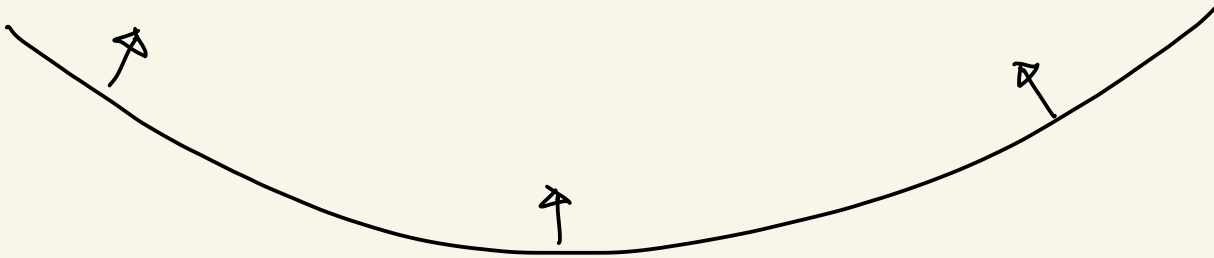
Asymptotic behavior of complete graphical curves under the curve shortening flow

Taehun Lee.



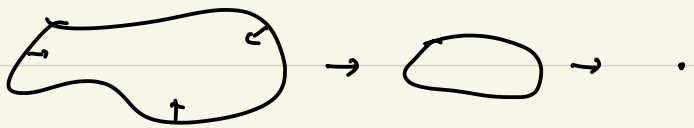
기하학적 흐름 겨울학교

January 17 ~ 19, 2022

Day 1 Graphness and curvature decay.

Reference Ecker-Huisken, '89 Annals.
"Mean curvature evolution of entire graphs"

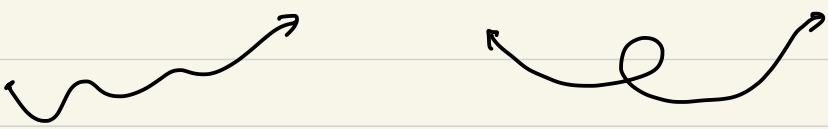
compact case:



in finite time.

non-compact case: CSF exists { in finite time. / for a long time

Consider a complete, non-compact, properly embedded curve γ in the plane $\mathbb{R}^2$.



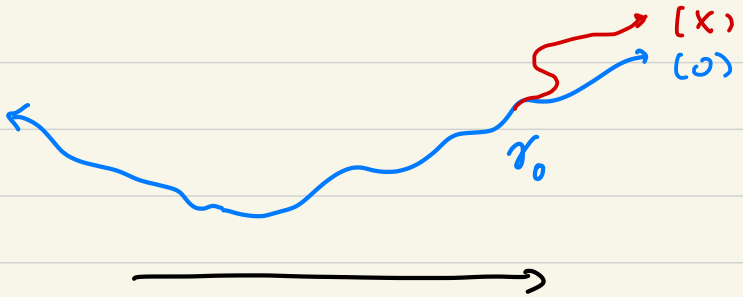
not complete. not embedded.

$$(CSF)\begin{cases} \frac{\partial}{\partial t}\gamma = \frac{\partial^2}{\partial s^2}\gamma & \text{in } \mathbb{R}\times(0,T) \\ \gamma(\cdot,0) = \gamma_0 \end{cases}$$

$\Gamma_0 = \gamma_0(\mathbb{R})$
$\Gamma_t = \gamma(\mathbb{R}, t)$.

$\gamma = \gamma(z,t)$
$\gamma_t = \frac{\partial}{\partial t}\gamma$
$\gamma_s = \frac{\partial}{\partial s}\gamma = \frac{1}{|\gamma_z|}\gamma_z$.

Initial data. γ_0: curve given by a graph.
(with linear growth and bounded curvature).



Fact CSF of γ_0 exists for all time ($T=\infty$)

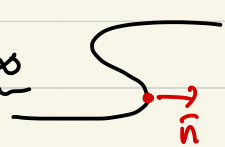
Goal: study the behavior of CSF
for large time t

Q. What is preserved? (or controlled)

① Preservation of graphness?

Quantify the graphness (w.r.t. x_2-axis).

