The following are the video link and slides for Dan Freed’s Lecture 2.

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Topics in Field Theory and Topological Phases of Matter

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Based on joint work with Mike Hopkins and Constantin Teleman

Lecture topics

Lecture 1: Moduli problems in quantum theory

Lecture 2: Symmetry in field theory

Lecture 3: Classification of invertible field theories

Lecture 4: The 2-dimensional Ising model

Lecture 5: Boundary field theories

A TFT is a homomorphism (symmetric monoidal functor)

$$F : \mathrm{Bord}^{\langle n-1 \rangle}_{n}(*) \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

($*$: tangential structure; n: "spacetime" dimension)

$$* \longrightarrow B\mathcal{O}_n$$

Example (G-gauge Theory) G finite group, $n=3$

M manifold

$\mathrm{Bun}_G(M)$ groupoid of principal G-bundles

$X : Y_0 \longrightarrow Y_1$ (3-mfld; Y_0, Y_1: closed 2-mflds)



Correspondence:

$$\mathrm{Bun}_G(X) \xrightarrow{r_0} \mathrm{Bun}_G(Y_0), \quad \mathrm{Bun}_G(X) \xrightarrow{r_1} \mathrm{Bun}_G(Y_1)$$

Linearize: Define $g_G(Y) = \mathrm{Map}(\mathrm{Bun}_G(Y), \mathbb{C})$

$$g_G(X) : g_G(Y_0) \longrightarrow g_G(Y_1)$$

$$g_G(X) = (r_1)_* \circ (r_0)^*$$

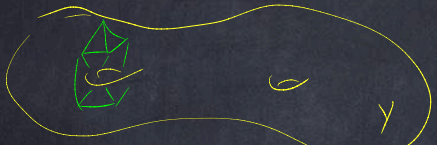
(finite path integral)

Ex: If $X : \phi^2 \longrightarrow \phi^2$ is a closed 3-mfld,

$$g_G(X) = \sum_{[\pi]} \frac{1}{\#\mathrm{Aut}(\pi)}$$

$P \xrightarrow{\pi} X$

Suppose Y closed 2-mfld with lattice/cw structure



Toric code: Hilbert space $\mathrm{Map}(\mathrm{Bun}_G(Y), \mathbb{C})$

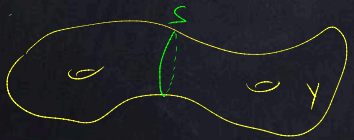
Hamiltonian H

Vacuum space $\mathrm{Map}(\mathrm{Bun}_G(Y), \mathbb{C})$

$\mathcal{G}_G$ is the low energy approximation to toric code

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Extended field theory



$\mathcal{G}_G : \mathrm{Bord}_{\langle 1,2,3\rangle} \longrightarrow \mathrm{Cat}_{\mathbb{C}}$

2-category



$\mathrm{Bun}_G(Y)$

r_0 r_1

$\mathrm{Bun}_G(S_0)$ $\mathrm{Bun}_G(S_1)$

$\mathcal{G}_G(S) = \mathrm{Vect}_{\mathbb{C}}(\mathrm{Bun}_G(S))$ vector bundles over $\mathrm{Bun}_G(S)$

$\mathcal{G}_G(Y) = (r_1)_* \circ (r_0)^*$

Ex: $\mathcal{G}_G(S^1)$ $\quad \mathrm{Bun}_G(S^1) = G/G$ $\quad$ G acting by conjugation









$$\mathcal{G}_G(S^1) = \mathrm{Vect}_G(G)$$

Go further: $\mathcal{G}_G : \mathrm{Bord}_{\langle 0,1,2,3\rangle} \longrightarrow \mathrm{Fus}_{\mathcal{C}}$

$\mathrm{Bord}_{\langle 0,1,2,3\rangle} = \mathrm{Bord}_3$

$\mathcal{G}_G(pt) = \mathrm{Vect}[G]$ vector bundles over G under convolution

Cobordism Hypothesis

Thm (Hopkins-Lurie, Lurie) $\mathcal{C}$ symmetric monoidal n-category

$$\mathrm{Fun}^{\otimes}(\mathrm{Bord}_n(n\text{-framed}), \mathcal{C}) \xrightarrow{\text{evaluate at pt}} \mathcal{C}$$

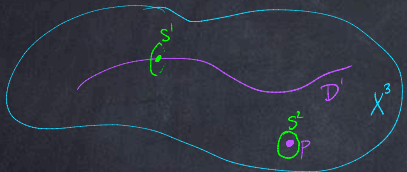
This is an equivalence onto subobject of $\mathcal{C}$:

- n-dualizable part of $\mathcal{C}$ (strong finiteness)
- only keep invertible morphisms of $\mathcal{C}$

Determines homotopy type of space of n-framed theories.

