The following are the video link and slides for Eric Rowell’s Lecture 1.

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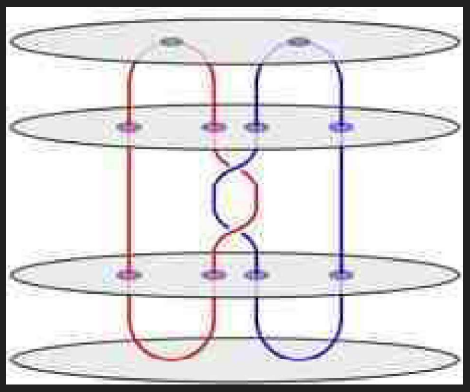
The Mathematical Foundations of Topological Quantum Computation

Anyons, Braids & Categories





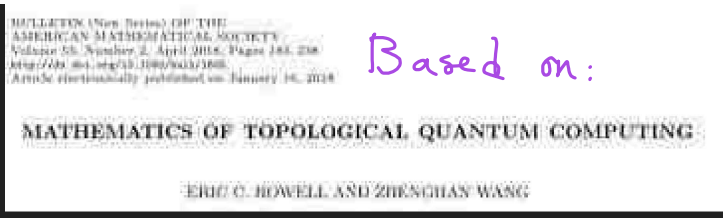
August 2–6, 2021



Eric Rowell



Outline of Lectures



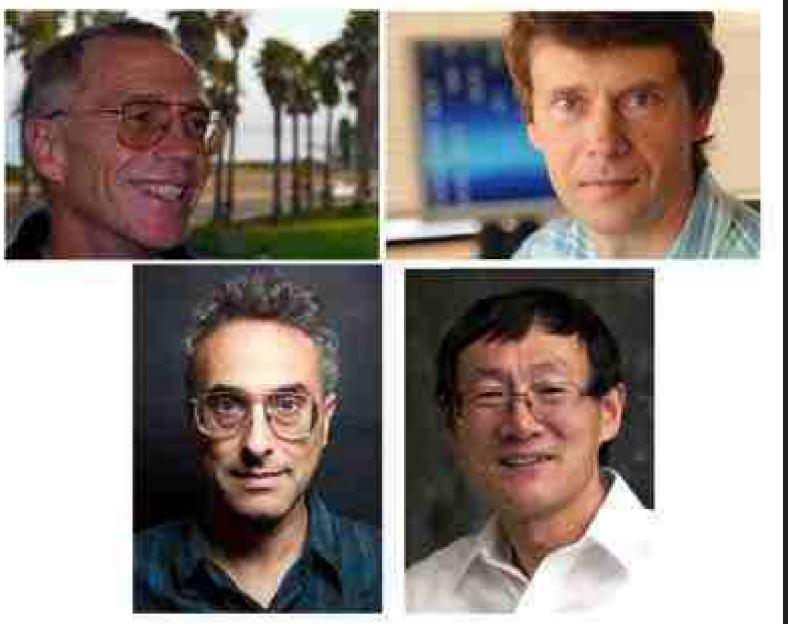
Based on:

- Quantum Mech. & Quantum Circuit Model
  TPMs, TQFTs, TQC, MTCs
- Computational Complexity & Universality
  Algorithms & Equiv. of Paradigms
- Flavors of Anyons
  Detecting Braiding only universality
- Towards a Periodic Table
  classification problems, Phase Transitions
- Further topics (time permitting) (suggestions...?)

Scrap Heap treasures: Measurement Assisted TQC, ECC

What is Quantum Computation?

From [Freedman-Kitaev-Larsen-Wang '03]



... any computational model based upon theoretical ability to create, manipulate, & measure Quantum states.

Idea: SQUEEZE classical information out of Quantum States.