$\langle \hat{n}, e_2 \rangle > 0$ or $0 < V := \langle \hat{n}, e_2 \rangle^{-1} < \infty$



Lemma $\partial_t v = \partial_{ss} v - k^2 v - 2\frac{\partial_s v^2}{v}$

pf) $v = \langle \hat{n}, e_2 \rangle^{-1}$.

$v_t = -\langle \hat{n}, e_2 \rangle^{-2} \langle \hat{n}_t, e_2 \rangle$.

$= -v^2 \langle \hat{n}_t, e_2 \rangle = v^2 k_s \langle \hat{t}, e_2 \rangle$.

$\hat{n}_t = a\hat{t} + b\hat{n} = -k_s \hat{t}$.

$b = \langle \hat{n}_t, \hat{n} \rangle = \frac{1}{2}(\underbrace{\langle \hat{n}, \hat{n} \rangle}_{=1})_t = 0$.

$a = \langle \hat{n}_t, \hat{t} \rangle = \langle \hat{n}, \hat{t} \rangle_t - \langle \hat{n}, \hat{t}_t \rangle$

$= -\langle \hat{n}, \partial_t \partial_s \gamma \rangle$ $\qquad \gamma_t = \gamma_{ss} = k n$.

$\partial_t \partial_s \gamma = \partial_t \left(\frac{1}{|\gamma_z|} \partial_z \gamma \right) = -\frac{1}{|\gamma_z|^2} \cdot \frac{\langle \gamma_z, \gamma_{tz} \rangle}{|\gamma_z|} \gamma_z + \frac{1}{|\gamma_z|} \partial_{tz} \gamma$.

$= -\frac{\langle \gamma_z, \partial_z (kn) \rangle}{|\gamma_z|^3} \gamma_z + \frac{1}{|\gamma_z|} \cdot \partial_z (k\hat{n})$.

$= -\langle \gamma_s, \partial_s (kn) \rangle \gamma_s + \partial_s (k\hat{n})$ $\quad (\gamma_s = \hat{t})$

$= -\langle \hat{t}, \partial_s k \hat{n} + k \partial_s \hat{n} \rangle \gamma_s + \partial_s (k\hat{n})$

$= -k \langle \hat{t}, -k\hat{t} \rangle \gamma_s + \partial_s \partial_t \gamma$.

$= \partial_s \partial_t \gamma + k^2 \gamma_s$

$\partial_t \partial_s \gamma = \partial_s \partial_t \gamma + k^2 \gamma_s$.

$\partial_s n = -k\hat{t}$

$a = -\langle \hat{n}, \partial_s (kn) + \underbrace{k^2 \gamma_s}_{\perp \hat{n}} \rangle = -k_s$

$$V_t = v^2 k_s \langle \hat{t}, e_2 \rangle$$

$$n_s = -k\hat{t}.$$

$$V_s = -v^2 \langle \hat{n}_s, e_2 \rangle := v^2 k \langle \hat{t}, e_2 \rangle.$$

$$V_{ss} = 2vk\langle \hat{t}, e_2 \rangle (v^2 k \langle \hat{t}, e_2 \rangle) + v^2 k_s \langle \hat{t}, e_2 \rangle + v^2 k \langle \hat{t}_s, e_2 \rangle. \qquad \hat{t}_s = k n$$

$$= \frac{2}{v} v_s^2 + V_t + v^2 k^2 \langle \hat{n}, e_2 \rangle.$$

$$V_t - V_{ss} = -\frac{2}{v} v_s^2 - k^2 v.$$

Maximum principle?

We would like to use maximum principle in PDE but the domain is non-compact.

$$\gamma : \mathbb{R} \times [0, \infty) \to \mathbb{R}^2 .$$

Remark It does not hold for the heat equation.

Tychonov's example:

$$\exists\, u \not\equiv 0 \text{ s.t. } \begin{cases} u_t = u_{xx} & \text{in } \mathbb{R} \times [0, \infty) \\ u(\cdot, 0) = 0 & \text{on } \mathbb{R}. \end{cases}$$

Thm A function $f = f(x,t)$ defined on a curve γ satisfies the inequality

$$f_t - f_{ss} \le a\, \partial_s f$$

for some constant a. If $a_0 = \sup_\gamma |a| < \infty$, then

$$\sup_{\Gamma_t} f \le \sup_{\Gamma_0} f$$

pf) Day 2

Cor v is bounded at $t=0 \Rightarrow v$ remains bounded by the same constant.

pf). $v_t - v_{ss} = -\frac{2}{v} v_s^2 - K^2 v \le 0.$

By max. principle, $\sup_{\Gamma_t} v \le \sup_{\Gamma_0} v < \infty.$

Cor Γ_0 is a graph w.r.t. $e_2 \Rightarrow \Gamma_t$: graph wrt e_2.