Rmk: There is a version for any tangential structure

Defects (operator)



$\mathbb{E}$: link of p is a 2-sphere surrounding p

$F(S^2)$ vector space of point defects

$F(S^1)$ 1-category of line defects

Rmk: Replace $F(S^k)$ by $\mathrm{Hom}(1, F(S^k))$

$F = g_G$

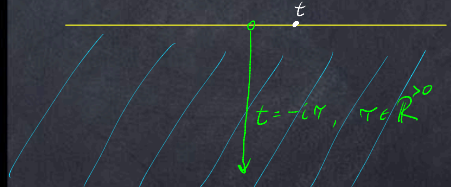
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Wick rotation and Symmetry

Quantum mechanics Hilbert space

Hamiltonian $H \geq 0$

$U_t = e^{-itH/\hbar}$ (unitary)



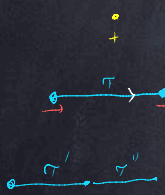
$C_\tau = e^{-\tau H/\hbar}$, $\tau > 0$ $C_t = e^{-itH/\hbar}$

Wick-rotated "time" evolution

holomorphic semigroup of contraction operators

Geometric representation:

$\mathrm{Bord}_{\langle 0,1\rangle}$ (oriented Riemannian) $\longrightarrow$ $t\mathrm{Vect}_{\mathbb{C}}$



$\longrightarrow$ Hilbert space

$\longrightarrow$ $e^{-\tau H/\hbar}$

$e^{-(\tau'+\tau'')H/\hbar} = e^{-\tau'' H/\hbar} \circ e^{-\tau' H/\hbar}$

Quantum Field Theory

$\mathbb{M}^n$ Minkowski spacetime — affine space, Lorentz metric, time orientation

$n \geq 2$



Wick rotation

$\mathbb{M}^n \rightsquigarrow \mathcal{D} \rightsquigarrow \mathbb{E}^n \rightsquigarrow X^n$

Minkowski — complex domain — Euclidean — Riemannian

Symmetry in QFT

The symmetry group of a relativistic QFT is a Lie group $\mathcal{H}_{1,n-1}$ and a homomorphism

$$\rho_n : \mathcal{H}_{1,n-1} \longrightarrow \mathcal{I}^{\uparrow}_{1,n-1}$$

whose image is either $\mathcal{I}^{\uparrow}_{1,n-1}$ or the identity component

$\ker \rho_n := K$ is assumed compact Lie — internal symmetries.

$$\begin{array}{ccc} K & = & K \\ \downarrow & & \downarrow \\ \mathbb{R}^{1,n-1} \subset \mathcal{H}_{1,n-1} & \longrightarrow & H_{1,n-1} \\ \downarrow & & \downarrow \\ \mathbb{R}^{1,n-1} \subset \mathcal{I}^{\uparrow}_{1,n-1} & \longrightarrow & O^{\uparrow}_{1,n-1} \end{array}$$

mod out translations

$$\begin{array}{ccccc} K & \subset & K(\mathbb{C}) & \supset & K \\ \downarrow & & \downarrow & & \downarrow \\ H_{1,n-1} & \subset & H_n(\mathbb{C}) & \supset & H_n \\ \downarrow & & \downarrow & & \downarrow \rho_n \\ O_{1,n-1} & \subset & O_n(\mathbb{C}) & \supset & O_n \end{array}$$

$M^n \rightsquigarrow \partial \rightsquigarrow \mathbb{E}^n$

- K didn't change under Wick rotation
- H_n compact Lie group
- $\rho_n : H_n \to O_n$ has image $SO_n \subset O_n$

Symmetry type of the theory

Thm (loosely) ❷

$$\begin{array}{ccccc} K & \longrightarrow & \widetilde{SH}_n & \longrightarrow & Spin_n \\ \| & & \downarrow & & \downarrow \\ K & \longrightarrow & H_n & \longrightarrow & O_n \end{array}$$

Then $\widetilde{SH}_n \cong Spin_n \times K$

$\Rightarrow$ distinguished elt $k_0 \in K$, $k_0^2 = 1$ ("$(-1)^F$")

❶ tells that H_n is constructed from $SH_n = \rho_n^{-1}(SO_n)$ by a pullback of a group extension.

Cor: If $n \geq 3$, then there exists a stabilization

$$\begin{array}{ccccccc} H_n & \longrightarrow & H_{n+1} & \longhookrightarrow & H_{n+2} & \longhookrightarrow & \cdots \\ \downarrow \rho_n & & \downarrow \rho_{n+1} & & \downarrow \rho_{n+2} & & \\ O_n & \longrightarrow & O_{n+1} & \longhookrightarrow & O_{n+2} & \longhookrightarrow & \cdots \end{array}$$

Can make bordism (multi) category of manifolds equipped with an $(H_n \rho_n)$-structure

Thm (loosely) ❷

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