How? First: Guiding Principles

Axioms of QM (closed quantum system)

- **Superposition:** state of a system A is a vector in a Hilbert space (of all states). denote $|\psi\rangle \in \mathcal{H}_A$. (up to phases)
- **Unitarity:** time evolution is by a unitary operator $|\psi_t\rangle = U_t |\psi_0\rangle$. (Shrödinger)
- **Entanglement:** Composite of A & B has Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \ni |\psi\rangle \otimes |\varphi\rangle$
- **Measurement:** Hermitian op. M, 1-dim'l eigenspaces $\mathbb{C}\{e_i\} = E(\lambda_i)$. $|\psi\rangle = \sum_i a_i |e_i\rangle \xrightarrow{M} |e_i\rangle$ w/ Prob $|a_i|^2$.

Prototype Quantum Computation (QCM)

$f: \{0,1\}^n \to \{0,1\}^n$ classical fnc. (bit)

n-qubit space

Goal: compute $f(N)$

1 encode: $|N\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle = |i_1 \cdots i_n\rangle \in (\mathbb{C}^2)^{\otimes n} = \mathcal{H}_n$

2 Process: $|N\rangle \xmapsto{U_f} \sum_J a_J |J\rangle$ (algorithm: find U_f)

3 Measure: get $|f(N)\rangle$ with prob. $|a_{f(N)}|^2 > \frac{1}{2}$

Ex: $f(N) :=$ smallest prime factor of N.

↪ Shor's algorithm.

Efficiency → output U_f classical alg. poly'l time.

Quantum Circuit Model

State space: $\mathcal{H}_n = (\mathbb{C}^2)^{\otimes n}$ array of n 2-level systems

Orthogonal basis: $\{|0\rangle, |1\rangle\} \to \{|i_1 \cdots i_n\rangle : i_j \in \{0,1\}\}$

Quantum Gates: $\mathcal{G} = \{G_1, \ldots, G_k\}$, $G_i \in U(\mathcal{H}_{n_i})$ (assume finite)

(physically implemented)

Eg. 1. CNOT $|i_1 i_2\rangle = |i_1 (i_1 + i_2)\rangle \leftrightarrow \left(\begin{array}{c|cc} I & 0 & \\ \hline 0 & \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \end{array}\right)$

2. $H \in U(\mathbb{C}^2)$: $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

3. $T|j\rangle = e^{\pi i j/4}|j\rangle$

---

n-qubit length l $\mathcal{G}$-circuits $\mathcal{G}_l(n)$: (a finite set)

$$\boxed{\prod_{j=1}^{l} U_j}, \quad U_j \in \{ I_2^{\otimes q} \otimes G_t \otimes I^{\otimes (n - n_t - q)} : G_t \in \mathcal{G} \}$$

n qubit circuits: $\mathcal{G}(n) = \bigcup_{l \geq 1} \mathcal{G}_l(n)$.

Defn $\mathcal{G}$ is universal if $SU(\mathcal{H}_n) \subseteq \overline{\langle \mathcal{G}(n) \rangle}$ $n >> 0$.

Efficiency: Kitaev-Solovay: $\mathcal{G}$ universal $\Rightarrow$ $\forall\ U \in SU(\mathcal{H}_n)$, $\varepsilon > 0$ $\exists\ l(\frac{1}{\varepsilon})$ poly in $\frac{1}{\varepsilon}$ & a circuit $X \in \mathcal{G}_l(n)$ s.t. $\|U - X\| < \varepsilon$.

Ex: $\{CNOT, H, T\}$ is universal.

End of story?

No! DECOHERENCE!

Environment interaction ⇝ Local Errors
(bit flips, phase shift...)

Classical soln: Redundancy... $A \mapsto AAAA$.

Quantum No clone thm ⇒ Redundancy

Shor: Quantum Ec: Encode $\{|0\rangle, |1\rangle\}$ in
(software...)
Entangled states in $(\mathbb{C}^2)^{\otimes 9}$. "Delocalize".

Expensive! 9 physical qubits ⇝ 1 logical qubit.

Alternative: encode in physical systems with
topological ("global") degrees of freedom.
(hardware...)

Non-local entanglement.

Freedman & Kitaev

Need: Topological Phases & Matter: TPMs
(Dan Freed's Talks!)