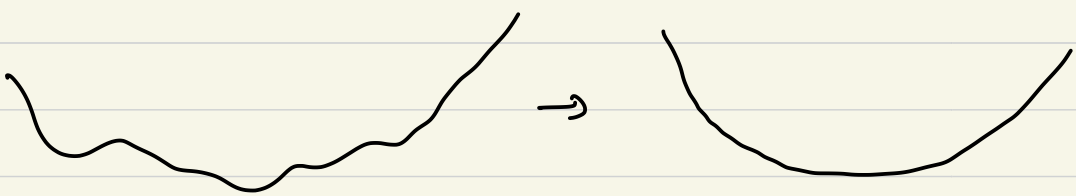
Def A curve has a linear growth $\langle x, u_{(n)} \rangle = r$

if $v \le c_1$ by some constant. $v = \sqrt{1+|Du|^2}$.

$v = \langle \hat{n}, e_2 \rangle^{-1}$.

Cor γ_0 has a linear growth $\Rightarrow \gamma(\cdot, t)$ also has a linear growth.

② Curvature bounds.



Lemma $(\partial_t - \partial_{ss})\kappa^2 = 2\kappa^4 - 2|\partial_s \kappa|^2$

pf) $\kappa = \langle \gamma_{ss}, \hat{n} \rangle$.

$\kappa_t = \langle \partial_{tss}\gamma, \hat{n} \rangle + \langle \gamma_{ss}, \hat{n}_t \rangle$

$\partial_{ts}\gamma = \partial_{st}\gamma + \kappa^2 \gamma_s$. ✦

$\kappa_t = \langle \partial_{sts}\gamma + \kappa^2 \gamma_{ss}, \hat{n} \rangle + \langle \kappa\hat{n}, -\kappa_s \hat{t} \rangle$.

$= \langle \partial_s(\partial_{st}\gamma + \kappa^2\gamma_s) + \kappa^2\gamma_{ss}, \hat{n} \rangle$.

$= \langle \partial_{sst}\gamma + \partial_s(\kappa^2)\gamma_s + 2\kappa^2\gamma_{ss}, \hat{n} \rangle$.

$\gamma_t = \gamma_{ss} = \kappa n$.

$= \langle \partial_{ss}(\kappa n) + \partial_s(\kappa^2)\hat{t} + 2\kappa^3\hat{n}, \hat{n} \rangle$.

$= \langle \kappa_{ss}\hat{n} + 2\kappa_s\hat{n}_s + \kappa\hat{n}_{ss}, \hat{n} \rangle + 2\kappa^3$.

$= \kappa_{ss} + \kappa \langle (-\kappa\hat{t})_s, \hat{n} \rangle + 2\kappa^3$.

$= \kappa_{ss} - \kappa^2 \langle \hat{t}_s, \hat{n} \rangle + 2\kappa^3$. ($\hat{t}_s = \kappa n$)

$= \kappa_{ss} - \kappa^3 + 2\kappa^3$.

$= \kappa_{ss} + \kappa^3$.

$\kappa_t = \kappa_{ss} + \kappa^3$. κ ↑

Cor If γ_0 has a linear growth and bounded curvature,

$$\sup_{\gamma_t} k^2 \leq \sup_{\gamma_0} k^2 v^2.$$

pf) • $(\partial_t - \partial_{s_t}) k^2 = 2k^4 - 2k_s^2$.

$(\partial_t - \partial_{s_t}) v = -\frac{2}{v} v_s^2 - k^2 v$.

