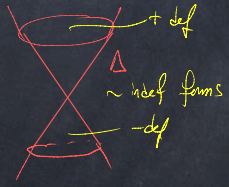
V 2 dim'l real vector space — inner product

$Q \in \mathrm{Sym}^2 V^*$ Symmetric bilinear forms $V \times V \to \mathbb{R}$

$\cup$

Δ degenerate forms

$\mathcal{M} = \mathrm{Sym}^2 V^* \setminus \Delta$ nondegenerate forms



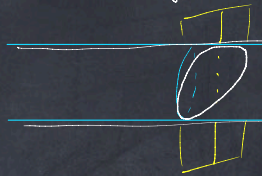
$\pi_0 \mathcal{M}$ = 3 points

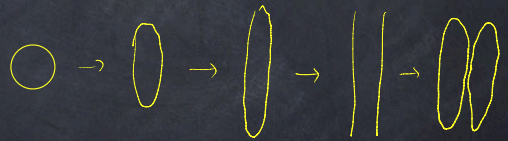
Rmk: $\pi_0 \mathcal{M}$ depends only on the homotopy type of $\mathcal{M}$.

② Moduli space of "1-dim'l metric shapes"

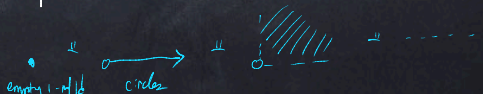


Riemannian 1-mfld





$\mathcal{M}$ = Compact Riemannian 1-mflds.



$\pi_0 \mathcal{M} \xrightarrow{\sim} \mathbb{Z}^{\geq 0}$

$X \longmapsto$ # components of X

deformation class

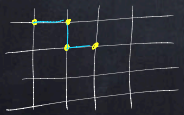
Rmk: We fix discrete parameters to define $\mathcal{M}$

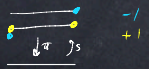
3) $\mathcal{M}$ "moduli space of quantum theories"

discrete parameters: dimension, symmetry type

$\pi_0 \mathcal{M}$ ~ phases of qtm theories of this type

Example of a discrete quantum system (Toric code Kitaev)





Y CW complex of dimension d (finite)

$Y^0 \subset Y^1 \subset Y^2 \subset \cdots$

$\mathrm{Bun}_{\mu_2}(Y, Y^0)$ = double covers of Y^1 together with a section over Y^0 (π, s)



$\mathcal{H} = \mathrm{Fun}(\mathrm{Bun}_{\mu_2}(Y, Y^0), \mathbb{C})$

$\mathcal{H}$

$v \in Y^0$ $\quad \varphi_v : \mathrm{Bun}_{\mu_2}(Y, Y^0) \longrightarrow \mathrm{Bun}_{\mu_2}(Y, Y^0)$, $(\pi, s) \longmapsto (\pi, s_v)$

$$s_v(v') = \begin{cases} s(v') & v' \neq v, \\ -s(v') & v' = v \end{cases} \qquad v' \in Y^0$$

$H_v \psi = \frac{1}{2}(\psi - \varphi_v^* \psi)$, $\quad \psi \in \mathcal{H}$

f 2-cell $\quad \mathrm{Hol}_{\partial f}(\pi) = \pm 1$

$H_f \psi = \frac{1}{2}\left(1 - \mathrm{Hol}_{\partial f}(\pi)\right)\psi$

$$H = \sum_v H_v + \sum_f H_f$$

$\operatorname{Spec} H \subset \mathbb{Z}^{\geq 0}$

$\ker H = \bigcap_v \ker H_v \cap \bigcap_f \ker H_f$

$= \operatorname{Fun}(\operatorname{Bun}_{\mu_2} Y, \mathbb{C})$

$= H^1(Y; \mu_2)$

Remks: 1) Δ phase transition

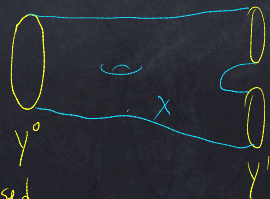
2) gapped quantum systems

3) Heuristic: homotopical information depends only on the long distance structure
= low energy

4) At low energy a (reasonable) gapped quantum system is well-approximated by a topological* field theory

1) Build a model
2) Prove Theorem
3) Apply back to model

Topology ~ bordism Fix $d \in \mathbb{Z}^{\geq 0}$.