- Ground state Degeneracy (on )
- Topological Invariance (local perterbations)
- Supports Anyons (point-like excitations)

Anyons generalize bosons & fermions:



t

$|\psi(r_2, r_1)\rangle = \pm |\psi(r_1, r_2)\rangle$

$|\psi(r_2, r_1)\rangle = e^{i\theta} |\psi(r_1, r_2)\rangle$ $\longrightarrow$ $\boxed{\theta = 0, \pi}$

Exist? In $\mathbb{R}^3$ or S^3: $\pi_1(\mathbb{R}^3 - \{p_1, \ldots, p_k\}) = 1$: ONLY ± 1

But in $\mathbb{R}^2$ or S^2: $\pi_1(\mathbb{R}^2 \setminus \{p_1, \ldots, p_k\}) \cong F_k$ free!

(Loop $\bigcirc$-like in $\mathbb{R}^3$ too...)

So we focus on 2D TPMs

Surfaces w/ ∂ + Quantum Mechanics

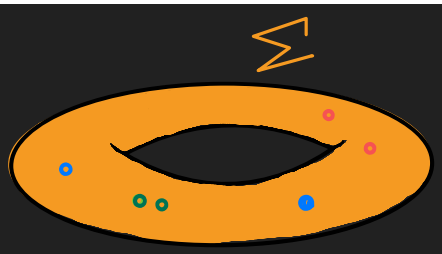
Label set of anyon types: $\mathcal{I} = \{0, 1, \ldots, k\}$

"colors"

Involution: $* : \mathcal{I} \to \mathcal{I}$ $0^* = 0$.

0: vacuum type

Surfaces may be oriented, ∂ comps labeled with anyons. $\bigcirc$ carry orientation & a base point...



superpos.

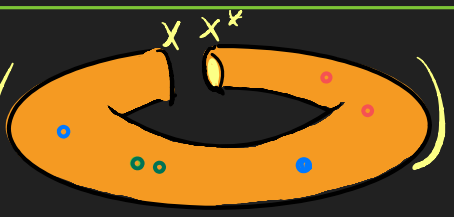
depends on Topology only!

$\mathcal{H}(\Sigma, \text{labels})$

$\mathcal{H}(\;\amalg\;) \cong \mathcal{H}(\;) \otimes \mathcal{H}(\;)$

Associative!

$\mathcal{H}(\;) \cong \bigoplus_{x} \mathcal{H}(\;)$



gluing, Feynman path integral, locality, sum of histories...

$\mathcal{H}(-\Sigma, \text{labels}) \cong \mathcal{H}(\Sigma, \text{labels})^*$

Orientation & duality compatible.

Topological invariance $\longrightarrow$ $MCG(\Sigma, \text{pts})$ reps etc...

Local Pieces : Disk Annulus Pants decomp.

$\mathcal{H}(\emptyset) = \mathbb{C}$ stability under $\amalg$

$\mathcal{H}($  $) \cong \delta_{x,0}\mathbb{C}$ no local creation

$\mathcal{H}($  $) \cong \delta_{a,b^*}\mathbb{C}$ particle/anti-particle pairs.

$\dim \mathcal{H}($  $) = N_{ab}^{c} \in \mathbb{N}$.

fusion Rules!



Upshot: 1. Modeled by 2D Unitary Top. Modular Functor

2. Labels $\rightsquigarrow$ Unitary Modular Tensor Category

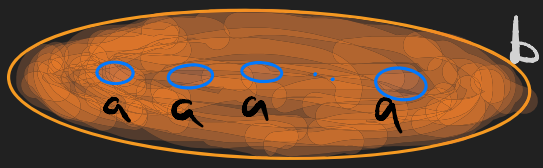
Reshetikhin-Turaev (2+1)TQFT.

Definition (Nayak, et al '08)

A system is in a topological phase if its low-energy effective field theory is a topological quantum field theory (TQFT).