• $(\partial_t - \partial_{s_t}) v^2 = -6 v_s^2 - 2k^2 v^2$.

$$(\partial_t - \partial_{s_t})(k^2 v^2) = k^2(-6v_s^2 - 2k^2 v^2) + v^2(2k^4 - 2k_s^2) - 2\partial_s(k^2)\partial_s(v^2).$$

$$= -6k^2 v_s^2 - 2k_s^2 v^2 - 2\partial_s(k^2)\partial_s(v^2).$$

$$= -2k^2 v_s^2 - 2k_s^2 v^2 - 4k^2 v_s^2 - 4kv\partial_s k \partial_s v - \frac{\partial_s(k^2 v^2)}{v^2}\partial_s v^2 + \frac{k^2}{v^2}\partial_s v^2 \partial_s v^2.$$

$$\leq -4k^2 v_s^2 - \frac{2v_s}{v}\partial_s(k^2 v^2) + 4k^2 v_s^2$$

$$= -\frac{2v_s}{v}\partial_s(k^2 v^2).$$

$$\sup_{\Gamma_t} (k^2 v^2) \leq \sup_{\Gamma_0} k^2 v^2 < \infty.$$

$$v = \langle \hat{n}, e_2 \rangle^{-1} \geq 1.$$

$$\sup_{\Gamma_t} k^2 \leq \sup_{\Gamma_t} k^2 v^2 \leq \sup_{\Gamma_0} k^2 v^2 < \infty.$$

Exercise γ_0 has a linear growth and bounded curvature.

i.e. $\sup_{\gamma_0} V \le C_1$, $\sup_{\gamma_0} K \le C_2$.

Show that $\sup_{\gamma_t} (t K^2 V^2) \le C$.

$\left(\sup_{\Gamma_t} K^2 \le C. \right)$

$t K^2 \le t K^2 V^2 \le C.$

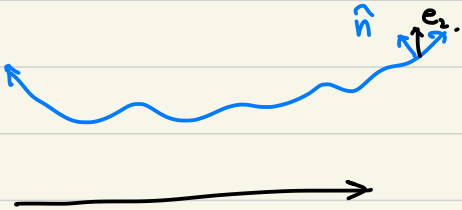
$K^2 \le \frac{C}{t}$. $\quad K \le \frac{C}{\sqrt{t}}$.

$K \to 0$ as $t \to \infty$. ⌟

Day 2 Rescaled solution and pf of max. principle.

Reference

- Ecker-Huisken, '89, Annals.
  "Mean curvature evolution of entire graphs"



① $v = \langle \hat{n}, e_2 \rangle^{-1} \leq C_1$.

② $k \leq C_2$. $(v \geq 1)$.

②′ $tk^2v^2 \leq C$.

$\Leftarrow$ $k \leq \frac{C}{\sqrt{t}}$.

pf) $(\partial_t - \partial_{ss})(tk^2v^2) \leq -\frac{2v_s}{v}\partial_s(tk^2v^2) + k^2v^2$. $\oplus$

$(\partial_t - \partial_{ss})\, v^2 = -6v_s^2 - 2k^2v^2$.

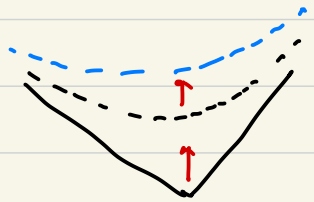
$(\partial_t - \partial_{ss})\left(tk^2v^2 + \frac{v^2}{2}\right) \leq \frac{2v_s}{v}\partial_s\left(tk^2v^2 + \frac{1}{2}v^2\right) - 8v_s^2$

$\leq \frac{2v_s}{v}\partial_s\left(tk^2v^2 + \frac{1}{2}v^2\right)$.

$\sup_{\Gamma_t}(tk^2v^2) \leq \sup_{\Gamma_t}\left(tk^2v^2 + \frac{v^2}{2}\right) \leq \sup_{\Gamma_0}\left(tk^2v^2 + \frac{v^2}{2}\right) = \frac{\sup_{\Gamma_0} v^2}{2}$

$\leq \frac{v_0^2}{2}$.

$\therefore k \le \frac{C}{\sqrt{t}}$.



Equality case

$$k \sim \frac{C}{\sqrt{t}}.$$

$$\gamma_t = k\hat{n} \sim \frac{C}{\sqrt{t}}$$

$$\int_0^{\infty} \frac{1}{\sqrt{t}}\,dt = 2\sqrt{t}\,\Big|_{t=0}^{\infty} = \infty.$$

The curve will move out to infinity.

→ We need rescaling.