Defn: Y^0, Y' are bordant if $\exists$ cpt X st. $\partial X = Y^0 \amalg Y'$.

closed d-manifold

addition = $\amalg$ (disjoint union)

Variation: Manifolds w/ tangential structure: orientation, Spin, framing, ...

Lashof: $\mathcal{X}_d$ pointed top space

$$\begin{array}{ccc} & & \mathcal{X}_d \\ & \nearrow^{\widetilde{\tau_d}} & \downarrow \\ TY & & \\ \nearrow & & \\ Y & \longrightarrow & BO_d \end{array} \qquad \tau_d : \mathcal{X}_d \to BO_d$$

Ex: $\rho_d : H_d \longrightarrow O_d$ homomorphism of Lie groups

$\mathcal{X}_d = BH_d$, $\downarrow B\rho_d$, BO_d

Ex: $H_d = SO_d$, $Spin_d$, 1

Bordism invariants: $\Omega_d^{\mathcal{X}_d} \longrightarrow \mathbb{Z}$ (Signature)

- A topological field theory (TFT) is a "categorified bordism invariant"

Domain: Bordism category $\mathrm{Bord}_{\langle n-1,n\rangle}(\mathcal{X})$ Milnor $n = d+1$

Y^{n-1} closed $(n-1)$-dmfld $\mathcal{X}$-mfld

$X^n : Y_0 \longrightarrow Y_1$ 1-morphism

Codomain: (algebraic) category

$\mathrm{Vect}_{\mathbb{K}}$ $(\mathbb{K} = \mathbb{C})$

Mod_R, Chain complexes, correspondences, ...

Defn: A TFT is a symmetric monoidal functor

$$F: \mathrm{Bord}^{(\mathfrak{X})}_{\langle n-1,n\rangle} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

$$\| \qquad\qquad \otimes$$

axiom system for quantum (nontopological) field theories.

Atiyah - Segal - Kontsevich

Rmk: 1) Collection of TFTs with fixed domain & codomain is a symmetric monoidal cat:

$F_1 \otimes F_2$, $\mathbb{1}$

$\Rightarrow$ Notion of invertible TFT.

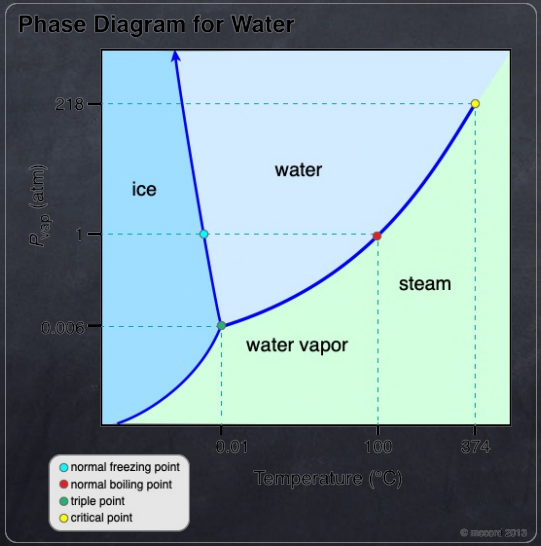
2) Choice of domain, codomain is important.

3) A variation of defn should be a good

4) We need to extend this definition to manifolds of dimension $< n-1$. Locality



5) This defn does not include unitarity


Phase Diagram for Water
218
1
0.006
P_{vap} (atm)
ice
water
steam
water vapor
0.01
100
374
Temperature (°C)
normal freezing point
normal boiling point
triple point
critical point
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References for Lecture 1

—, "Lectures on Field Theory and Topology", Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, Number 133, American Mathematical Society, Providence, RI, 2019, pdf at `https://web.ma.utexas.edu/users/dafr/`

—, *The cobordism hypothesis*, Bull. Amer. Math. Soc., **50** (2013), no. 1, 57–92, `arXiv:1210.5100`.

Jacob Lurie, *On the classification of topological field theories*, Current developments in mathematics, 2008, Int. Press, Somerville, MA, 2009, pp. 129–280. `arXiv:0905.0465`.