Physical Features/constraints $\longleftrightarrow$ Axioms

Anyons in a disk:



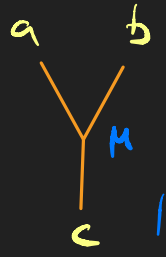
$: (D^2, b; a, \dots, a)$

$\mathrm{Hom}(X_b, X_a^{\otimes n}) \longleftrightarrow \mathcal{H}(D^2, b; a, \dots a)$

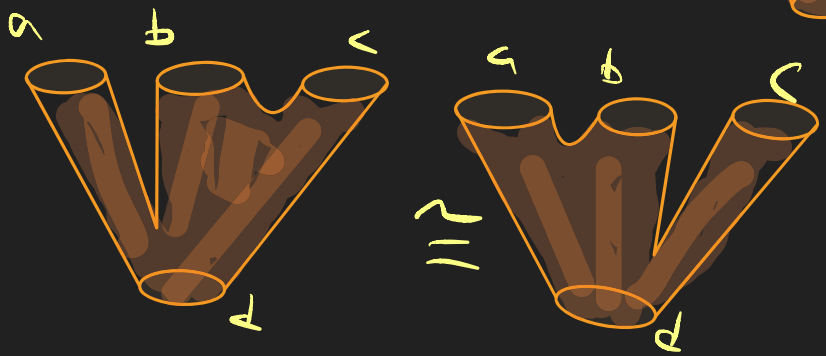
$\mathrm{Hom}(X_c, X_a \otimes X_b)$:



basis:



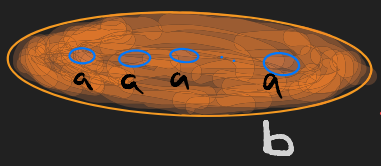
$1 \le \mu \le N_{ab}^{c}$



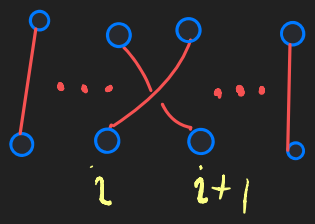
associativity

Braiding / Exchange Statistics

THE GATES

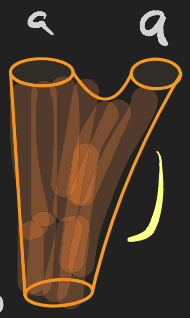
$MCG($ 

$) \cong B_n$: braid gp on n strands



$\longmapsto \sigma_i$

$\mathcal{H}($ 

$) \cong \mathcal{H}($ 

$)$

Yields unitary B_n rep. $\rho_{a,n} = \bigoplus_b \rho_{b,a,n}$ on $\bigoplus_b \mathcal{H}(D^2; b, a, \dots a)$

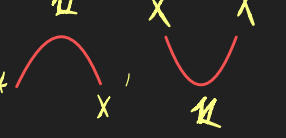
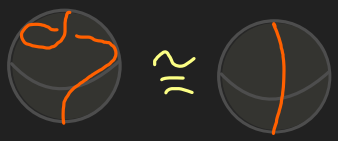
Ground state Degeneracy (multi-dimensionality)

$\mathcal{H}($  $) \cong \bigoplus_{i \in \mathcal{I}} \mathcal{H}($ 

$) \cong \mathbb{C}^{|\mathcal{I}|}$

Modular Category $\mathcal{C}$ (over $\mathbb{C}$)

Ribbon diagram graphical calculus

- **Monoidal**: $(\otimes, \mathbb{1}, \alpha)$:  axioms
- **Semisimple**: $X \cong \bigoplus_i X_i^{\oplus m_i}$
- **Linear**: $\mathrm{Hom}(X, Y) \in \mathrm{Vec}_{\mathbb{C}}$
- **Rigid**: $X^* \otimes X \to \mathbb{1} \to X \otimes X^*$ 
- **Braided**: $X \otimes Y \overset{c}{\cong} Y \otimes X$  axioms
- **Finite Rank**: $|\mathrm{Irr}(\mathcal{C})| < \infty$
- **Spherical**: $\theta_X : X \cong X$ 
- **Non-degenerate**: $S_{ab} =$  invertible