(3) rescaling.

- example : heat equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} u.$$

$$\tilde{u}(x,t) = c\,u(ax, bt).$$

$$\partial_t \tilde{u} = cb\,\partial_t u(ax, bt)$$

$$\partial_{xx} \tilde{u} = ca^2\,\partial_{xx} u(ax, bt).$$

$$b \sim a^2.$$

$$u(ax, a^2 t).$$

- CSF

$$\begin{cases} \tilde{\gamma}(\cdot,\tau) = a(t)\,\gamma(\cdot,t) \\ \tau = b(t). \quad \to \quad \frac{d\tau}{dt} = b'. \end{cases}$$

Then $\tilde{\gamma} = a\gamma$, $\tilde{v} = \langle \hat{n}, e_2 \rangle^{\sim} = v$

$$\tilde{\kappa} = \frac{\kappa}{a}.$$

$$\partial_\tau \tilde{r} = \frac{\partial(ar)}{\partial t} \cdot \frac{dt}{d\tau} = (a'\gamma + a\gamma')\frac{1}{b'}.$$

$$= \frac{a'}{ab'}\tilde{r} + \frac{a}{b'}\kappa n$$

$$= \frac{a'}{ab'}\tilde{r} + \frac{a^2}{b'}\tilde{\kappa}\tilde{n}$$

$$\frac{a'}{ab'} \sim \frac{a^2}{b'} \sim 1$$

Note that $\begin{cases} a \searrow, & a(0) = 1. \\ b \nearrow, & b(0) = 1. \end{cases}$

$$a' = ka^3, \quad b' = la^2.$$

$$a' = -a^3, \quad b' = a^2.$$

$$\frac{da}{a^3} = -dt \Rightarrow -\frac{1}{2a^2} + \frac{1}{2} = -t.$$

$$a = \frac{1}{\sqrt{2t+1}}. \quad b' = \frac{1}{2t+1} \Rightarrow b = \frac{1}{2}\log(2t+1).$$

$\tau = \frac{1}{2}\log(2t+1)$ $\qquad \tilde{\gamma}(\tau) = \frac{1}{\sqrt{2t+1}}\gamma(t)$.

$\partial_\tau \tilde{\gamma} = \tilde{k}\tilde{n} - \tilde{\gamma}$.

- $\underline{\tau = -\log(-t)}$ (for compact case)

- $\tilde{r} = \frac{1}{\sqrt{2t+1}} r$. $\quad \tilde{V} = V \le C_1$.

$$\tilde{k} = \frac{k}{a} = \sqrt{2t+1}\, k$$

$\partial_\tau \tilde{\gamma} \to 0$. as $\tau \to \infty$. $\quad \tilde{k} \sim C$.

$\tilde{k} = \sqrt{2t+1}\, k$. $\Rightarrow k \sim \frac{C}{\sqrt{2t+1}}$

$\sim \frac{C}{\sqrt{t}}$.

Goal: behavior of $\tilde{\Gamma}_\tau$ for large τ.

possible candidate: If $\tilde{r}_\tau \to 0$ as $\tau \to \infty$,

$\tilde{r}_\tau = \tilde{k}\tilde{n} - \tilde{\gamma}$

At $\tau = \infty$, $\quad \tilde{k}\tilde{n} = \tilde{\gamma}$.

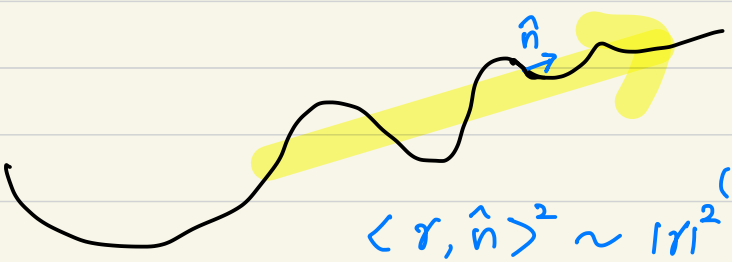
or $\tilde{\gamma}^\perp = \tilde{k}$. (elliptic eq).

Main Thm γ_0 satisfies $V \le C_1$, $k \le C_2$ at $t=0$.

In addition, $(*)\cdots$ $\langle \gamma, \hat{n} \rangle^2 \le C_3 (1+|\gamma|^2)^{1-\delta}$ $\delta \in (0,1)$ at $t=0$.

Then, the normalized solution $\tilde{\gamma}_z$ converges to a limiting curve $\tilde{\gamma}_\infty$ satisfying $\gamma_\infty^\perp = k$.

Rmk $(*)$ ~ small oscillation at infinity.



$\langle \gamma, \hat{n} \rangle^2 \sim |\gamma|^2 \not\le C(1+|\gamma|^2)^{1-\delta}$ $(*)$

Rmk $\tilde{\gamma}$ asymptotically converges to (expanding) self-similar solution.

$\gamma_0^\perp = k$ $\quad r = \sqrt{2t+1}\,\gamma_0$: sol of CSF.

④ Proof of Maximum Principle.

Recall.

$$\begin{cases} (\partial_t - \partial_{ss}) f \le a f_s. \\ a_0 := \sup_\gamma a < \infty. \end{cases} \Rightarrow \sup_{\Gamma_t} f \le \sup_{\Gamma_0} f.$$

- Backward heat kernel:

  For $(x_0, t_0) \in \mathbb{R}^2 \times \mathbb{R}$.

$$\rho(x,t) = \frac{1}{(4\pi(t_0-t))^{1/2}} \cdot e^{-\frac{|x_0-x|^2}{4(t_0-t)}}, \quad t < t_0.$$

- $\rho_t + \rho_{ss} = \rho\left(\frac{\langle x_0 - x, k\rangle}{t_0-t} - \frac{|(x_0-x)^\perp|^2}{4(t_0-t)^2}\right)$.

- Monotonicity formula. $f \ge 0$.

$$\frac{d}{dt}\left(\int_{\Gamma_t} f\rho\, ds\right) = \int_{\Gamma_t} (f_t - f_{ss})\rho\, ds$$

$$- \int_{\Gamma_t} f\rho \left| k + \frac{1}{2t}(x-x_0)^\perp \right|^2 ds$$

$$\le \int_{\Gamma_t} (f_t - f_{ss})\rho\, ds$$

pf $k := \sup_{\Gamma_0} f < \infty$.

Define $f_k = \max(f - k, 0)$. $f_k = 0$ at $t = 0$.

Goal : $f_k = 0$. ($\Leftrightarrow f \le k$).

$f_t - f_{ss} \le a f_s$.

$$\Rightarrow (f_k^2)_t - (f_k^2)_{ss} = 2 f_k \left((f_k)_t - (f_k)_{ss} \right) - 2 (\partial_s f_k)^2 .$$

$$\le 2 f_k \cdot a (f_k)_s - 2 (\partial_s f_k)^2 .$$

$$\le \frac{1}{2} (a f_k)^2 + 2 (\partial_s f_k)^2 - 2 (\partial_s f_k)^2 .$$

(by AM-GM).

$$\le \frac{1}{2} a_0^2 (f_k)^2 .$$

Apply the monotonicity formula to $(f_k)^2$.

$$\frac{d}{dt} \left(\int_{\Gamma_t} \underbrace{f_k^2}_{\| \; \because Q(f)} \rho \, ds \right) \le \int_{\Gamma_t} \left(\partial_t (f_k^2) - \partial_{ss} (f_k^2) \right) \rho \, ds$$

$$\le \frac{a_0^2}{2} \int_{\Gamma_t} f_k^2 \rho \, ds .$$

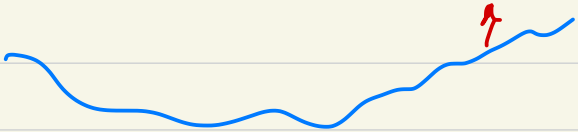
$$\frac{dQ}{dt} \le \frac{a_0^2}{2} Q. \Rightarrow \quad Q(t) \le Q(0) \cdot e^{\frac{a_0^2}{2} t}.$$

$$Q(0) = \int_{\Gamma_t} f_k^2 \rho \, ds = 0. \Rightarrow Q(t) = 0.$$

$$Q(t) = \int_{\Gamma_t} f_k^2 \rho \, ds.$$

$\therefore f_k = 0.$ $\qquad \sup_{\Gamma_t} f \le \sup_{\Gamma_0} f.$

Day 3 : Asymptotic behavior (pf of main thm).



$\gamma_t \sim \frac{C}{\sqrt{t}}$.

$$\begin{cases} \tilde{\gamma}(z) = \frac{1}{\sqrt{2t+1}}\,\gamma(t). \\ z = \frac{1}{2}\log(2t+1). \end{cases}$$

Main Thm γ_0 satisfies

1) $v = \langle \hat{n}, e_2 \rangle^{-1} \le C_1$.
2) $k \le C_2$.
3) $\langle \gamma, n \rangle^2 \le C_3 (1 + |\gamma|^2)^{1-\delta}$ $0 < \delta \le 1$.

Then the normalized curve $\tilde{\gamma}_s$ converges for $\tau \to \infty$ to a limiting curve $\tilde{\gamma}_\infty$ satisfying $\tilde{K} = \langle \tilde{\gamma}, \tilde{n} \rangle$.

Rmk 3) is necessary. $\exists$ a curve that oscillates at infinity without asymptotic convergence.

Known results :

1) $\tilde{v} \le C$. for $t > 0$.
2) $\tilde{K} \le C$ for $t > 0$.

Goal: $\tilde{k} - \langle \tilde{\gamma}, \tilde{n} \rangle \to 0$ as $z \to \infty$.

Claim: $(\tilde{k} - \langle \tilde{\gamma}, \tilde{n} \rangle)^2 \le C e^{-\beta z}$ for some $\beta > 0$. (locally).

What is eq. of $(\tilde{k} - \langle \tilde{\gamma}, \tilde{n} \rangle)^2 e^{\beta z}$?

σ denotes the arc length of $\tilde{\gamma}$.

Lemma (Evolution equations)

1) $(\partial_z - \partial_{\sigma\sigma}) \langle \tilde{\gamma}, \tilde{n} \rangle = \tilde{k}^2 \langle \tilde{\tau}, \tilde{n} \rangle + 2\tilde{k} - \langle \tilde{\tau}, \tilde{n} \rangle$.

2) $(\partial_z - \partial_{\sigma\sigma}) \tilde{k} = \tilde{k}^3 + \tilde{k}$.

3) $(\partial_z - \partial_{\sigma\sigma}) \tilde{v}^2 = -2\tilde{k}^2 \tilde{v}^2 - 6\tilde{v}_s^2$.

$$\tilde{\gamma}_z = \tilde{k}\tilde{n} - \tilde{\tau}.$$

pf is left to the audience.

Lemma $\langle \gamma, n \rangle^2 \le C(t) \cdot (1 + |\gamma|^2)^{1-\delta}$

$\langle \tilde{\gamma}, \tilde{n} \rangle^2 \le C(s) (1 + |\tilde{\gamma}|^2)^{1-\delta}$.

pf) $(\partial_t - \partial_{ss})\langle r,n\rangle^2 = 2\langle r,n\rangle^2 k^2 + 4k\langle r,\hat{n}\rangle - 2|\langle r,n\rangle_s|^2.$

$\leq C\left(\langle r,n\rangle^2 + 1\right) - 2|\langle r,n\rangle_s|^2.$

$Q = \dfrac{\langle r,n\rangle^2}{(1+|r|^2)^{1-\delta}}$

Define $\eta = 1 + |r|^2 + 2t.$

$\eta_t = 2\langle r, r_t\rangle + 2 = 2k\langle r,n\rangle + 2.$

$\eta_s = 2\langle r, r_s\rangle.$

$\eta_{ss} = 2\langle r, \underbrace{r_{ss}}_{=kn}\rangle + 2\langle \underbrace{r_s}_{=\hat{t}}, r_s\rangle.$

$= 2k\langle r,n\rangle + 2.$

$\eta_t - \eta_{ss} = 0.$

$(\partial_t - \partial_{ss})\eta^{\delta-1} \leq 0.$

$(\partial_t - \partial_{ss})Q = \langle r,n\rangle^2 \cdot (\partial_t - \partial_{ss})\eta^{\delta-1}$

$+ \eta^{\delta-1}(\partial_t - \partial_{ss})\langle r,n\rangle^2.$

$- 2\,\partial_s \eta^{\delta-1}\,\partial_s\langle r,n\rangle^2.$

$\leq C(Q+1).$

$(\partial_t - \partial_{ss})(\underbrace{Qe^{-Ct} - Ct}) \leq 0.$

$Q \leq (C + Ct)e^{Ct}.$

$$(\partial_t - \partial_{\sigma\sigma})\left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle\right) = \tilde{k}^3 + \tilde{k} + \langle \tilde{r}, \tilde{n} \rangle - \tilde{k}^2 \langle \tilde{r}, \tilde{n} \rangle - 2\tilde{k}.$$

$$= (\tilde{k}^2 - 1)\left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle\right).$$

$$(\partial_t - \partial_{\sigma\sigma})\left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle\right)^2 = 2(\tilde{k}^2 - 1)\left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle\right)^2 - 2\left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle\right)_\sigma^2$$

Define $f = \left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle\right)^2 e^{\beta t}$.

$$(\partial_t - \partial_{\sigma\sigma}) f = e^{\beta t}\left(2(\tilde{k}^2 - 1)\left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle\right)^2 - 2\left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle\right)_\sigma^2\right) + \beta e^{\beta t}\left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle\right)^2.$$

$$= (2\tilde{k}^2 - 2 + \beta) f - 2e^{\beta t}\left(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle_\sigma\right)^2.$$

$$\left(k \sim \frac{C}{\sqrt{t}} \overset{?}{\Rightarrow} \sqrt{t}\, k \approx \tilde{k} \overset{?}{\sim} C.\right)$$

$$(\partial_t - \partial_{\sigma\sigma})(f v^2) = (-2 + \beta)(f v^2) - \frac{2}{v}\partial_\sigma v \, \partial_\sigma (f v^2).$$

$(-2+\beta) \le 0.$

$$\partial_\sigma v = \partial_\sigma \langle \tilde{n}, e_2 \rangle = -v^2 \langle \tilde{n}_\sigma, e_2 \rangle = v^2 k \langle \tilde{t}, e_2 \rangle, \quad \tilde{n}_\sigma = -k\tilde{t}$$

$$\left|\frac{2}{v}\partial_\sigma v\right| \le 2\tilde{v}\tilde{k} \le C.$$

Take $\beta < 2$, and apply the max. principle to $f v^2$.

Then
$$\sup_{\tilde{P}_\varepsilon} (\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle)^2 v^2 e^{\beta z}$$
$$\leq \sup_{\tilde{P}_0} (\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle)^2 v^2 \underbrace{e^{\beta z}}_{=1.}$$
$$= \sup_{P_0} (k - \langle r, n \rangle)^2 v^2 .$$

not bounded.

Are we happy ? No.

$$Q := \frac{(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle)^2 \tilde{v}^2 e^{\beta z}}{(1 + \alpha |\tilde{r}|^2)^{1-\delta}} \cdot \leq \sup_{\Gamma_0} Q.$$

$$\leq 2k^2 + 2\langle r, n \rangle^2.$$

$$= \sup_{\Gamma_0} \frac{(k - \langle r, n \rangle)^2 v^2}{(1 + \alpha |r|^2)^{1-\delta}}$$

$$\leq \sup_{\Gamma_0} C k^2 v^2 + \frac{C \langle r, n \rangle^2 v^2}{(1 + \alpha |r|^2)^{1-\delta}}$$

$$(\tilde{k} - \langle \tilde{r}, \tilde{n} \rangle)^2 \leq C e^{-\beta z} (1 + \alpha |\tilde{r}|^2)^{1-\delta}$$

exponentially fast convergence on compact sets.

Thus we need to show that

$$Q_z - Q_{\sigma\sigma} \leq a \cdot Q_\sigma.$$

- $\eta = 1 + \alpha |\tilde{r}|^2.$

  $\eta_z - \eta_{\sigma\sigma} = -2\alpha (1 + |r|^2)$

  $(\partial_z - \partial_{\sigma\sigma}) \eta^{\varepsilon - 1} = 2(1 - \varepsilon) \eta^{\varepsilon - 1}.$

$$g = f v^2 \eta^{\varepsilon - 1}$$

$$(\partial_z - \partial_\sigma)\, g \le a\, g_v + (\beta + C\sqrt{\alpha} - 2\varepsilon)\, g.$$

take small α, β s.t. $\beta + C\sqrt{\alpha} < 2\varepsilon$

$$\le a \cdot g_\sigma.$$